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Perfect i.i.d Processes

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This note proves a theorem about i.i.d. i.e. independent and indentically distributed processes, when the index space is a measure space. The statement of the problem corresponding to the theorem proved in this paper appears in [1], in which the concept of a sample distribution limit corresponds to the concept of a perfect i.i.d process in this paper.
Theorems proved in this theme, regarding existing and non-existence, have been shown in the economics literature, when the index set is $[0, 1]$, in [2], [3], [4], [5]. The approach taken in this paper is perhaps, surprisingly elementary. We may apply standard measure extension theorems to show existence. These may be found in [6], [7].

Keywords: Index space; probability space; measurable functions; IID process.

1 Model

Suppose $(R, \mathcal{R}, \rho)$ is a probability space that we will call the state space; and $(P, \mathcal{P}, \pi)$ be a probability space called the index space. The following definitions convey the prime theme of the paper.

Definition 1.1. *A setting is defined as a pair $< (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) >$ consisting of a state space and an index space.*

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Definition 1.2. *A measure-preserving transformation is any measurable map $\psi : P \to R$ such that*

$$(\forall B \in \mathcal{R})(\pi(\{p : \psi(p) \in B\}) = \rho(B)). \tag{1.1}$$

Definition 1.3. *A setting $< (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) >$ is said to admit a perfect i.i.d process if there exists a probability space $< \Omega, \mathcal{F}, \mathbb{P} >$ and measurable functions $\{X_p\}_{p \in P}$ where $X_p : \Omega \to R$ such that*

1. *For any finite $P' \subset P$ and collection $\{B_p\}_{p \in P'} \subseteq \mathcal{R}$ we have that*

$$\mathbb{P}(\cap_{p \in P'}\{X_p \in B_p\}) = \prod_{p \in P'} \rho(B_p).$$

2. *There exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 1$ and*

$$A \subseteq \{\omega \in \Omega : X_p(\omega) \text{ is measure-preserving in } p\}.$$

Definition 1.4. *An index space $(P, \mathcal{P}, \pi)$ will be called fine if*

1. *For every $p \in P$, $\{p\} \in \mathcal{P}$.*
2. *For every $p \in P$, $\pi(\{p\}) = 0$.*

It follows immediately that any fine index space $(P, \mathcal{P}, \pi)$ is uncountably infinite. The following is the main theorem of the paper.

Theorem 1.1. *Let $< (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) >$ be a setting. Suppose that the index space $(P, \mathcal{P}, \pi)$ is fine. Further, suppose that there exists a measure-preserving transformation $\psi : P \to R$. Then, the setting $< (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) >$ admits a perfect i.i.d process.*

Proof. The proof proceeds in a few steps.

Step 1 : We argue that given a measure-preserving transformation $\psi : P \to R$; a countable subset $\hat{P} \subseteq P$; and any function $\hat{\psi} : \hat{P} \to R$, the map $\psi' : P \to R$ defined as

$$\psi'(p) = \begin{cases} \hat{\psi}(p) & \text{if } p \in \hat{P} \\ \psi(p) & \text{otherwise} \end{cases} \tag{1.2}$$

is also a measure-preserving transformation. This is true since the probability space $(P, \mathcal{P}, \pi)$ is assumed to be fine. As $\mathcal{P}$ includes all singleton sets, it follows that $\hat{P} \in \mathcal{P}$. Hence, ψ' is measurable. Further, since singletons have zero probability according to the probability measure π, implying that $\pi(\hat{P}) = 0$ (due to countable additivity), it follows that ψ' is also a measure-preserving transformation.

Step 2 : We now define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define Ω as

$$\Omega := \{\psi : P \to R : \psi \text{ is measure-preserving}\}.$$

By assumption, we have that $\Omega \neq \emptyset$. For a finite subset $\hat{P} \subseteq P$ and collection of sets $\{B_p\}_{p \in \hat{P}} \subseteq \mathcal{R}$, define the set

$$< \hat{P}, \{B_p\}_{p \in \hat{P}} > := \{\psi \in \Omega : (\forall p \in \hat{P})(\psi(p) \in B_p)\}.$$

The collection of all such sets is defined as

$$\mathcal{S} := \Big\{ < \hat{P}, \{B_p\}_{p \in \hat{P}} >: \text{ finite } \hat{P} \subseteq P \text{ and collection } \{B_p\}_{p \in \hat{P}} \subseteq \mathcal{R} \Big\}.$$

We show that $\mathcal{S}$ is a semi-ring (see [8]). Further, we show that the following set function $\mathbb{P}'$ defines a measure on $\mathcal{S}$

$$\mathbb{P}'(< \hat{P}, \{B_p\}_{p \in \hat{P}} >) = \prod_{p \in \hat{P}} \rho(B_p).$$

We first prove that $\mathcal{S}$ is a semi-ring. This follows simply from the following facts.

1. Suppose that we have a set of the form $< \hat{P}, \{B_p\}_{p\in\hat{P}} >$ such that $B_p = \emptyset$ for some $p \in \hat{P}$. This immediately implies that $< \hat{P}, \{B_p\}_{p\in\hat{P}} >= \emptyset \in \mathcal{S}$.
2. Suppose that $< \hat{P}, \{B_p\}_{p\in\hat{P}} >, < \hat{P}', \{B'_p\}_{p\in\hat{P}'} > \in \mathcal{S}$. Then, it is simple to prove that

   $$< \hat{P}, \{B_p\}_{p\in\hat{P}} > \cap < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >=< \hat{P}\cup\hat{P}', \{B_p\cap B'_p\}_{p\in\hat{P}\cap\hat{P}'}\cup\{B_p\}_{p\in\hat{P}\setminus\hat{P}'}\cup\{B'_p\}_{p\in\hat{P}'\setminus\hat{P}} > .$$

   Hence, we have shown that $< \hat{P}, \{B_p\}_{p\in\hat{P}} > \cap < \hat{P}', \{B'_p\}_{p\in\hat{P}'} > \in \mathcal{S}$.
3. Suppose that $< \hat{P}, \{B_p\}_{p\in\hat{P}} >, < \hat{P}', \{B'_p\}_{p\in\hat{P}'} > \in \mathcal{S}$. We wish to show that $< \hat{P}, \{B_p\}_{p\in\hat{P}} > \setminus < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >$ may be represented as a finite union of pairwise disjoint sets in $\mathcal{S}$. Note that

   $$< \hat{P}, \{B_p\}_{p\in\hat{P}} > \setminus < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >=< \hat{P}, \{B_p\}_{p\in\hat{P}} > \cap(\Omega\setminus < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >).$$

   Then, it follows that

   $$\Omega\setminus < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >= \bigcup_{Q\subseteq\hat{P}';Q\neq\emptyset} < \hat{P}', \{\Omega\setminus B'_p\}_{p'\in Q} \cup \{B'_{p'}\}_{p'\in\hat{P}'\setminus Q} >,$$

   which is a finite union of disjoint sets in $\mathcal{S}$. Hence, from 2., we have indeed shown that it is the case $< \hat{P}, \{B_p\}_{p\in\hat{P}} > \setminus < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >$ is a finite union of disjoint sets in $\mathcal{S}$ as it may be represented as

   $$< \hat{P}, \{B_p\}_{p\in\hat{P}} > \setminus < \hat{P}', \{B'_p\}_{p\in\hat{P}'} >$$

   $$= \bigcup_{Q\subseteq\hat{P}';Q\neq\emptyset} < \hat{P}, \{B_p\}_{p\in\hat{P}} > \cap < \hat{P}', \{\Omega\setminus B'_p\}_{p'\in Q} \cup \{B'_{p'}\}_{p'\in\hat{P}'\setminus Q} > .$$

   and we have proved that $\mathcal{S}$ is a semiring.

We show that $\mathbb{P}'$ defines a measure on $\mathcal{S}$. Suppose, we have a countable collection of pairwise disjoint sets $\{< \hat{P}^i, \{B^i_p\}_{p\in\hat{P}^i} >\}_{i=1}^{\infty} \subseteq \mathcal{S}$ and a set $< \hat{P}, \{B_p\}_{p\in\hat{P}} > \in \mathcal{S}$ such that the following holds

$$< \hat{P}, \{B_p\}_{p\in\hat{P}} >= \bigcup_{i=1}^{\infty} < \hat{P}^i, \{B^i_p\}_{p\in\hat{P}^i} > . \tag{1.3}$$

We prove it also holds that

$$\mathbb{P}'(< \hat{P}, \{B_p\}_{p\in\hat{P}} >) = \bigcup_{i=1}^{\infty} \mathbb{P}'(< \hat{P}^i, \{B^i_p\}_{p\in\hat{P}^i} >).$$

We prove this as follows. Define the set $\hat{P}^* = \hat{P} \cup (\cup_{i=1}^{\infty}\hat{P}^i)$. Since $\hat{P}^*$ is a countable union of finite sets, it is at most countable. We denote the probability space $(\otimes_{p\in\hat{P}^*}R, \otimes_{p\in\hat{P}^*}\mathcal{R}, \otimes_{p\in\hat{P}^*}\rho)$ as the product measure space where $\otimes_{p\in\hat{P}^*}R = \{\hat{\psi} : \hat{\psi} : \hat{P}^* \to R\}$ is the product space corresponding to R with index set $\hat{P}^*$; $\otimes_{p\in\hat{P}^*}\mathcal{R}$ is the product σ-field ; $\otimes_{p\in\hat{P}^*}\rho$ is denoted as the associated product measure (see [9]).

Define the map $T : \mathcal{S} \to \otimes_{p\in\hat{P}^*}\mathcal{R}$ as

$$T(< \hat{P}', \{B_p\}_{p\in\hat{P}'} >) = \{\hat{\psi} : \hat{P}^* \to R : (\forall p \in \hat{P}' \cap \hat{P}^*)(\hat{\psi}(p) \in B_p)\}.$$

Hence, it follows that $T(<\hat{P}', \{B_p\}_{p\in\hat{P}}>) \in \otimes_{p\in\hat{P}^*}\mathcal{R}$. If we have $\hat{P}' \subseteq \hat{P}^*$, then by the definition of the product measure space $(\otimes_{p\in\hat{P}^*}R, \otimes_{p\in\hat{P}^*}\mathcal{R}, \otimes_{p\in\hat{P}^*}\rho)$, we have that :

$$\otimes_{p\in\hat{P}^*}\rho(T(<\hat{P}', \{B_p\}_{p\in\hat{P}'}>)) = \prod_{p\in\hat{P}'}\rho(B_p) = \mathbb{P}'(<\hat{P}', \{B_p\}_{p\in\hat{P}'}>).$$

Hence, from countable additivity of the product measure $\otimes_{p\in\hat{P}^*}$ and by applying equality 1.3, to prove that $\mathbb{P}'$ defines a measure on $\mathcal{S}$, it suffices to show that for the pairwise disjoint finite collection of sets given by $\{<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>\}_{i=1}^{\infty} \subseteq \mathcal{S}$, the following holds :

$$T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) = \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>).$$

We prove that $T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) \subseteq \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. Suppose that we have that it is the case that $\hat{\psi} \in T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. By equality 1.3, we have that $\hat{\psi} \in T(<\hat{P}, \{B_p\}_{p\in\hat{P}}>)$. Then, from Step 1, it follows that there exists a measure-preserving transformation ψ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^*$. Hence, by equality 1.3 we get that $\psi \in <\hat{P}, \{B_p\}_{p\in\hat{P}}> = \cup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>$. This implies there exists a $\hat{P}^i$ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^i$. Hence, this shows that $\hat{\psi} \in \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. We have, hence established that $T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) \subseteq \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$.

This way, appropriately we may also that prove that, $\bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) \subseteq T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. Suppose $\hat{\psi} \in \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. Then, there exists a $\hat{P}^i$ such that $\hat{\psi} \in T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. From Step 1 again, it follows that there exists a measure-preserving transformation ψ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}$. This means that $\psi \in <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>$. By equality 1.3, this implies $\psi \in <\hat{P}, \{B_p\}_{p\in\hat{P}}>$. Hence, $\hat{\psi} \in T(<\hat{P}, \{B_p\}_{p\in\hat{P}}>)$, which means $\hat{\psi} \in T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$.

Hence, $\mathbb{P}'$ defines a measure on $\mathcal{S}$.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is then completely defined as $\mathcal{F} := \sigma(\mathcal{S})$ and $\mathbb{P}$ is defined to be the extension of the measure $\mathbb{P}'$ on the defined σ-field $\mathcal{F}$ by the Caratheodory Extension Theorem.

Step 3 : We finish the proof of the theorem. We have defined the appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $p \in P$, define $X_p(\psi) := \psi(p)$ and $A := \Omega$.

This completes the proof of the theorem.

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Competing Interests

Author has declared that no competing interests exist.

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