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Research Article

Two-Dimensional Fredholm Integro-Differential Equation with Singular Kernel and Its Numerical Solutions

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In this paper, we introduce the nonlinear Fredholm integro-differential equation of the second kind with singular kernel in two-dimensional NT-DFIDE. Furthermore, we study this new equation numerically. The existence of a unique solution of the equation is proved. The numerical results of NT-DFIDE are obtained by the following methods: Toeplitz matrix method (TMM) and product Nystrom method (PNM). The given applications showed the efficiency of these methods.

1. Introduction

There are many well-written texts on the theory and applications of integral equations in different sciences. From 1960 to the present day, many new numerical methods have been developed for the solution of many types of integral equations, such as the Toeplitz matrix method, the product Nystrom method, the Galerkin method, the Runge-Kutta method, and the Block by block method (see Linz [1], Baker et al. [2], and Delves and Mohamed [3]). Authors, in [4], used the iterative method based on quadrature formula to solve T-DNFIEs. Fattahzadeh, in [5], solved T-DLFIE and NFIE of the first kind based on the Haar wavelet. Authors, in [6], solved T-DIE of the first kind by multistep method. Pashazadeh Atabakan et al. [7] solved linear FIDEs using the well-known Chebyshev-Gauss-Lobatto collocation points. Rabbani and B. Zarali, in [8], solved a system of LIDEs with initial conditions by using modified decomposition method. Arqub et al., in [9], solved the FIDE numerically in a reproducing kernel Hilbert space. Pandey, in [10], studied the LFIDEs numerically. Erfanian and Zeidabadi, in [11], solved the NFIDE numerically. Saadatmandi and Dehghan, in [12], discussed the higher-order LFIDDE with variable coefficients. In [13], Al-Bugami studied the numerical algorithm for the solution of nonlinear two-dimensional Volterra integral equation arising from torsion problem. In [14], Al-Bugami studied N-FIDE of the second kind in two dimensions with continuous kernel. The authors, in [15–18], studied the different types of integral equations with singular kernel numerically. In [19], Khajehnasiri studied the numerical solution of nonlinear 2D Volterra–Fredholm integro-differential equations by two-dimensional triangular function. The authors, in [20], used bivariate Jacobi polynomials for solving Volterra partial integro-differential equations with the weakly singular kernel.

The N-FIDE of the second kind in two dimensions with singular kernel has application as the contact problem in the theory of elasticity with singular kernel, where the coefficient bed of the compressible materials is neglected.

Consider

$$Z''(m,n) + A(m,n)Z'(m,n) + B(m,n)Z(m,n) = Q(m,n) - \lambda\int_a^b\int_c^d L(m,n,t,s)\gamma(t,s,Z(t,s))\text{dtds}. \tag{1}$$

Under the boundary conditions,

$$Z(a,c) = q_1 r_1, \quad Z(b,d) = q_2 r_2. \tag{2}$$

Z is the unknown function, which represents solution of the NT-DIDE (1). Z' and Z'' are the first and second derivatives of Z, respectively. Also, λ is a constant. $A(m, n)$, $B(m, n) \in C[a, b] \times C[c, d]$ with its derivatives. $p(m, n, t, s)$ is the singular kernel.

Integrating (1), twice, and then letting $m = b$ and $n = d$, Equation (1) reduces to

$$Z(m, n) = f(m, n) + \lambda \int_a^b \int_c^d p((m-t), (n-s)) \gamma(t, s, Z(t, s)) \text{dtds}. \tag{3}$$

Equation (3) represents T-DFIDE in the nonlinear case.

2. Existence of a Solution of NT-DFIDE

Consider the following conditions, by using the Picard method:

(i) The kernel $p((m-n), (t-s)) \in C([a, b] \times C[c, d])$ and satisfies the discontinuity conditions

$$\left[\int_a^b \int_c^d |p((m-t), (n-s))|^2 \text{dtds}\right]^{1/2} = A < \infty (A \text{ is a constant}) \tag{4}$$

(ii) $f(m, n) \in C[a, b] \times C[c, d]$, and its norm is defined as

$$\|f(m, n)\| = \max_{m,n \in J} \int_a^b \left[\int_c^d f^2(m, n) \text{dm}\right]^{1/2} \text{dn} = M, J = C[a, b] \times C[c, d] \tag{5}$$

(iii) The unknown function $Z(m, n)$ satisfies the Lipschitz condition with respect to its argument and its normal is defined as

$$\|Z(m, n)\| = \left[\int_a^b \int_c^d |Z(m, n)|^2 \text{dmdn}\right]^{1/2} \leq C \|Z\|_2 \tag{6}$$

Theorem 1. *The solution of the T-DFIE with singular kernels exists and is unique under the condition*

$$|\lambda| < \frac{|\mu|}{A}. \tag{7}$$

We state the following lemmas.

Lemma 2. *The infinite series $\sum_{i=0}^{\infty} \psi_i(m, n)$ is uniformly convergent to a continuous solution function $Z(m, n)$.*

Proof. We construct the sequence of the functions $Z_n(m, n)$ as

$$\mu Z_{n*}(m, n) = f(m, n) + \lambda \int_a^b \int_c^d p(m-t, n-s) Z_{n*-1}(t, s) \text{dtds}, \tag{8}$$

with

$$Z_0(m, n) = f(m, n), \tag{9}$$

$$\psi_{n*}(m, n) = Z_{n*}(m, n) - Z_{n*-1}(m, n), \tag{10}$$

where

$$Z_n(m, n) = \sum_{i=0}^{n^*} \psi_i(m, n), n^* = 1, 2, \cdots (\psi_0(m, n) = f(m.n)). \tag{11}$$

Using the properties of the modulus, formula (10) takes the form

$$\|\psi_{n*}(m, n)\| \leq \left|\frac{\lambda}{\mu}\right| \left\| \int_a^b \int_c^d p(m-t, n-s)(Z_{n*-1}(t, s) - Z_{n*-2}(t, s)) \text{dtds} \right\|. \tag{12}$$

With the aid of formula (10), we have

$$\|\psi_{n*}(m, n)\| \leq \left|\frac{\lambda}{\mu}\right| \|\psi_{n*-1}(s, t)\| \left\| \int_a^b \int_c^d p(m-t, n-s) \text{dtds} \right\|. \tag{13}$$

Then,

$$\|\psi_{n*}(m, n)\| \leq \frac{1}{|\mu|} A |\lambda| \|\psi_{n*-1}(t, s)\|, \tag{14}$$

which takes the form

$$\|\psi_{n*}(m, n)\| \leq \alpha \|\psi_{n*-1}\|, \left(\alpha = \frac{1}{|\mu|} A |\lambda| < 1\right). \tag{15}$$

Let, in (12), $n = 1$, we get

$$\|\psi_1(m, n)\| \leq \left|\frac{\lambda}{\mu}\right| \left\| \left(\int_a^b \int_c^d p^2(m-t, n-s) \text{dtds} \right)^{1/2} \max_{m,n \in J} \int_a^b \int_c^d \left(\psi_0^2(t, s) dt\right)^{1/2} \text{ds} \right\|. \tag{16}$$

Using (i) and (ii), we have

$$\|\psi_1(m, n)\| \leq \frac{1}{|\mu|} \text{AM} |\lambda| \leq \alpha M. \tag{17}$$

So, by the mathematical induction method, we get

$$\|\psi_{n*}(m, n)\| \leq \alpha^{n*} M, n^* = 0, 1, 2, \cdots \tag{18}$$

Then, we can write

$$Z(m,n)=\sum_{i=0}^{\infty}\psi_i(m,n). \tag{19}$$

The series (19) is uniformly convergent since the terms $\psi_i(m,n)$ are dominated by

$$\begin{aligned} E_{yz}&=\frac{1}{2}\left(\frac{\partial\psi}{\partial y}+x\right)\alpha(x,t),\\ E_{xz}&=\frac{1}{2}\left(\frac{\partial\psi}{\partial x}-y\right)\alpha(x,t)\quad(2.26). \end{aligned} \tag{20}$$

□

Lemma 3. *A continuous function $\phi(x,y)$ represents a unique solution of Equation (3).*

Proof. We first prove that $\phi(x,y)$ defined by (19) satisfies Equation (3), and then, we set

$$Z(m,n)=Z_{n'}(m,n)+q_{n'}(m,n),\ \left(q_{n'}(m,n)\longrightarrow 0 \text{ as } n'\longrightarrow\infty\right). \tag{21}$$

Then, we get

$$\begin{aligned} Z(m,n)-q_{n'}(m,n)&=\frac{1}{\mu}f(m,n)+\frac{\lambda}{\mu}\int_a^b\int_c^d p(m-t,n-s)\\ &\quad\cdot(Z(t,s)-q_{n-1}(t,s))\text{dtds}. \end{aligned} \tag{22}$$

Then, we have

$$\begin{aligned} &\max_{m,n\in J}\left|Z(m,n)-\frac{1}{\mu}f(m,n)-\frac{\lambda}{\mu}\int_a^b\int_c^d p(m-t,n-s)Z(t,s)\text{dtds}\right|\\ &\le\max_{m,n\in J}|q_{n'}(m,n)|+\frac{|\lambda|}{|\mu|}\int_a^b\int_c^d p(m-t,n-s)\max_{m,n\in J}|q_{n'-1}(t,s)|\text{dtds}. \end{aligned} \tag{23}$$

The previous inequality takes the form

$$\begin{aligned} &\left\|Z(m,n)-\frac{1}{\mu}f(m,n)-\frac{\lambda}{\mu}\int_a^b\int_c^d p(m-t,n-s)\text{dtds}\,Z(t,s)\text{dtds}\right\|\\ &\le\|q_{n'}(m,n)\|+\alpha\|q_{n'-1}(t,s)\|,\ \left(\alpha=\frac{1}{|\mu|}A|\lambda|\right). \end{aligned} \tag{24}$$

To show that $\phi(x,y)$ is the only solution, we assume that $\phi(x,y)$ is also a continuous solution of (3); then, we get

$$\left|Z(m,n)-\tilde{Z}(m,n)\right|\le\int_a^b\int_c^d|p(m-t,n-s)|\left|Z(t,s)-\tilde{Z}(t,s)\right|\text{dtds}. \tag{25}$$

Equation (25) leads to

$$\left\|Z(m,n)-\tilde{Z}(m,n)\right\|\le\alpha\left\|Z(t,s)-\tilde{Z}(t,s)\right\|,\ \left(\alpha=\frac{A}{|\mu|}|\lambda|<1\right). \tag{26}$$

Since $\int_{x_0}^{x_1}f(x)dx=(h(E-1)/\ln(E))f_0$ (1.74), then (26) is true only if $\phi(x,y)=\phi(x,y)$; that is, the solution of (3) is unique. □

3. Numerical Processors for Solving NT-DFIDE

3.1. TMM. The numerical experiments are prepared to illustrate these considerations, and the estimating error is calculated.

Consider the linear integral Equation (3), and let the domain of integration $\Omega=[a,b]\times[c,d]$.

$$Z(m,n)=f(m,n)+\lambda\int_a^b\int_c^d p((m-t),(n-s))\gamma(t,s,Z(t,s))\text{dtds}. \tag{27}$$

We can write the integral term of Equation (27) as

$$\begin{aligned} &\int_a^b\int_c^d p((m-t),(n-s))\gamma(t,s,Z(t,s))\text{dtds}\\ &=\sum_n{}^{\prime=-N}N-1\sum_m{}^{\prime=-M}Mp((m-t),(n-s))\gamma(t,s,Z(t,s))\text{dtds}, \end{aligned} \tag{28}$$

where in $\int_{x_0}^{x_1}f(x)dx=h/12[5f_0+8f_1-f_2]+(h^4/24)f_0^3(\delta)$, $\delta\in(x_0,x_1)$ (1.77), we get

$$\begin{aligned} &\int_{n'h}^{n'h+h}\int_{m'h}^{m'h+h}p((m-t),(n-s))\gamma(t,s,Z(t,s))\text{dtds}\\ &=A_{n'm'}(m,n)Z\left(n'h,m'h\right)+B_{n'm'}\left(n'h+h,m'h+h\right)+R, \end{aligned} \tag{29}$$

where $A_{n'm'}(m,n)$ and $B_{n'm'}(m,n)$ are two arbitrary functions to be determined and R is the error.

Then, we put $Z(t,s)=1.1,ts$ in Equation (29). If the error R is assumed negligible, then we obtain

$$A_{n'm'}(m,n)=\frac{1}{h}\left[\frac{\left(n'h+h\right)\left(m'h+h\right)I}{\left(n'h+m'h+h\right)}-\frac{J}{\left(n'h+m'h+h\right)}\right], \tag{30}$$

$$B_{n'm'}(m,n)=\frac{1}{h}\left[\frac{J}{\left(n'h+m'h+h\right)}-\frac{\left(n'h\right)\left(m'h\right)I}{\left(n'h+m'h+h\right)}\right]. \tag{31}$$

Hence, Equation (28) becomes

$$\begin{aligned}&\int_{-a}^{a}\int_{-a}^{a}Z(m-t,n-s)\gamma(t,s,Z(t,s))dtds\\&=\sum_{n'=-N}^{N-1}\sum_{m'=-M}^{M-1}\left[A_{n',m'}(m,n)\gamma\left(n'h,m'h,Z\left(n'h,m'h\right)\right)\right.\\&\quad\left.+B_{n',m'}(m,n)\right]\gamma\left(n'h,m'h,Z\left(n'h,m'h\right)\right)\\&=\sum_{n'=-N}^{N-1}\sum_{n'=-N}^{N-1}A_{n',m'}(m,n)\gamma\left(n'h,m'h,Z\left(n'h,m'h\right)\right)\\&\quad+\sum_{n'=-N}^{N}\sum_{m'=-M}^{M}B_{(n'-1)(m'-1)}(m,n)\gamma\left(n'h,m'h,Z\left(n'h,m'h\right)\right)\\&=\sum_{n'=-N}^{N}\sum_{m'=-M}^{M}D_{n',m'}(m,n)\gamma\left(n'h,m'h,Z\left(n'h,m'h\right)\right),\end{aligned} \tag{32}$$

where

$$D_{n',m'}(m,n)=\begin{cases}A_{-N}(m,n)\ n'=m'=-N,\\A_{n'}(m,n)+B_{n'-1}(m,n) & -N<n'=m'<N,\\B_{N-1}(m,n) & n'=m'=N\quad.\end{cases} \tag{33}$$

Thus, the integral Equation (3) becomes

$$\begin{aligned}Z(m,n)-\lambda\sum_{n'=-N}^{N}\sum_{m'=-M}^{M}D_{n',m'}(m,n)\\\cdot\gamma\left(n'h,m'h,Z\left(n'h,m'h\right)\right)=f(m,n).\end{aligned} \tag{34}$$

If we put $m=kh$, $n=lh$, then we get

$$\begin{aligned}&Z_{k,l}-\lambda\sum_{n'=-N}^{N}\sum_{m'=-M}^{M}D_{n',m'kl}\gamma\left(n'h,m'h,Z_{n'm'}\right)\\&=f_{kl}-N\le k\le N,-M\le l\le M.\end{aligned} \tag{35}$$

3.2. The PNM. Consider the T-DFIE of the second kind.

$$Z(m,n)=f(m,n)+\lambda\int_a^b\int_c^d p((m-t),(n-s))\gamma(t,s,Z(t,s))\text{dtds}. \tag{36}$$

When the kernel $p((m-t),(n-s))$ is a singular term, we can often factor out the singularity in k by writing

$$p((m-t),(n-s))=\tilde{k}((m-t),(n-s))k((m-t),(n-s)), \tag{37}$$

where $\tilde{k}((m-t),(n-s)), k((m-t),(n-s))$ are badly behaved and well-behaved functions of their arguments, respectively. We rewrite (36) in the form

$$\begin{aligned}Z(m,n)=f(m,n)+\lambda\int_a^b\int_c^d\tilde{k}((m-t),(n-s))\\\cdot k((m-t),(n-s))\gamma(t,s,Z(t,s))\text{dtds}.\end{aligned} \tag{38}$$

We approximate the integral term in (38) when $m=m_i$, $n=n_i$ by

$$\begin{aligned}&\int_a^b\int_a^b\tilde{k}(m_i-t,n_i-s)k(m_i-t,n_i-s)\\&\cdot\gamma(t,s,Z(t,s))\text{dtds}\approx\sum_{j=0}^{N}\sum_{l=0}^{M}w_{ij}w_{il}k\left(m_i-t_j,n_i-s_l\right)\\&\cdot\gamma\left(t_j,s_l,Z\left(t_j,s_l\right)\right),\end{aligned} \tag{39}$$

where w_{ij} and w_{il} are the weights. We may write

$$\begin{aligned}&\int_a^b\int_a^b\tilde{k}(m_i-t,n_i-s)k(m_i-t,n_i-s)\gamma(t,s,Z(t,s))\\&\cdot\text{dtds}\approx\sum_{j=0}^{N-2/2}\sum_{l=0}^{M-2/2}\int_{t_{2j}}^{t_{2j+2}}\int_{s_{2l}}^{s_{2l+2}}\tilde{k}(m_i-t,n_i-s)\\&\cdot k(m_i-t,n_i-s)\text{dtds},\end{aligned} \tag{40}$$

where $m_i=t_i=n_i=s_i=a+\text{ih}$, $i=0,1,\cdots,N$ with $h=(b-a)/N$ and N even. Now, if we approximate the nonsingular part of the integrand, we find

$$\begin{aligned}&\int_a^b\int_a^b p(t_i-t,s_i-s)\tilde{k}(t_i-t,s_i-s)\gamma(t,s,Z(t,s))\text{dtds}\\&=\sum_{j=0}^{N-2/2}\sum_{l=0}^{M-2/2}\int_{t_{2j}}^{t_{2j+2}}\int_{s_{2l}}^{s_{2l+2}}\tilde{k}(t_i-t,s_i-s)\\&\quad\times\left\{\frac{(t_{2j+1}-t)(s_{2l+1}-s)(t_{2j+2}-t)(s_{2l+2}-s)}{(2h^2)(2h^2)}k\right.\\&\quad\cdot\left(t_i-t_{2j},s_i-s_{2l}\right)\gamma\left(t,s,Z\left(t_{2j},s_{2l}\right)\right)\\&\quad+\frac{(t-t_{2j})(s-s_{2l})(t_{2j+2}-t)(s_{2l+2}-s)}{(h^2)(h^2)}k\\&\quad\cdot\left(t_i-t_{2j+1},s_i-s_{2l+1}\right)\gamma\left(t,s,Z\left(t_{2j+1},s_{2l+1}\right)\right)\\&\quad+\frac{(t-t_{2j})(s-s_{2l})(t-t_{2j+1})(s-s_{2l+1})}{(2h^2)(2h^2)}k\\&\quad\left.\cdot\left(t_i-t_{2j+2},v_i-v_{2l+2}\right)\gamma\left(t,s,Z\left(t_{2j+2},s_{2l+2}\right)\right)\right\}\text{dtds}\\&=\sum_{j=0}^{N}\sum_{l=0}^{M}w_{ij}w_{il}k\left(t_i-s_j,t_i-s_l\right)\gamma\left(t,s,Z\left(t_i,s_l\right)\right),\end{aligned} \tag{41}$$

Table 1: Numerical values and absolute error values by using TMM and PNM, $n = 20$, at linear case $k = 1$.

λ	m	n	u_{Exact}	TMM		PNM	
				U_{TMM}	$\text{Error}_{\text{TMM}}$	U_{PNM}	$\text{Error}_{\text{PNM}}$
	-1.0	-1.0	1.0000	0.99120	0.008797	0.98999	0.010000
	-0.8	-0.8	0.6400	0.62784	0.012154	0.62790	0.012094
	-0.6	-0.6	0.3600	0.35051	0.009489	0.34924	0.010750
	-0.4	-0.4	0.1600	0.15502	0.004974	0.15224	0.007754
	-0.2	-0.2	0.0400	0.03874	0.001256	0.03590	0.004092
0.01	0	0	0.0000	0.00229	0.000229	0.00006	0.000060
	0.2	0.2	0.0400	0.03874	0.001256	0.43951	0.003951
	0.4	0.4	0.1600	0.15502	0.004974	0.16759	0.007599
	0.6	0.6	0.3600	0.35051	0.009489	0.37048	0.010488
	0.8	0.8	0.6400	0.62784	0.012154	0.65197	0.011978
	1.0	1.0	1.0000	0.99120	0.008797	1.00991	0.009916
	-1.0	-1.0	1.0000	0.99912	0.000878	0.99899	0.001001
	-0.8	-0.8	0.6400	0.63879	0.001212	0.63879	0.001209
	-0.6	-0.6	0.3600	0.35905	0.000945	0.35892	0.001076
	-0.4	-0.4	0.1600	0.15950	0.000495	0.15922	0.000776
	-0.2	-0.2	0.0400	0.03987	0.000124	0.03958	0.000411
0.001	0	0	0.0000	0.00002	0.000023	0.67×10^{-5}	0.67×10^{-5}
	0.2	0.2	0.0400	0.03987	0.000124	0.04039	0.000393
	0.4	0.4	0.1600	0.15950	0.000495	0.16075	0.000759
	0.6	0.6	0.3600	0.35905	0.000945	0.36104	0.001048
	0.8	0.8	0.6400	0.63878	0.001212	0.64119	0.001197
	1.0	1.0	1.0000	0.99912	0.000878	1.00099	0.000991

where $t_j = jh, t_{j+1} = (j+1)h, s_j - s_{j+1} = s_l - s_{l+1} = -h,$ and the weight functions $w_{ij} w_{il}$ are given by

$$\begin{aligned} w_{i,0} w_{i,0} &= \frac{1}{4h^2} \int_{t_0}^{t_2} \int_{s_0}^{s_2} k(t_i - t, s_i - s)(t_1 - t) \\ &\quad \cdot (s_1 - s)(t_2 - t)(s_2 - s)\text{dtds}, \\ w_{i,2j+1} w_{i,2l+1} &= \frac{1}{h^4} \int_{t_{2j}}^{t_{2j+2}} \int_{s_{2l}}^{s_{2l+2}} k(t_i - t, s_i - s)\left(t - t_{2j}\right) \\ &\quad \cdot (s - s_{2l})\left(t_{2j+2} - t\right)(s_{2l+2} - s)\text{dtds}, \\ w_{i,2j} w_{i,2l} &= \frac{1}{4h^4} \int_{t_{2j-2}}^{t_{2j}} \int_{s_{2l-2}}^{s_{2l}} k(t_i - t, s_i - s)\left(t - t_{2j-2}\right) \\ &\quad \cdot \left(s - s_{2j-2}\right)\left(t - t_{2j-1}\right)\left(s - s_{2j-1}\right)\text{dtds} \\ &\quad + \frac{1}{4h^4} \int_{t_{2j}}^{t_{2j+2}} \int_{s_{2l}}^{s_{2l+2}} k(t_i - t, s_i - s)\left(t_{2j+1} - t\right) \\ &\quad \cdot \left(s_{2j+1} - s\right)\left(t_{2j+2} - t\right)\left(s_{2j+2} - s\right)\text{dtds}, \\ w_{i,N} w_{i,M} &= \frac{1}{4h^4} \int_{t_{N-2}}^{t_N} \int_{s_{M-2}}^{s_M} k(t_i - t, s_i - s)(t - t_{N-2}) \\ &\quad \cdot (s - s_{M-2})(t - t_{N-1})(s - s_{M-1})\text{dtds}. \end{aligned} \tag{42}$$

4. Numerical Problems

4.1. Application For a Logarithmic Kernel. Consider

$$\begin{aligned} &Z''(m,n) + AZ'(m,n) + BZ(m,n) \\ &\quad = f(m,n) - \lambda \int_{-1}^{1} \int_{-1}^{1} \ln |m - t| \ln |n - s| (Z(t,s))^k \text{dtds}. \end{aligned} \tag{43}$$

Under the boundary conditions,

$$\begin{aligned} Z(-1,-1) &= 1, \\ Z(1,1) &= 1. \end{aligned} \tag{44}$$

The exact solution is $Z(m,n) = m.n$; if we set $k = 1$ in (43), we get

$$\begin{aligned} &Z''(m,n) + AZ'(m,n) + BZ(m,n) \\ &\quad = f(m,n) - \lambda \int_{-1}^{1} \int_{-1}^{1} \ln |m - t| \ln |n - s| (Z(t,s)) \text{dtds}, \end{aligned} \tag{45}$$

which is called the LT-DFIDE, and if we set $k \geq 2$ in (43), we obtained the NT-DFIDE, of the second kind, with $\lambda = 0.01, 0.001, A = (-2/m + n), B = 1$. We solve Equation (43)

Table 2: Numerical values and absolute error values by using TMM and PNM, $n = 20$, at nonlinear case $k = 2$.

λ	m	n	u_{Exact}	TMM		PNM	
				U_{TMM}	$\text{Error}_{\text{TMM}}$	U_{PNM}	$\text{Error}_{\text{PNM}}$
0.01	-1.0	-1.0	1.0000	0.99939	0.000605	0.998146	0.0018535
	-0.8	-0.8	0.6400	0.63839	0.001604	0.637179	0.0028200
	-0.6	-0.6	0.3600	0.35881	0.001182	0.357505	0.0024945
	-0.4	-0.4	0.1600	0.15939	0.000605	0.158220	0.0017792
	-0.2	-0.2	0.0400	0.03962	0.000375	0.038720	0.0012790
	0	0	0.0000	0.00026	0.000260	0.001018	0.0010188
	0.2	0.2	0.0400	0.03962	0.000375	0.038870	0.0011290
	0.4	0.4	0.1600	0.15939	0.000605	0.158457	0.0015420
	0.6	0.6	0.3600	0.35881	0.001182	0.357663	0.0021336
	0.8	0.8	0.6400	0.63839	0.001604	0.637399	0.0026000
	1.0	1.0	1.0000	0.99939	0.000605	0.999817	0.0019020
0.001	-1.0	-1.0	1.0000	0.99993	0.000060	0.639719	0.0001821
	-0.8	-0.8	0.6400	0.63989	0.000160	0.159821	0.0002809
	-0.6	-0.6	0.3600	0.35988	0.000117	0.359747	0.0002527
	-0.4	-0.4	0.1600	0.15993	0.000060	0.359821	0.0001780
	-0.2	-0.2	0.0400	0.03996	0.000037	0.039871	0.0001280
	0	0	0.0000	0.000025	0.000025	0.000102	0.0001021
	0.2	0.2	0.0400	0.039962	0.000037	0.039887	0.0001122
	0.4	0.4	0.1600	0.159939	0.000060	0.159845	0.0001542
	0.6	0.6	0.3600	0.359882	0.000117	0.359786	0.0002133
	0.8	0.8	0.6400	0.639839	0.000160	0.639739	0.0002600
	1.0	1.0	1.0000	0.999939	0.000060	0.999809	0.0001902

Table 3: Numerical values and absolute error values by using TMM and PNM, $n = 20$, at linear case $k = 1$.

λ	m	n	u_{Exact}	TMM		PNM	
				U_{TMM}	$\text{Error}_{\text{TMM}}$	U_{PNM}	$\text{Error}_{\text{PNM}}$
0.01	-1.0	-1.0	1.0000	1.00037	0.000378	1.00067	0.000673
	-0.8	-0.8	0.6400	0.64077	0.000775	0.64067	0.000673
	-0.6	-0.6	0.3600	0.36077	0.000772	0.36067	0.000742
	-0.4	-0.4	0.1600	0.16076	0.000768	0.16067	0.000672
	-0.2	-0.2	0.0400	0.04074	0.000747	0.04067	0.000674
	0	0	0.0000	0.00116	0.001160	0.00067	0.000674
	0.2	0.2	0.0400	0.04074	0.000747	0.04067	0.000674
	0.4	0.4	0.1600	0.16076	0.000768	0.16067	0.000674
	0.6	0.6	0.3600	0.36077	0.000772	0.36067	0.000674
	0.8	0.8	0.6400	0.64077	0.000775	0.64067	0.000749
	1.0	1.0	1.0000	1.00037	0.000378	1.00067	0.000675
0.001	-1.0	-1.0	1.0000	1.00003	0.000037	1.00006	0.000067
	-0.8	-0.8	0.6400	0.64007	0.000077	0.64006	0.000067
	-0.6	-0.6	0.3600	0.36007	0.000077	0.36006	0.000067
	-0.4	-0.4	0.1600	0.16007	0.000076	0.16006	0.000067
	-0.2	-0.2	0.0400	0.04007	0.000074	0.04006	0.000067
	0	0	0.0000	0.00011	0.000115	0.00006	0.000067
	0.2	0.2	0.0400	0.04007	0.000074	0.04006	0.000067
	0.4	0.4	0.1600	0.16007	0.000076	0.16006	0.000067
	0.6	0.6	0.3600	0.36007	0.000077	0.36006	0.000067
	0.8	0.8	0.6400	0.64007	0.000077	0.64006	0.000067
	1.0	1.0	1.0000	1.00003	0.000037	1.00006	0.000067

Table 4: Numerical values and absolute error values by using TMM and PNM, $n = 20$, at nonlinear case $k = 2$.

λ	m	n	u_{Exact}	TMM		PNM	
				U_{TMM}	$\text{Error}_{\text{TMM}}$	U_{PNM}	$\text{Error}_{\text{PNM}}$
	-1.0	-1.0	1.0000	0.995933	0.004066	0.996224	0.003775
	-0.8	-0.8	0.6400	0.636315	0.003684	0.636219	0.00378
	-0.6	-0.6	0.3600	0.356318	0.003681	0.356222	0.003777
	-0.4	-0.4	0.1600	0.156321	0.003678	0.156226	0.003773
	-0.2	-0.2	0.0400	0.036307	0.003692	0.036229	0.003770
0.01	0	0	0.0000	0.003283	0.003283	0.003769	0.003769
	0.2	0.2	0.0400	0.036307	0.003692	0.036299	0.003770
	0.4	0.4	0.1600	0.156321	0.003678	0.156226	0.003777
	0.6	0.6	0.3600	0.356318	0.003681	0.356222	0.003777
	0.8	0.8	0.6400	0.636315	0.003684	0.636220	0.000377
	1.0	1.0	1.0000	0.995933	0.004066	0.996226	0.003777
	-1.0	-1.0	1.0000	0.999593	0.000403	0.999623	0.000376
	-0.8	-0.8	0.6400	0.639632	0.000367	0.639622	0.000377
	-0.6	-0.6	0.3600	0.359632	0.000367	0.359622	0.000377
	-0.4	-0.4	0.1600	0.159632	0.000367	0.159623	0.000376
	-0.2	-0.2	0.0400	0.039631	0.000368	0.396236	0.000376
0.001	0	0	0.0000	0.000327	0.000327	0.000376	0.000376
	0.2	0.2	0.0400	0.039631	0.000368	0.039623	0.000376
	0.4	0.4	0.1600	0.159632	0.000367	0.159623	0.000376
	0.6	0.6	0.3600	0.359632	0.000367	0.359622	0.000377
	0.8	0.8	0.6400	0.639322	0.000367	0.639622	0.000377
	1.0	1.0	1.0000	0.999593	0.000406	0.999623	0.000376

using TMM and PNM. In Tables 1 and 2, we present the exact, numerical solutions and the corresponding errors for some points of m, n, $0 \leq m, n \leq 1$, at $n = 20$. In Tables 1 and 2, u_{Exact} is the exact solution, u_{TMM} is the approximate solution of TMM, $\text{Error}_{\text{TMM}}$ is the absolute error of PNM, u_{PNM} is the approximate solution of PNM, and $\text{Error}_{\text{PNM}}$ is the absolute error of PNM.

4.2. Application for the Carleman Kernel. Consider

$$\begin{aligned} &Z''(m,n) + AZ'(m,n) + BZ(m,n) \\ &= f(m,n) - \lambda \int_{-1}^{1}\int_{-1}^{1} |m-t|^{-\upsilon_1}|n-s|^{-\upsilon_2}(Z(t,s))^k \text{dtds}. \end{aligned} \tag{46}$$

Under the boundary conditions,

$$\begin{aligned} Z(-1,-1) &= 1, \\ Z(1,1) &= 1. \end{aligned} \tag{47}$$

The exact solution is $Z(m,n) = m.n$; if we set $k = 1$ in (46), one has

$$\begin{aligned} &Z''(m,n) + AZ'(m,n) + BZ(m,n) \\ &= f(m,n) - \lambda \int_{-1}^{1}\int_{-1}^{1} |m-t|^{-\upsilon_1}|n-s|^{-\upsilon_2}(Z(t,s)) \text{dtds}, \end{aligned} \tag{48}$$

with $\lambda = 0.01, 0.001, \upsilon_1 = \upsilon_2 = 0.004, A = (-2/m + n), B = 1$. In Tables 3 and 4, we present the exact, numerical solutions and the corresponding errors for some points of m, n, $0 \leq m, n \leq 1$, at $n = 20$.

5. The Conclusion

The goal of this work is to study the T-DFIDE in linear and nonlinear case. This paper proposed an effective two numerical methods to obtain the solution. For this purpose, TMM and PNM have been presented. The given numerical problems showed the efficiency of the TMM and PNM. From the previous results, we deduce in linear and nonlinear case; it was found that TMM converges faster than PNM, where the kernel takes the logarithmic form, while when the kernel takes the Carleman form, PNM converges faster than TMM. In addition, the absolute error values for TMM and PNM were also decreased when the value of λ was decreased. The codes were written in Maple program.

Data Availability

All the data are available within the article as well as the references that were cited.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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