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Extended Local Convergence for the Chebyshev Method under the Majorant Condition

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Short Research Article

Abstract

In this article, we present the study on local convergence behaviour of Chebyshev's method, which is a third order iterative method used to solve a non-linear system in Banach space locale. In contrast to the earlier works, we establish the convergence using restricted-majorant conditions. As a result, we get better convergence radius and more tighter error estimates in comparison to the previous researches. Suitable numerical examples complement the theory.

Keywords: Non-linear equations; Fréchet derivative; local convergence; banach space.

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1 Introduction

Finding a locally unique solution of the system of non-linear equations of the form

$$F(x) = 0 \tag{1.1}$$

is a major problem with extensive applications in the field of mathematical and engineering sciences. In most of the cases non-linear equations and systems arising from mathematical modeling of physical systems do not have exact solutions. Because of this problem, scientists and researchers have focused on proposing iterative methods for solving non-linear systems. Newton's method is a popular iterative process for dealing with non-linear equations. Many novel, higher-order iterative strategies for dealing with nonlinear equations have been studied and are currently being used in recent years[1, 2, 3, 4, 5, 6, 7, 8, 9]. Also, numerous research articles have been published recently which deals with the study of local and semi-local convergence properties of various iterative methods and also give results on computable convergence domain and estimates on error bounds[2, 3, 4, 5, 6, 7, 8, 9].

Chebyshev's method is a well-known iterative method for solving equations of type (1.1) which is cubically convergent to the root x^*. The method is given as

$$x_{n+1} = x_n - [I + \mathfrak{L}_F(x_n)]\beta_n F(x_n), \quad n \geq 0 \tag{1.2}$$

where

$$\beta_n = F'(x_n)^{-1},$$

$\mathfrak{L}_F(x_n)$ is defined by

$$\mathfrak{L}_F(x_n) = \frac{1}{2}\beta_n F^{''}(x_n)\beta_n F(x_n), \quad x_n \in B_1,$$

$F : \Omega \subset B_1 \to B$ is a continuous Fréchet differentiable non-linear operator, Ω is a non-empty open convex set, B_1 and B are Banach spaces.

The method (1.2) has been well studied by distinct researchers for its local and semi-local convergence properties using different strategies such as majorizing sequences, recurrent relations and many more as given in [8, 10, 11, 12, 9, 13]. The convergence properties of the Chebyshev's method used for finding multiple polynomial roots were studied by [14] and [15]. A new type of majorant conditions were introduced by [6, 7] and used to study the local and semi-local convergence behaviour of the cubically convergent Halley's method. Recently, [5] followed the same criteria as in [6] and used the majorant functions to study the local convergence analysis of the method (1.2).

In the present work, we have provided better convergence domains than in [5] by using a new type of restricted majorant conditions. It is worth noting that the articles in [5, 6, 7] did not provide such formulas and locations either. The originality and novelty of our work are derived from this fact. The same advantages can be obtained if our methodology is applied to other single or multi-step methods using the inverses of divided differences or derivatives along the same lines.

The other contents of this material can be summarized as follows: Section 2 discusses the development of majorant functions for the method (1.2). Section 3 discusses the local convergence properties of the presented method (1.2). Numerical testing of convergence outcomes are placed in Section 5. Concluding remarks are also stated.

2 Majorant Functions

Let us consider that there exists $x^* \in \Omega$ such that $L = F'(x^*)^{-1} \in L(B_1, B)$ with $F(x^*) = 0$. Suppose $R > 0$ be such that $U(x^*, R) \subset \Omega$.

Some majorant conditions are introduced and compared with each other.

Definition 2.1. The operator $F^{''}$ satisfies the center-majorant condition on the ball $U(x^*, R)$ if there exists a function $h_0 : [0, R) \to \mathbb{R}$ with $h_0 \in C^2\,[0, R)$ such that

$$\|L(F^{''}(x) - F^{''}(x^*))\| \leq h_0^{''}(\|x - x^*\|) - h_0^{''}(0) \tag{2.1}$$

for all $x \in U(x^*, R)$.

Suppose that the equation

$$h_0^{''}(t) = 0 \tag{2.2}$$

has a smallest solution $r \in (0, R)$.

Definition 2.2. The operator $F^{''}$ satisfies the restricted-majorant condition on the ball $U(x^*, r)$ if there exists a function $h : [0, r) \to \mathbb{R}$ with $h \in C^2\,[0, r)$ such that

$$\|L(F^{''}(y) - F^{''}(x))\| \leq h^{''}(\|y - x\| + \|x - x^*\|) - h^{''}(\|x - x^*\|) \tag{2.3}$$

for all $x, y \in U(x^*, r)$, $\|y - x\| + \|x - x^*\| < r$.

Definition 2.3. The operator $F^{''}$ satisfies the majorant condition on the ball $U(x^*, R)$ if there exists a function $h_1 : [0, R) \to \mathbb{R}$ with $h_1 \in C^2\,[0, r)$ such that

$$\|L(F^{''}(y) - F^{''}(x))\| \leq h_1^{''}(\|y - x\| + \|x - x^*\|) - h_1^{''}(\|x - x^*\|) \tag{2.4}$$

for all $x, y \in U(x^*, r)$; $\|y - x\| + \|x - x^*\| < R$.

Remark 2.1. It follows by these definitions that for all $t \in [0, r)$

$$h_0(t) \leq h_1(t) \tag{2.5}$$

$$h(t) \leq h_1(t), \tag{2.6}$$

since $r \in (0, R)$. The functions h_0 and h_1 were used in [5] to show the local convergence of the method (1.2). Notice that $h_0 = h_0(\Omega)$, $h_1 = h_1(\Omega)$, but $h = h(\Omega, [0, r))$. It follows that the tighter function h can replace h_1 in all the results in [5]. That is why we report only the extended results, omit the proofs and focus on the effect of this modifications in the numerical examples. It is expected that the convergence radius will be at least as large and the error distances $\|x_n - x^*\|$ at least as tight, since the estimates (2.5) and (2.6) hold.

The following conditions are used:

- (A_1) $h_0^{''}(0) > 0$, $h_0^{'}(0) = -1$, $h^{''}(0) \geq 0$.
- (A_2) $h_0^{''}$ is convex in $[0, r)$, $h_0^{''}$ and $h^{''}$ are strictly increasing in $[0, r)$.
- (A_3) $h_0^{'}$ has zeroes in $(0, r)$.
  Denote by ρ_0 the smallest such zero.
- (A_4) $\|LF^{''}(x^*)\| \leq h_0^{''}(0)$.

The following auxiliary results are needed.

Lemma 2.1. *Suppose that the conditions (A_1)-(A_4) hold. Then, the following assertions hold:*

- *(i)* $h_0^{'}$ *is strictly increasing and strictly convex in* $[0, r)$
  and
- *(ii)* $h_0^{'}(t) \in (-1, 0)$ *for all* $t \in (0, r)$.

Proof. Simply replace the function h_1 by the function h in the proof of the Lemma 2.1 in [5]. □

Lemma 2.2. *Suppose that $\|x-x^*\| \leq t < r$ and (2.1) holds on the interval $[0,r)$. Then, the following assertions*

$$\|F'(x)^{-1}F'(x^*)\| \leq -\frac{1}{h_0'(\|x-x^*\|)} \leq -\frac{1}{h_0'(t)}$$

and

$$\|F'(x^*)^{-1}F''(x)\| \leq h_0''(\|x-x^*\|) \leq h_0''(t).$$

Proof. The proof is given in the Lemma 2.2 in [5] with h_1 replacing h_0. □

Notice that the weaker assertion (2.1) is used in Lemma 2.2. If (2.4) is used as in [5], then we get the less tight estimate

$$\|F'(x)^{-1}F'(x^*)\| \leq -\frac{1}{h_1(\|x-x^*\|)} \leq -\frac{1}{h_1(t)}.$$

3 Local Convergence

3.1 Analysis

Define the functions on the interval $[0,r)$ by

$$\begin{aligned}
\delta_1(t) &= (2+h_0'(t))h_0''(t)t - 2(h_0'(t))^2, \\
\delta_2(t) &= \frac{(2+h_0'(t))h_0''(t)t}{2(h_0'(t))^2}, \\
\delta_3(t) &= \frac{1}{1-\delta_2(t)}, \\
\delta(t) &= -\frac{1}{2}\delta_2(t)\frac{h_0''(t)}{h_0'(t)}\left[\frac{2\delta_2(t)+\delta_2(t)^2}{t} - \frac{1}{2}\frac{h_0''(t)}{h_0'(t)}\right] \\
\text{and} \quad \delta_4(t) &= \delta(t)t^2 - 1.
\end{aligned}$$

It was shown in [5] that the functions δ_1 has a smallest solution $r_1 \in (0,\rho_0)$ and that the function δ_4 has a smallest solution $r^* \in (0,r_1)$. Hence, we reach the local convergence result for the method (1.2).

Theorem 3.1. *Suppose that the conditions (A_1)-(A_4) hold. Then, the method (1.2) for $x_0 \in U(x^*,r^*)$ is well defined in the ball $U(x^*,r^*)$, remains in $U(x^*,r^*)$ for all $n \geq 0$ and converges to x^*. Moreover, the following assertions hold*

$$\|x_{n+1}-x^*\| \leq \delta(r^*)\|x_n-x^*\|^3 \text{ for all } n = 0,1,2,\ldots. \tag{3.1}$$

Remark 3.1. The special cases in Section 5 extend immediately along the same lines.

Our technique determines a ball $U(x^*,r)$ where the iterates lie that is more precise than the ball $U(x^*,R)$ used in [5]. This also allows the construction of the restricted function h that replaces the function h_1 in [5] leading to the aforementioned advantages.

Next, we present a second way of replacing h_1 by a tighter function that may give better results.

Definition 3.1. The operator F' satisfies the center-majorant condition on the ball $U(x^*,R)$ if there exists a function $\bar{h}_0 : [0,R) \to \mathbb{R}$ with $\bar{h}_0 \in C^2[0,R]$ such that

$$\|L(F''(x)-F''(x^*))\| \leq \bar{h}_0'(\|x-x^*\|) - \bar{h}_0'(0). \tag{3.2}$$

Suppose that the equation

$$\bar{h}_0(t) = 0 \tag{3.3}$$

has a smallest solution $\bar{r} \in (0, R)$. Set

$$\rho = min\{r, \bar{r}\}. \tag{3.4}$$

Moreover, suppose that (2.3) holds but for some function $\bar{h}^{''}$ as $h^{''}$ but defined on the interval $[0, \rho)$.
If

$$\bar{r} \leq r \text{ and } \bar{h}^{''}(t) \leq h^{''}(t) \quad \text{for all } t \in [0, \rho)\,, \tag{3.5}$$

then the obtained results can be rewritten with $\rho, \bar{h}$ replacing r, h and the new radii and error bounds shall be at least as better than our earlier ones.

It is worth noticing that the iterates x_n exist and belong in $U(x^*, r_0)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq -\frac{1}{\bar{h}_0'(\|x - x^*\|)} \leq -\frac{1}{\bar{h}_0'(t)} \tag{3.6}$$

for $\|x - x^*\| \leq t < \rho_0$ by using (3.2) instead (2.1).

Remark 3.2. If the function $h_0(t)$ is replaced by $\bar{h}_0(t)$ in the conditions (A_1)-(A_4), and also in the definitions of the functions $\delta_1, \delta_2, \delta_3, \delta$ and δ_4, we then denote the resultant functions as $\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3, \bar{\delta}$ and $\bar{\delta}_4$ respectively. Moreover, r, r_1 and r^* are replaced by $\rho, \bar{r}_1$ and $\bar{r}^*$.

Thus we arrive at the following theorem:

Theorem 3.2. *Suppose that the conditions (A_1)-(A_4) hold. Then, the method (1.2) for $x_0 \in U(x^*, \bar{r}^*)$ is well defined in the ball $U(x^*, \bar{r}^*)$, remains in $U(x^*, \bar{r}^*)$ for all $n \geq 0$ and converges to x^*. Moreover, the following assertions hold*

$$\|x_{n+1} - x^*\| \leq \bar{\delta}(\bar{r}^*)\|x_n - x^*\|^3 \text{ for all } n = 0, 1, 2, \ldots. \tag{3.7}$$

That is, Chebyshev's method (1.2) is cubically convergent to x^*.

4 Discussion

The local convergence of the Chebyshev method has also been discussed in [5], where it was achieved using majorant type conditions on the operator $F^{''}$. In the present paper, we deal with the local convergence of the Chebyshev method and extend the convergence domain using restricted majorant conditions. These extensions become possible without additional conditions and under weaker conditions than in [5]. This allows the determination of a more precise region where the iterates are located. The confirmation of our convergence findings can be observed from the numerical illustrations in Section 5. Also, it is worth noticing that this approach does not take into account the method itself. Hence, we can extend the same technique to obtain the same benefits on other iterative methods using inverses of linear operators. The method discussed in this paper find applications in the problems listed in [16, 17, 18, 19, 20, 21].

5 Numerical Illustrations

Example 5.1. *Choose $B_1 = B = \mathbb{R}$, $\Omega = \bar{U}(x^*, R) = \bar{U}(0, 1)$. Define F on Ω by*

$$F(x) = (e^{x_1} - 1, \frac{e-1}{6}x_2^3 + x_2, \frac{x_3^3}{6} + x_3)^T$$

for $x = (x_1, x_2, x_3)^T$. *We have that* $x^* = (x_1^*, x_2^*, x_3^*)^T = (0,0,0)^T$ *is a zero of* F. *We obtain the first and second Fréchet derivative as follows:*

$$F'(x) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & \frac{(e-1)}{2}u_2^2 + 1 & 0 \\ 0 & 0 & \frac{u_3^2}{2} + 1 \end{bmatrix}$$

and

$$F^{''}(x) = \left[\begin{array}{ccc|ccc|ccc} e^{u_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (e-1)u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_3 \end{array}\right].$$

Now, we can observe that $F'(x^*) = F'(x^*)^{-1} = diag(1,1,1)$. *Thus, we have* $L = F'(x^*)^{-1} = diag(1,1,1)$ *and* $\|L\| = 1$. *Now, we can easily calculate the parameter values as:*

$$\|F'(x^*)^{-1}(F^{''}(x) - F^{''}(x^*))\| \leq (e-1)\|x - x^*\|,$$

so we have $h_0^{''}(t) = (e-1)t$, $r = \frac{1}{e-1} < R = 1$. *Also,*

$$\|F'(x^*)^{-1}(F^{''}(y) - F^{''}(x))\| \leq e^{\frac{1}{e-1}}\|y - x\|,$$

hence we get $h^{''}(t) = e^{\frac{1}{e-1}}t < h_1^{''}(t) = et$.

Therefore, we get

$$r = 0.581977, \quad r_1 = 0.320365 \quad \textit{and} \quad r^* = 0.308453.$$

Now, using the center-majorant condition on F', *we get*

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq e^{\frac{1}{e-1}}\|y - x\|.$$

Thus, we have $\bar{h}^{''}(t) = e^{\frac{1}{e-1}} = h^{''}(t)$ *and* $r = \bar{r} = \rho = \frac{1}{e-1}$. *Hence, we get*

$$\rho = 0.581977, \quad \bar{r}_1 = 0.385386, \quad \textit{and} \quad \bar{r}^* = 0.371669.$$

Example 5.2. *Next, we consider the non-linear integral equation of the Hammerstein-type given by*

$$F(x)(\gamma) = x(\gamma) - 5H(x)(\gamma),$$

where H *is any function such that*

$$H'(x)(\gamma) = \gamma \int_0^1 \phi x^3(\phi) d\phi$$

defined on $B_1 = B = C[0,1]$, *the space of all continuous functions on the interval* $[0,1]$ *and let* $\Omega = \bar{U}(x^*, R) = \bar{U}(0,1)$. *Then, we get the Fréchet derivative* F' *as*

$$F'(x(\chi))(\gamma) = \chi(\gamma) - 15\gamma \int_0^1 \phi x^2(\phi)\chi(\phi) d\phi \quad \textit{for} \quad \textit{all} \quad x \in \Omega.$$

We can observe that $x^* = x^*(\gamma) = 0$ *is solution of* $F(x)$. *Then, by applying the conditions* (A_1)*-*(A_4), *we have* $h_0^{''}(t) = 7.5t$. *Also, observe that* $r = 0.579796 < R = 1$. *Moreover, we get* $h^{''}(t) = 2t < h_1^{''}(t) = 15t$. *Therefore, we get*

$$r = 0.579796, \quad r_1 = 0.157198, \quad \textit{and} \quad r^* = 0.151331.$$

Again, by using the center-majorant condition on F', *we get* $\bar{h}^{''}(t) = h^{''}(t) = 2t$ *and* $r = \bar{r} = \rho = 0.579796$. *Thus, we get*

$$\bar{r}_1 = 0.193937 \quad \textit{and} \quad \bar{r}^* = 0.187046.$$

6 Conclusions

The local convergence behaviour of the Chebyshev's method under restricted majorant conditions is presented in this paper. On comparing the results obtained in the numerical illustrations, we can conclude that new technique presented in our work determines a more precise convergence ball than obtained in earlier works. Also, we can observe that more tighter error estimates are obtained in comparison to the earlier works. The technique used in the analysis does not really depend on the method. Therefore, it can be used on other single and multi-step methods using inverses of divided differences or derivatives along the same lines. This will be the topic of our future research.

Competing Interests

Authors have declared that no competing interests exist.

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