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Research Article

Abundant Exact Soliton Solutions of the (2 + 1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation Based on the Jacobi Elliptic Function Ideas

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The Heisenberg ferromagnetic spin chain equation (HFSCE) is very important in modern magnetism theory. HFSCE expounded the nonlinear long-range ferromagnetic ordering magnetism. Also, it depicts the characteristic of magnetism to many insulating crystals as well as interaction spins. Moreover, the ferromagnetism plays a fundamental role in modern technology and industry and it is principal for many electrical and electromechanical devices such as generators, electric motors, and electromagnets. In this article, the exact solutions of the nonlinear (2 + 1)-dimensional HFSCE are successfully examined by an extended modified version of the Jacobi elliptic expansion method (EMVJEEM). Consequently, much more new Jacobi elliptic traveling wave solutions are found. These new solutions have not yet been reported in the studied models. For the study models, the new solutions are singular solitons not yet observed. Additionally, certain interesting 3D and 2D figures are performed on the obtained solutions. The geometrical representation of the HFSCE provides the dynamical information to explain the physical phenomena. The results will be significant to understand and study the (2 + 1)-dimensional HFSCE. Therefore, further studying EMVJEEM may help researchers to seek for more soliton solutions to other nonlinear differential equations.

1. Introduction and Main Results

The investigation of traveling wave solutions of nonlinear evolution equations (NLEEs) plays a key role in the study of the internal mechanisms of complex phenomena. In the last few decades, we have made imperative developments and found in the literature many powerful and skilled methods to obtain analytical traveling wave solutions, such as electromagnetism, liquid mechanics, atomic materials, complex physics, electrical engineering, optical fibers, and geochemistry [1–8]. As a result, a large number of mathematicians and physicists tried to invent various methods to obtain solutions to such equations. About describing various complex phenomena in the field of NLEEs, soliton theory plays a crucial role in a number of nonlinear models. Different scientists have studied the dynamics of solitons in different models. For instance, Belić [9] investigated analytical light bullet solutions of the generalized (3 + 1)-dimensional nonlinear Schrödinger equation in 2007. In recent years, Kumar et al. have made great achievements in the study of nonlinear differential equations of water wave models by using Lie symmetry analysis [10, 11].

In 2011, Sabry et al. interpreted three-dimensional ion-acoustic envelope soliton excitations in electron-positron-ion magnetoplasmas through the derivation of three-dimensional nonlinear Schrödinger equation. [12] Helal and Seadawy applied function transformation methods to the D-dimensional nonlinear Schrödinger equation with damping and diffusive terms. [13] In 2012, Giulini and Groβardt [14] derived the Schrödinger-Newton equation for spherically symmetric gravitational fields in a WKB-like expansion in 1/*c* from the Einstein-Klein-Gordon and Einstein-Dirac system. Kumar et al. [15] obtained exact space-time periodic traveling wave solutions of the generalized (3 + 1)-dimensional cubic-quandt nonlinear Schrödinger equation with spatial distribution coefficients. In 2017, Seadawy and Lu [16] derived the

exact bright, dark, and bright-dark solitary wave soliton solutions of the generalized higher order nonlinear NLS equation by using the amplitude ansatz method. The powerful sine-Gordon expansion method was utilized to search for the solutions to some important nonlinear mathematical models arising in nonlinear sciences by Bulut et al. [17]. Zayed et al. [18] investigated the soliton solutions to the nonlinear Schrödinger equation with fourth-order dispersion and dual power law nonlinearity.

In modern magnetic theory, the $(2+1)$-dimension HFSCE is considered as one of the very important equations to explain the dynamics of nonlinear magnets. The $(2+1)$-dimensional HFSCE which is a suitable equation representing many insulating magnetic crystal properties and explaining spin-long ferromagnetic ordered interactions is of striking interest in the soliton theory. The soliton solutions for the $(2+1)$-dimensional HFSC equation are characterized by high quality and qualitative studies for a lot of phenomena and processes in various fields such as ferromagnetic materials, nonlinear optics, and optical fibers. In the meantime, the Heisenberg model of ferromagnetic spin chains with various magnetic interactions associated with nonlinear evolution equations exhibiting a neat and tidy behavior in the classical and semiclassical continuum limits [19]. Inhomogeneous exchange interactions are also good candidates for activating spin reversal processes in ferromagnets [20]. In 2014, Latha and Vasanthi [21] applied the modified Kudryashov and Darboux transformation method for construction of exact traveling wave solutions. In 2017, Inc et al. [22] applied the complex envelope function and the generalized tanh methods to obtain the resolution of the soliton solutions of the two-dimensional HFSC equation. In 2018, Ma et al. [23] utilized an improved F-expansion method (Exp-function method) and the Jacobi elliptic method, respectively, to explore soliton solutions of the 2D-HFSC equation. In 2018, Li and Ma [24] used the Hirota bilinear method and chose proper polynomial functions in bilinear forms, the one-order rogue waves solution and its existence condition were obtained. In 2020, the different wave structures of the 2D-HFSC equation were investigated by Osman et al. [25] via the new extended FAN subequation method. In 2020, Bashar and Islam [26] implemented the modified simple equation (MSE) and improve F-expansion method to find the exact solutions of the HFSCE. In 2021, Hosseini et al. [27] investigated the HFSCE by the Jacobi elliptic functions (JEFs) method. In 2022, Sahoo and Tripathy [28] used the modified Khater method to study the HFSCE for the new exact solitary solutions.

Considering the $(2+1)$-dimensional HFSCE [25, 27] which is defined as

$$i\frac{\partial V(x,y,t)}{\partial t}+\beta_1\frac{\partial^2 V(x,y,t)}{\partial x^2}+\beta_2\frac{\partial^2 V(x,y,t)}{\partial y^2}+\beta_3\frac{\partial^2 V(x,y,t)}{\partial y\partial x}-\beta_4 V(x,y,t)|V(x,y,t)|^2=0, \tag{1}$$

here

$$\beta_1=\delta^4(\gamma+\gamma_2),\ \beta_2=\delta^4(\gamma_1+\gamma_2),\ \beta_3=2\delta^4\gamma_2,\ \beta_4=2\delta^4 B. \tag{2}$$

Here the complex-valued function $V(x,y,t)$ signifies the wave propagation, x, y are the spatial variables, and t is the time variable. The lattice parameter is represented as δ with the interaction coefficients $\gamma, \gamma_1, \gamma_2$, while the anisotropic parameter [25, 27] is denoted as B. In this section, we present a mathematical analysis of the proposed model. The complex wave transformation which has been taken into account to seek for the solitary solutions of HFSCE is of the next form:

$$V(x,y,t)=\Phi(\xi)e^{i\theta}, \tag{3}$$

where the amplitude is denoted as $\Phi(\xi)$ with $\xi=x+y-\chi t$ and $\theta=-lx+ky+\omega t+\varepsilon_2$ is the corresponding phase component. By (3), we deduce that:

$$V_t=\left[-\chi\Phi'+i\omega\phi\right]e^{i\theta}, \tag{4}$$

$$V_{xx}=\left[\phi''-2il\phi'-l^2\phi\right]e^{i\theta}, \tag{5}$$

$$V_{yy}=\left[\phi''-2ik\phi'-k^2\phi\right]e^{i\theta}, \tag{6}$$

$$V_{xy}=\left[\phi''-i(l-k)\phi'+lk\right]e^{i\theta}. \tag{7}$$

Inserting (3) and (4)–(7) into (1), the real part,

$$(\beta_1+\beta_2+\beta_3)\Phi''-\left(\omega+\beta_1 l^2+\beta_2 k^2-\beta_3 kl\right)\Phi-\beta_4\Phi^3=0, \tag{8}$$

and the imaginary part,

$$\chi=2\beta_1 l-2\beta_2 k-\beta_3 k+\beta_3 l, \tag{9}$$

here the $\beta_1, \beta_2, \beta_3, k$, and l are all nonzero parameters.

In 2021, based on the ideas of the Jacobi elliptic functions, Yang and Zhang [29] used the unified F-expansion method to study the Korteweg-De Vries partial differential equations. In 2021, Ünal et al. [30] also investigated the exact solutions of space-time fractional symmetric regularized long wave equation using ideas of JEFs.

Twelve kinds JEFs are available in literature [31]. Basic JEFs are expressed as $sn\xi = sn(\xi; m)$ or $cn\xi = cn(\xi; m)$ or $dn\xi = dn(\xi; m)$ and other basic JEFs such as sd, cd, nd, sc, nc, dc, ns, cs, and ds. In addition, if $m=0$ and $m=1$, then the JEFs turn into trigonometric and hyperbolic functions. Here, m is the modulus and is a complex number. If m is real, it can be arranged $0<m^2<1$.

2. The Extend Modified Version of the Jacobi Elliptic Expansion Method

Based on the ideas of [30], we consider the following form of Equation (8), and we give the details of the extend modified

version of the Jacobi elliptic expansion method (EMVJEEM) and employ this method to seek for new and more general traveling wave exact solutions to Equation (8).

Step 1. Regarding that the solution of Equation (8) can be expressed by a polynomial in K as follows:

$$\Phi(\xi) = \sum_{j=-N}^{N} a_j K^j(\xi), \tag{10}$$

in Equation (8), $K = K(\xi)$ is satisfied the following differential equations:

$$\left(K'\right)^2(\xi) = pK^4(\xi) + qK^2(\xi) + r, \tag{11}$$

where p, q, and r are arbitrary constants. N can be determined by the uniform equilibrium term between the highest derivative and the nonlinear term appearing in (8). All the solutions $K(\xi)$ of (11) are listed in Table 1.

Step 2. Taking (10) into (8) and using (11), (8) is converted into another polynomial in K. Calculating all the coefficients of the polynomial to zero produces the system algebraic equations for $a_1,\cdots, a_N, p, q$, and r.

Step 3. The constants $a_1,\cdots, a_N$ can be obtained by solving the system of algebraic equations obtained in Step 3. Since (11) may have the following many possible solutions. Thus, the exact solutions for given (8) can be derived. Here:

Step 4. Putting the inverse transform T^{-1} into the solutions $\Phi(\xi)$ $(\xi = x + y - \chi t)$, we can get all exact solutions $V(x, y, t) = \Phi(x, y, t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$ of the original Equation (8).

Remark 1. In 2021, Hosseini et al. [27] only considered the four cases in Table 1 (such as Cases 1, 2, 9, and 10). Obviously, we consider a broader scenario in Table 1, and therefore, more new solutions will be obtained in this paper.

3. Employing the EMVJEEM to Equation (8)

Taking into account the homogeneous equilibrium term between Φ'' and Φ^2 in (8), it is easy to deduce $N = 1$. The solution of (9) can be listed as follows:

$$\Phi(\xi) = a_1 K + a_0 + b_1 \frac{1}{K}, \tag{12}$$

here a_0, a_1, and b_1 are unknown constants and will be determined later.

By using (12) and (11) and collecting all terms with the same power K together, we deduce

$$\Phi'(\xi) = a_1 K'(\xi) - b_1 K^{-2}(\xi)K'(\xi), \tag{13}$$

$$\Phi''(\xi) = 2pa_1 K^3(\xi) + a_1 qK(\xi) + b_1 qK^{-1}(\xi) + 2b_1 rK^{-3}(\xi). \tag{14}$$

Substituting (14) into (9), we obtain

$$\begin{aligned}&(\beta_1 + \beta_2 + \beta_3)\left(2pa_1K^3 + a_1qK + b_1qK^{-1} + 2b_1rK^{-3}\right)\\ &+ \left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right)\left(a_0 + a_1K + b_1\frac{1}{K}\right)\\ &- \beta_4\left(a_0 + a_1K + b_1\frac{1}{K}\right)^3 = 0.\end{aligned} \tag{15}$$

Sorting out all terms with the same power of K together, we obtain

$$\begin{aligned}&\left(2pa_1 + ca_1^3\right)K^3 + \left((\beta_1 + \beta_2 + \beta_3)a_1q + a_1\left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right)\right.\\ &\left. - 3\beta_4a_0^2a_1 - 3\beta_4a_1^2b_1\right)K + \left((\beta_1 + \beta_2 + \beta_3)b_1q\right.\\ &\left. + b_1\left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right) - 3\beta_4a_0^2b_1 - 3\beta_4a_1b_1^2\right)\frac{1}{K}\\ &+ \left(2b_1r(\beta_1 + \beta_2 + \beta_3) - \beta_4b_1^3\right)\frac{1}{K^3} + \left(a_0\left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right)\right.\\ &\left. - \beta_4a_0^3 - 6a_0a_1b_1\beta_4\right) - 3a_0a_1^2\beta_4K^2 - 3\beta_4a_0b_1^2\frac{1}{K^2} = 0.\end{aligned} \tag{16}$$

For the same term of function K, we are extracting their unknown coefficients and setting them to zero to get the next equation:

$$\begin{cases} a_0\left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right) - \beta_4a_0^3 - 6a_0a_1\beta_1\beta_4 = 0,\\ (\beta_1 + \beta_2 + \beta_3)a_1q + a_1\left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right) - 3\beta_4a_0^2a_1 - 3\beta_4a_1^2b_1 = 0,\\ (\beta_1 + \beta_2 + \beta_3)b_1q + \left(\beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right)b_1 - 3\beta_4a_0^2b_1 - 3\beta_4a_1b_1^2 = 0,\\ 3a_0a_1^2\beta_4 = 0,\\ 3\beta_4a_0b_1^2 = 0,\\ 2b_1r(\beta_1 + \beta_2 + \beta_3) - \beta_4b_1^3 = 0,\\ 2pa_1 - \beta_4a_1^3 = 0. \end{cases} \tag{17}$$

Solving this system of equations, the unknown coefficients are found:

$$\begin{aligned} a_0 &= 0,\\ a_1 &= \pm\sqrt{\frac{2p}{\beta_4}},\\ b_1 &= \pm\sqrt{\frac{2r(\beta_1 + \beta_2 + \beta_3)}{\beta_4}}, \end{aligned} \tag{18}$$

and $\left((\beta_1 + \beta_2 + \beta_3)q + \beta_3kl - \omega - \beta_1l^2 - \beta_2k^2\right)^2 = 36pr(\beta_1 + \beta_2 + \beta_3)$. From (3), we know that $\Phi(\xi)$ is the real function. Hence, $K(\xi)$ takes the real cases in Table 1, the forms as

Table 1: Basic structure.

	p	q	r	K
1	m^2	$-(1+m)^2$	1	$\pm sn\xi, \pm cd\xi$
2	1	$-(1+m)^2$	m^2	$\pm ns\xi, \pm dc\xi$;
3	$-m^2$	$-(1+m)^2$	-1	$\pm isn\xi, \pm icd\xi$
4	-1	$-(1+m)^2$	$-m^2$	$\pm ins\xi, \pm idc\xi$
5	1	$2-m^2$	$1-m^2$	$\pm cs\xi, \pm idn\xi$
6	$1-m^2$	$2-m^2$	1	$\pm sc\xi, \pm ind\xi$
7	1	$2-m^2$	m^2-1	$\pm ics\xi, \pm dn\xi$
8	m^2-1	$2-m^2$	-1	$\pm isc\xi, \pm nd\xi$
9	$1-m^2$	$2m^2-1$	$-m^2$	$\pm nc\xi, \pm imsd\xi$
10	$-m^2$	$2m^2-1$	$1-m^2$	$\pm cn\xi, \pm \frac{i}{m} ds\xi$
11	m^2-1	$2m^2-1$	m^2	$\pm inc\xi, \pm msd\xi$
12	m^2	$2m^2-1$	m^2-1	$\pm icn\xi, \pm \frac{1}{m} ds\xi$
13	m^4-m^2	$2m^2-1$	1	$\pm \frac{i}{m} nc\xi, \pm sd\xi$
14	1	$2m^2-1$	m^4-m^2	$\pm imcn\xi, \pm ds\xi$
15	$-m^4+m^2$	$2m^2-1$	-1	$\pm \frac{1}{m} nc\xi, \pm isd\xi$
16	-1	$2m^2-1$	$-m^4+m^2$	$\pm \frac{1}{m} cn\xi, \pm ids\xi$
17	$\frac{1}{4}$	$\frac{1+m^2}{2}$	$\frac{(1-m^2)^2}{4}$	$ds\xi \pm cs\xi, -ds\xi \mp cs\xi, i(mcn\xi \pm dn\xi), -i(mcn\xi \pm dn\xi)$
18	$\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1}{4}$	$\frac{sn\xi}{dn\xi \pm cn\xi}, \frac{-sn\xi}{dn\xi \pm cn\xi}, \frac{i}{mcn\xi \pm dn\xi}, \frac{-1}{mcn\xi \pm dn\xi}$
19	$-\frac{1}{4}$	$\frac{1+m^2}{2}$	$-\frac{(1-m^2)^2}{4}$	$i(ds\xi \pm cs\xi), -i(ds\xi \pm cs\xi), mcn\xi \pm dn\xi, -mcn\xi \mp dn\xi$
20	$-\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$-\frac{1}{4}$	$\frac{isn\xi}{dn\xi \pm cn\xi}, \frac{-isn\xi}{dn\xi \pm cn\xi}, \frac{1}{mcn\xi \pm dn\xi}, \frac{-1}{mcn\xi \pm dn\xi}$
21	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$nc\xi \pm sc\xi, -nc\xi \mp sc\xi, i(msd\xi \pm nd\xi), -i(msd\xi \pm nd\xi)$
22	$\frac{m^2-1}{4}$	$\frac{1+m^2}{2}$	$\frac{m^2-1}{4}$	$i(nc\xi \pm sc\xi), -i(nc\xi \pm sc\xi), msd\xi \pm nd\xi, -msd\xi \mp nd\xi$

Table 1: Continued.

	p	q	r	K
23	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$ns\xi \pm ds\xi, -ns\xi \mp ds\xi, dc\xi \pm \sqrt{1-m^2}nc\xi, -dc\xi \mp \sqrt{1-m^2}nc\xi$
24	$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{sn\xi}{1 \pm dn\xi}, \frac{-sn\xi}{1 \pm dn\xi}, \frac{cn\xi}{dn\xi \pm \sqrt{1-m^2}}, \frac{-cn\xi}{dn\xi \pm \sqrt{1-m^2}}$
25	$-\frac{1}{4}$	$\frac{m^2-2}{2}$	-m^4/4	$i(ns\xi \pm ds\xi), -i(ns\xi \mp ds\xi), i\left(dc\xi \pm \sqrt{1-m^2}nc\xi\right), -i\left(dc\xi \mp \sqrt{1-m^2}nc\xi\right)$
26	$-\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$-\frac{1}{4}$	$\frac{isn\xi}{1 \pm dn\xi}, \frac{-isn\xi}{1 \pm dn\xi}, \frac{icn\xi}{dn\xi \pm \sqrt{1-m^2}}, \frac{-icn\xi}{dn\xi \pm \sqrt{1-m^2}}$
27	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$sn\xi \pm icn\xi, -sn\xi \mp icn\xi, cd\xi \pm i\sqrt{1-m^2}sd\xi, -cd\xi \mp i\sqrt{1-m^2}sd\xi$
28	$-\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$-\frac{m^2}{4}$	$cn\xi \pm isn\xi, -cn\xi \mp isn\xi, \sqrt{1-m^2}sd\xi \pm icd\xi, -cd\xi \mp i\sqrt{1-m^2}sd\xi$
29	$\frac{1}{4}$	$\frac{1-m^2}{2}$	$\frac{1}{4}$	$ns\xi \pm cs\xi, -ns\xi \mp cs\xi, msn\xi \pm idn\xi, -msn\xi \mp idn\xi, dc\xi \pm \sqrt{1-m^2}sc\xi, -dc\xi \mp \sqrt{1-m^2}sc\xi mcd\xi \pm i\sqrt{1-m^2}nd\xi, -mcd\xi \mp i\sqrt{1-m^2}nd\xi$
30	$-\frac{1}{4}$	$\frac{1-m^2}{2}$	$-\frac{1}{4}$	$i(ns\xi \pm cs\xi), -i(ns\xi \mp cs\xi), dn\xi \pm imsn\xi, -dn\xi \mp imsn\xi, i\left(dc\xi \pm \sqrt{1-m^2}sc\xi\right), -i\left(dc\xi \pm \sqrt{1-m^2}sc\xi\right)\sqrt{1-m^2}nd\xi \pm imcd\xi, -\sqrt{1-m^2}nd\xi \mp imcd\xi$

follows:

$$\Phi(\xi) = a_1 K(\xi) + b_1 \frac{1}{K(\xi)} = \pm\sqrt{\frac{2p}{\beta_4}} K(\xi) \pm \sqrt{\frac{2r(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{K(\xi)}. \tag{19}$$

Case 1. If $p = m^2$, $q = -(1+m^2)$, and $r = 1$, then $K(\xi) = \pm sn\xi$ or $\pm cd\xi$,

$$\Phi_{31}(x+y-\chi t) = \pm m\sqrt{\frac{2}{\beta_4}} sn(x+y-\chi t) \pm \sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{sn(x+y-\chi t)}, \tag{20}$$

or

$$\Phi_{32}(x+y-\chi t) = \pm m\sqrt{\frac{2}{\beta_4}} cd(x+y-\chi t) \pm \sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{cd(x+y-\chi t)};$$

$$V_{31}(x+y-\chi t) = \Phi_{31}(x+y-\chi t) e^{i(-lx+ky+\omega t+\varepsilon_2)} == \pm\left(m\sqrt{\frac{2}{\beta_4}} sn(x+y-\chi t) \pm \sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{sn(x+y-\chi t)} \right) e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{21}$$

or

$$V_{32}(x+y-\chi t) = \Phi_{32}(x+y-\chi t) e^{i(-lx+ky+\omega t+\varepsilon_2)} = \pm\left(m\sqrt{\frac{2}{\beta_4}} cd(x+y-\chi t) \pm \sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{cd(x+y-\chi t)} \right) e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{22}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2) - (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = 36m^2(\beta_1+\beta_2+\beta_3)$.

Case 2. If $p = 1$, $q = -(1+m^2)$, and $r = m^2$, then $K(\xi) = \pm ns\xi$ or $\pm dc\xi$,

$$\Phi_{33}(x+y-\chi t) = \pm\sqrt{\frac{2}{\beta_4}} ns(x+y-\chi t) \pm m\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{ns(x+y-\chi t)}, \tag{23}$$

or

$$\Phi_{34}(x+y-\chi t) = \pm\sqrt{\frac{2}{\beta_4}} dc(x+y-\chi t) \pm m\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{dc(x+y-\chi t)};$$

$$V_{33}(x+y-\chi t) = \Phi_{33}(x+y-\chi t) e^{i(-lx+ky+\omega t+\varepsilon_2)}; = \left(\pm\sqrt{\frac{2}{\beta_4}} ns(x+y-\chi t) \pm m\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{ns(x+y-\chi t)} \right) e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{24}$$

or

$$V_{34}(x+y-\chi t) = \Phi_{34}(x+y-\chi t) e^{i(-lx+ky+\omega t+\varepsilon_2)} = \pm\left(\sqrt{\frac{2}{\beta_4}} dc(x+y-\chi t) \pm m\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{dc(x+y-\chi t)} \right) e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{25}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2) - (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = 36m^2(\beta_1+\beta_2+\beta_3)$.

Case 3. If $p = 1$, $q = 2 - m^2$, and $r = 1 - m^2$, then $K(\xi) = \pm cs\xi$,

$$\Phi_{35}(x+y-\chi t) = \pm\sqrt{\frac{2}{\beta_4}} cs(x+y-\chi t) \pm \sqrt{\frac{2(1-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{cs(x+y-\chi t)};$$

$$V_{35}(x+y-\chi t) = \Phi_{35}(x+y-\chi t) e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm\sqrt{\frac{2}{\beta_4}} cs(x+y-\chi t) \pm \sqrt{\frac{2(1-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}} \frac{1}{cs(x+y-\chi t)} \right) e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{26}$$

and $((\beta_1+\beta_2+\beta_3)(2-m^2) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = 36(1-m^2)(\beta_1+\beta_2+\beta_3)$.

Case 4. If $p = 1 - m^2$, $q = 2 - m^2$, and $r = 1$, then $K(\xi) = \pm sc\xi$,

$$\Phi_{36}(x+y-\chi t) = \pm\sqrt{\frac{2(1-m^2)}{\beta_4}}sc(x+y-\chi t) \pm \sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{sc(x+y-\chi t)};$$

$$V_{36}(x+y-\chi t) = \Phi_{36}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm\sqrt{\frac{2(1-m^2)}{\beta_4}}sc(x+y-\chi t) \pm \sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{sc(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{27}$$

and $((\beta_1+\beta_2+\beta_3)(2-m^2) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = 36(1-m^2)(\beta_1+\beta_2+\beta_3)$.

Case 5. If $p = 1$, $q = 2 - m^2$, and $r = m^2 - 1$, then $K(\xi) = \pm dn\xi$,

$$\Phi_{37}(x+y-\chi t) = \pm\sqrt{\frac{2}{\beta_4}}dn(x+y-\chi t) \pm \sqrt{\frac{2(m^2-1)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{dn(x+y-\chi t)};$$

$$V_{37}(x+y-\chi t) = \Phi_{37}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm\sqrt{\frac{2}{\beta_4}}dn(x+y-\chi t) \pm \sqrt{\frac{2(m^2-1)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{dn(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{28}$$

and $((\beta_1+\beta_2+\beta_3)(2-m^2) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = 36(m^2-1)(\beta_1+\beta_2+\beta_3)$.

Case 6. If $p = m^2 - 1$, $q = 2 - m^2$, and $r = -1$, then $K(\xi) = \pm nd\xi$,

$$\Phi_{38}(x+y-\chi t) = \pm\sqrt{\frac{2(m^2-1)}{\beta_4}}nd(x+y-\chi t)$$

$$V_{38}(x+y-\chi t) = \Phi_{38}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm\sqrt{\frac{2(m^2-1)}{\beta_4}}nd(x+y-\chi t) \pm \sqrt{\frac{-2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{nd(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{29}$$

and $((\beta_1+\beta_2+\beta_3)(2-m^2) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = -36(m^2-1)(\beta_1+\beta_2+\beta_3)$.

Case 7. If $p = 1 - m^2$, $q = 2m^2 - 1$, and $r = -m^2$, then $K(\xi) = \pm nc\xi$,

$$\Phi_{39}(x+y-\chi t) = \pm\sqrt{\frac{2(1-m^2)}{\beta_4}}nc(x+y-\chi t) \pm \sqrt{\frac{-2m^2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{nc(x+y-\chi t)};$$

$$V_{39}(x+y-\chi t) = \Phi_{39}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm\sqrt{\frac{2(1-m^2)}{\beta_4}}nc(x+y-\chi t) \pm \sqrt{\frac{-2m^2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{nc(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{30}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = -36(1-m^2)m^2(\beta_1+\beta_2+\beta_3)$.

$$V_{39}(x+y-\chi t) = \Phi_{39}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm\sqrt{\frac{2(1-m^2)}{\beta_4}}nc(x+y-\chi t) \pm \sqrt{\frac{-2m^2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{nc(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{31}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = -36(1-m^2)m^2(\beta_1+\beta_2+\beta_3)$.

Case 8. If $p = -m^2$, $q = 2m^2 - 1$, and $r = 1 - m^2$, then $K(\xi) = \pm cn\xi$,

$$\Phi_{310}(x+y-\chi t) = \pm m\sqrt{\frac{-2}{\beta_4}}cn(x+y-\chi t) \pm \sqrt{\frac{2(1-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{cn(x+y-\chi t)};$$

$$V_{310}(x+y-\chi t) = \Phi_{310}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} = \left(\pm m\sqrt{\frac{-2}{\beta_4}}cn(x+y-\chi t) \pm \sqrt{\frac{2(1-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{cn(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{32}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1) + (\beta_3 kl - \omega - \beta_1 l^2 - \beta_2 k^2))^2 = -36(1-m^2)m^2(\beta_1+\beta_2+\beta_3)$.

Case 9. If $p = m^2 - 1$, $q = 2m^2 - 1$, and $r = m^2$, then $K(\xi) = \pm msd\xi$,

$$\Phi_{311}(x+y-\chi t) = \pm\sqrt{\frac{2(m^2-1)}{\beta_4}}msd(x+y-\chi t) \pm m\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{msd(x+y-\chi t)};$$

$$V_{311}(x+y-\chi t)=\Phi_{311}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{2(m^2-1)}{\beta_4}}msd(x+y-\chi t)\right.$$
$$\left.\pm m\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{msd(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{33}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=36(m^2-1)m^2(\beta_1+\beta_2+\beta_3)$.

Case 10. If $p=m^2$, $q=2m^2-1$, and $r=m^2-1$, then $K(\xi)=\pm(1/m)ds\xi$,

$$\Phi_{312}(x+y-\chi t)=\pm\sqrt{\frac{2}{\beta_4}}ds(x+y-\chi t)$$
$$\pm\sqrt{\frac{2(m^2-1)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{m}ds(x+y-\chi t);$$

$$V_{312}(x+y-\chi t)=\Phi_{312}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{2}{\beta_4}}ds(x+y-\chi t)\right.$$
$$\left.\pm\sqrt{\frac{2(m^2-1)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{m}ds(x+y-\chi t)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{34}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=36m^2(m^2-1)(\beta_1+\beta_2+\beta_3)$.

Case 11. If $p=m^4-m^2$, $q=2m^2-1$, and $r=1$, then $K(\xi)=\pm sd\xi$,

$$\Phi_{313}(x+y-\chi t)=\pm\sqrt{\frac{2(m^4-m^2)}{\beta_4}}sd(x+y-\chi t)$$
$$\pm\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{sd(x+y-\chi t)};$$

$$V_{313}(x+y-\chi t)=\Phi_{313}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{2(m^4-m^2)}{\beta_4}}sd(x+y-\chi t)\right.$$
$$\left.\pm\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{sd(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{35}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=36(m^4-m^2)(\beta_1+\beta_2+\beta_3)$.

Case 12. If $p=1$, $q=2m^2-1$, and $r=m^4-m^2$, then $K(\xi)=\pm ds\xi$,

$$\Phi_{314}(x+y-\chi t)=\pm\sqrt{\frac{2}{\beta_4}}ds(x+y-\chi t)\pm\sqrt{\frac{2(m^4-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{ds(x+y-\chi t)};$$
$$V_{314}(x+y-\chi t)=\Phi_{314}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)=\left(\pm\sqrt{\frac{2}{\beta_4}}ds(x+y-\chi t)\pm\sqrt{\frac{2(m^4-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{ds(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}},\tag{36}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=36(m^4-m^2)(\beta_1+\beta_2+\beta_3)$.

Case 13. If $p=-m^4+m^2$, $q=2m^2-1$, and $r=-1$, then $K(\xi)=\pm(1/m)nc\xi$,

$$\Phi_{315}(x+y-\chi t)=\pm\sqrt{\frac{2(1-m^2)}{\beta_4}}nc(x+y-\chi t)$$
$$\pm\sqrt{\frac{-2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{m}{nc(x+y-\chi t)};$$

$$V_{315}(x+y-\chi t)=\Phi_{315}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{2(1-m^2)}{\beta_4}}nc(x+y-\chi t)\right.$$
$$\left.\pm\sqrt{\frac{-2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{m}{nc(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{37}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=36(m^4-m^2)(\beta_1+\beta_2+\beta_3)$.

Case 14. If $p=-1$, $q=2m^2-1$, $r=m^2-m^4$, then $K(\xi)=\pm mcn\xi$,

$$\Phi_{316}(x+y-\chi t)=\pm m\sqrt{\frac{-2}{\beta_4}}cn(x+y-\chi t)$$
$$\pm\sqrt{\frac{2(1-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{cn(x+y-\chi t)};$$

$$V_{316}(x+y-\chi t)=\Phi_{316}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm m\sqrt{\frac{-2}{\beta_4}}cn(x+y-\chi t)\right.$$
$$\left.\pm\sqrt{\frac{2(1-m^2)(\beta_1+\beta_2+\beta_3)}{\beta_4}}\frac{1}{cn(x+y-\chi t)}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{38}$$

and $((\beta_1+\beta_2+\beta_3)(2m^2-1)+(\beta_3 kl-\omega-\beta_1 l^2-\beta_2 k^2))^2 = 36(m^4-m^2)(\beta_1+\beta_2+\beta_3)$.

Case 15. If $p=1/4$, $q=(1+m^2)/2$, and $r=(1-m^2)^2/4$, then $K(\xi)=ds\xi \pm cs\xi$ or $-ds\xi \mp cs\xi$,

$$\Phi_{317}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(ds(x+y-\chi t)\pm cs(x+y-\chi t)) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)(1-m^2)^2}{2\beta_4}}\frac{1}{(ds(x+y-\chi t)\pm cs(x+y-\chi t))}, \tag{39}$$

or

$$\Phi_{318}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(-ds(x+y-\chi t)\mp cs(x+y-\chi t)) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)(1-m^2)^2}{2\beta_4}}\frac{1}{(-ds(x+y-\chi t)\mp cs(x+y-\chi t))}; \tag{40}$$

$$V_{317}(x+y-\chi t)=\Phi_{317}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} =\left(\pm\sqrt{\frac{1}{2\beta_4}}(ds(x+y-\chi t)\pm cs(x+y-\chi t)) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)(1-m^2)^2}{2\beta_4}}\frac{1}{(ds(x+y-\chi t)\pm cs(x+y-\chi t))}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{41}$$

or

$$V_{318}(x+y-\chi t)=\Phi_{318}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} =\left(\pm\sqrt{\frac{1}{2\beta_4}}(-ds(x+y-\chi t)\mp cs(x+y-\chi t)) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)(1-m^2)^2}{2\beta_4}}\frac{1}{(-ds(x+y-\chi t)\mp cs(x+y-\chi t))}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{42}$$

and $((\beta_1+\beta_2+\beta_3)((1+m^2)/2)+(\beta_3 kl-\omega-\beta_1 l^2-\beta_2 k^2))^2 = (9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 16. If $p=(1-m^2)^2/4$, $q=(1+m^2)/2$, and $r=1/4$, then $K(\xi)=sn\xi/dn\xi \pm cn\xi$ or $-sn\xi/dn\xi \pm cn\xi$,

$$\Phi_{319}(x+y-\chi t)=\pm\sqrt{\frac{(1-m^2)^2}{2\beta_4}}\left(\frac{sn(x+y-\chi t)}{dn(x+y-\chi t)\pm cn(x+y-\chi t)}\right) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{dn(x+y-\chi t)\pm cn(x+y-\chi t)}{sn(x+y-\chi t)}\right), \tag{43}$$

or

$$\Phi_{320}(x+y-\chi t)=\pm\sqrt{\frac{(1-m^2)^2}{2\beta_4}}\left(\frac{-sn(x+y-\chi t)}{dn(x+y-\chi t)\pm cn(x+y-\chi t)}\right) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{dn(x+y-\chi t)\pm cn(x+y-\chi t)}{-sn(x+y-\chi t)}\right); \tag{44}$$

$$V_{319}(x+y-\chi t)=\Phi_{319}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} =\left(\pm\sqrt{\frac{(1-m^2)^2}{2\beta_4}}\left(\frac{sn(x+y-\chi t)}{dn(x+y-\chi t)\pm cn(x+y-\chi t)}\right) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{dn(x+y-\chi t)\pm cn(x+y-\chi t)}{sn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{45}$$

or

$$V_{320}(x+y-\chi t)=\Phi_{320}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)} =\left(\pm\sqrt{\frac{(1-m^2)^2}{2\beta_4}}\left(\frac{-sn(x+y-\chi t)}{dn(x+y-\chi t)\pm cn(x+y-\chi t)}\right) \pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{dn(x+y-\chi t)\pm cn(x+y-\chi t)}{-sn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{46}$$

and $((\beta_1+\beta_2+\beta_3)((1+m^2)/2)+(\beta_3 kl-\omega-\beta_1 l^2-\beta_2 k^2))^2 = (9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 17. If $p=-1/4$, $q=(1+m^2)/2$, and $r=-(1-m^2)^2/4$, then $K(\xi)=mcn\xi \pm dn\xi$ or $-mcn\xi \mp dn\xi$,

$$\Phi_{321}(x+y-\chi t)=\pm\sqrt{\frac{-1}{2\beta_4}}(mcn(x+y-\chi t)\pm dn(x+y-\chi t)) \pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}} \cdot\left(\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right), \tag{47}$$

or

$$\Phi_{322}(x+y-\chi t)=\pm\sqrt{\frac{-1}{2\beta_4}}(-mcn(x+y-\chi t)\mp dn(x+y-\chi t)) \pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}} \cdot\left(\frac{1}{-mcn(x+y-\chi t)\mp dn(x+y-\chi t)}\right);$$

$$V_{321}(x+y-\chi t)=\Phi_{321}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{-1}{2\beta_4}}(mcn(x+y-\chi t)\pm dn(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{48}$$

or

$$V_{322}(x+y-\chi t)=\Phi_{322}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{-1}{2\beta_4}}(-mcn(x+y-\chi t)\mp dn(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{-mcn(x+y-\chi t)\mp dn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{49}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 18. If $p=-(1/4)$, $q=(1+m^2)/2$, and $r=-(1-m^2)^2/4$, then $K(\xi)=mcn\xi\pm dn\xi$ or $-mcn\xi\mp dn\xi$,

$$\Phi_{323}(x+y-\chi t)=\pm\sqrt{\frac{-1}{2\beta_4}}(mcn(x+y-\chi t)\pm dn(x+y-\chi t))$$
$$\pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right),\tag{50}$$

or

$$\Phi_{324}(x+y-\chi t)=\pm\sqrt{\frac{-1}{2\beta_4}}(-mcn(x+y-\chi t)\mp dn(x+y-\chi t))$$
$$\pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{-mcn(x+y-\chi t)\mp dn(x+y-\chi t)}\right);\tag{51}$$

$$V_{323}(x+y-\chi t)=\Phi_{323}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{-1}{2\beta_4}}(mcn(x+y-\chi t)\pm dn(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{52}$$

or

$$V_{324}(x+y-\chi t)=\Phi_{324}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{-1}{2\beta_4}}(-mcn(x+y-\chi t)\mp dn(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{-(1-m^2)^2(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{-mcn(x+y-\chi t)\mp dn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{53}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 19. If $p=-((1-m^2)^2/4)$, $q=(1+m^2)/2$, and $r=-(1/4)$, then $K(\xi)=1/mcn\xi\pm dn\xi$ or $-(1/mcn\xi\pm dn\xi)$,

$$\Phi_{325}(x+y-\chi t)=\pm\sqrt{\frac{-(1-m^2)^2}{2\beta_4}}\left(\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right)$$
$$\pm\sqrt{\frac{-(\beta_1+\beta_2+\beta_3)}{2\beta_4}}(mcn(x+y-\chi t)\pm dn(x+y-\chi t)),\tag{54}$$

or

$$\Phi_{326}(x+y-\chi t)=\pm\sqrt{\frac{-(1-m^2)^2}{2\beta_4}}$$
$$\cdot\left(-\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right)$$
$$\pm\sqrt{\frac{-(\beta_1+\beta_2+\beta_3)}{2\beta_4}}(-(mcn(x+y-\chi t)\pm dn(x+y-\chi t)));$$

$$V_{325}(x+y-\chi t)=\Phi_{325}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{-(1-m^2)^2}{2\beta_4}}\right.$$
$$\cdot\left(\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right)$$
$$\left.\pm\sqrt{\frac{-(\beta_1+\beta_2+\beta_3)}{2\beta_4}}(mcn(x+y-\chi t)\pm dn(x+y-\chi t))\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{55}$$

or

$$V_{326}(x+y-\chi t)=\Phi_{326}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{-(1-m^2)^2}{2\beta_4}}\left(-\frac{1}{mcn(x+y-\chi t)\pm dn(x+y-\chi t)}\right)\right.$$
$$\pm\sqrt{\frac{-(\beta_1+\beta_2+\beta_3)}{2\beta_4}}(-(mcn(x+y-\chi t)$$
$$\left.\pm dn(x+y-\chi t))\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{56}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 20. If $p=(1-m^2)/4$, $q=(1+m^2)/2$, and $r=(1-m^2)/4$, then $K(\xi)=nc\xi\pm sc\xi$ or $-nc\xi\mp sc\xi$,

$$\Phi_{327}(x+y-\chi t)=\pm\sqrt{\frac{(1-m^2)}{2\beta_4}}(nc(x+y-\chi t)\pm sc(x+y-\chi t))$$
$$\pm\sqrt{\frac{(1-m^2)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{nc(x+y-\chi t)\pm sc(x+y-\chi t)}\right),\tag{57}$$

or

$$\Phi_{328}(x+y-\chi t)=\pm\sqrt{\frac{(1-m^2)}{2\beta_4}}(-nc(x+y-\chi t)\mp sc(x+y-\chi t))$$
$$\pm\sqrt{\frac{(1-m^2)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{-nc(x+y-\chi t)\mp sc(x+y-\chi t)}\right);\tag{58}$$

$$V_{327}(x+y-\chi t)=\Phi_{327}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{(1-m^2)}{2\beta_4}}(nc(x+y-\chi t)\pm sc(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{(1-m^2)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{nc(x+y-\chi t)\pm sc(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{59}$$

or

$$V_{328}(x+y-\chi t)=\Phi_{328}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{(1-m^2)}{2\beta_4}}(-nc(x+y-\chi t)\mp sc(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{(1-m^2)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{-nc(x+y-\chi t)\mp sc(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{60}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 21. If $p=(m^2-1)/4$, $q=(1+m^2)/2$, and $r=(m^2-1)/4$, then $K(\xi)=msd\xi\pm nd\xi$ or $-msd\xi\mp nd\xi$,

$$\Phi_{329}(x+y-\chi t)=\pm\sqrt{\frac{(m^2-1)}{2\beta_4}}(msd(x+y-\chi t)\pm nd(x+y-\chi t))$$
$$\pm\sqrt{\frac{(m^2-1)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{msd(x+y-\chi t)\pm nd(x+y-\chi t)}\right),\tag{61}$$

or

$$\Phi_{330}(x+y-\chi t)=\pm\sqrt{\frac{(m^2-1)}{2\beta_4}}(-msd(x+y-\chi t)\mp nd(x+y-\chi t))$$
$$\pm\sqrt{\frac{(m^2-1)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{-msd(x+y-\chi t)\mp nd(x+y-\chi t)}\right);\tag{62}$$

$$V_{329}(x+y-\chi t)=\Phi_{329}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{(m^2-1)}{2\beta_4}}(msd(x+y-\chi t)\pm nd(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{(m^2-1)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{msd(x+y-\chi t)\pm nd(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{63}$$

or

$$V_{330}(x+y-\chi t)=\Phi_{330}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{(m^2-1)}{2\beta_4}}(-msd(x+y-\chi t)\mp nd(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{(m^2-1)(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{-msd(x+y-\chi t)\mp nd(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{64}$$

and $((\beta_1+\beta_2+\beta_3)(1+m^2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9(m^2-1)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case 22. If $p=1/4$, $q=(m^2-2)/2$, and $r=m^4/4$, then $K(\xi)=ns\xi\pm ds\xi$ or $-ns\xi\mp ds\xi$, or $dc\xi\pm\sqrt{1-m^2}nc\xi$, or $-dc\xi\mp\sqrt{1-m^2}nc\xi$

$$\Phi_{331}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(ns(x+y-\chi t)\pm ds(x+y-\chi t))$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{ns(x+y-\chi t)\pm ds(x+y-\chi t)}\right),\tag{65}$$

or

$$\Phi_{332}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(-ns(x+y-\chi t)\mp ds(x+y-\chi t))$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{-ns(x+y-\chi t)\mp ds(x+y-\chi t)}\right),\tag{66}$$

or

$$\Phi_{333}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}\left(dc(x+y-\chi t)\pm\sqrt{1-m^2}nc(x+y-\chi t)\right)$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{dc(x+y-\chi t)\pm\sqrt{1-m^2}nc(x+y-\chi t)}\right),\tag{67}$$

or

$$\Phi_{334}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}\left(-dc(x+y-\chi t)\mp\sqrt{1-m^2}nc(x+y-\chi t)\right)$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\cdot\left(\frac{1}{-dc(x+y-\chi t)\mp\sqrt{1-m^2}nc(x+y-\chi t)}\right);\tag{68}$$

$$V_{331}(x+y-\chi t)=\Phi_{331}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}(ns(x+y-\chi t)\pm ds(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{ns(x+y-\chi t)\pm ds(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{69}$$

or

$$V_{332}(x+y-\chi t)=\Phi_{332}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}(-ns(x+y-\chi t)\mp ds(x+y-\chi t))\right.$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{-ns(x+y-\chi t)\mp ds(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{70}$$

or

$$V_{333}(x+y-\chi t)=\Phi_{333}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}\left(dc(x+y-\chi t)\pm\sqrt{1-m^2}nc(x+y-\chi t)\right)\right.$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{dc(x+y-\chi t)\pm\sqrt{1-m^2}nc(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{71}$$

or

$$V_{334}(x+y-\chi t)=\Phi_{334}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}\left(-dc(x+y-\chi t)\mp\sqrt{1-m^2}nc(x+y-\chi t)\right)\right.$$
$$\pm\sqrt{\frac{m^4(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{1}{-dc(x+y-\chi t)\mp\sqrt{1-m^2}nc(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{72}$$

and $((\beta_1+\beta_2+\beta_3)(m^2-2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9m^4/4)(\beta_1+\beta_2+\beta_3)$.

Case 23. If $p=m^4/4$, $q=(m^2-2)/2$, and $r=1/4$, then $K(\xi)=sn\xi/1\pm dn\xi$ or $-sn\xi/1\pm dn\xi$, or $cn\xi/dn\xi\pm\sqrt{1-m^2}$, or $-cn\xi/dn\xi\pm\sqrt{1-m^2}$

$$\Phi_{335}(x+y-\chi t)=\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{sn(x+y-\chi t)}{1\pm dn(x+y-\chi t)}\right)$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1\pm dn(x+y-\chi t)}{sn(x+y-\chi t)}\right),\tag{73}$$

or

$$\Phi_{336}(x+y-\chi t)=\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{-sn(x+y-\chi t)}{1\pm dn(x+y-\chi t)}\right)$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1\pm dn(x+y-\chi t)}{-sn(x+y-\chi t)}\right),\tag{74}$$

or

$$\Phi_{337}(x+y-\chi t)=\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{cn(x+y-\chi t)}{dn(x+y-\chi t)\pm\sqrt{1-m^2}}\right)$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{dn(x+y-\chi t)\pm\sqrt{1-m^2}}{cn(x+y-\chi t)}\right),\tag{75}$$

or

$$\Phi_{338}(x+y-\chi t)=\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{-cn(x+y-\chi t)}{dn(x+y-\chi t)\pm\sqrt{1-m^2}}\right)$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{dn(x+y-\chi t)\pm\sqrt{1-m^2}}{-cn(x+y-\chi t)}\right);\tag{76}$$

$$V_{335}(x+y-\chi t)=\Phi_{335}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{sn(x+y-\chi t)}{1\pm dn(x+y-\chi t)}\right)\right.$$
$$\left.\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1\pm dn(x+y-\chi t)}{sn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{77}$$

or

$$V_{336}(x+y-\chi t)=\Phi_{336}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{-sn(x+y-\chi t)}{1\pm dn(x+y-\chi t)}\right)\right.$$
$$\left.\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1\pm dn(x+y-\chi t)}{-sn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{78}$$

or

$$V_{337}(x+y-\chi t)=\Phi_{337}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$\cdot=\left(\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{cn(x+y-\chi t)}{dn(x+y-\chi t)\pm\sqrt{1-m^2}}\right)\right.$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{dn(x+y-\chi t)\pm\sqrt{1-m^2}}{cn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{79}$$

or

$$V_{338}(x+y-\chi t)=\Phi_{338}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{m^4}{2\beta_4}}\left(\frac{-cn(x+y-\chi t)}{dn(x+y-\chi t)\pm\sqrt{1-m^2}}\right)\right.$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}$$
$$\left.\cdot\left(\frac{dn(x+y-\chi t)\pm\sqrt{1-m^2}}{-cn(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{80}$$

and $((\beta_1+\beta_2+\beta_3)(m^2-2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=(9m^4/4)(\beta_1+\beta_2+\beta_3)$.

Case 24. If $p=1/4$, $q=(1-2m^2)/2$, and $r=1/4$, then $K(\xi)=ns\xi\pm cs\xi$ or $-ns\xi\mp cs\xi$, or $dc\xi\pm\sqrt{1-m^2}sc\xi$, or $-dc\xi\mp\sqrt{1-m^2}sc\xi$

$$\Phi_{339}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(ns(x+y-\chi t)\pm cs(x+y-\chi t))$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1}{ns(x+y-\chi t)\pm cs(x+y-\chi t)}\right),\tag{81}$$

or

$$\Phi_{340}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(-ns(x+y-\chi t)\mp cs(x+y-\chi t))\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\cdot\left(\frac{1}{-ns(x+y-\chi t)\mp cs(x+y-\chi t)}\right),\tag{82}$$

or

$$\Phi_{341}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}\left(dc(x+y-\chi t)\pm\sqrt{1-m^2}sc(x+y-\chi t)\right)\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\cdot\left(\frac{1}{dc(x+y-\chi t)\pm\sqrt{1-m^2}sc(x+y-\chi t)}\right),\tag{83}$$

or

$$\Phi_{342}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}\left(-dc(x+y-\chi t)\pm\sqrt{1-m^2}sc(x+y-\chi t)\right)\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\cdot\left(\frac{1}{-dc(x+y-\chi t)\pm\sqrt{1-m^2}sc(x+y-\chi t)}\right);\tag{84}$$

$$V_{339}(x+y-\chi t)=\Phi_{339}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}(ns(x+y-\chi t)\pm cs(x+y-\chi t))\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1}{ns(x+y-\chi t)\pm cs(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{85}$$

or

$$V_{340}(x+y-\chi t)=\Phi_{340}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}(-ns(x+y-\chi t)\mp cs(x+y-\chi t))\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1}{-ns(x+y-\chi t)\mp cs(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{86}$$

or

$$V_{341}(x+y-\chi t)=\Phi_{341}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}\left(dc(x+y-\chi t)\pm\sqrt{1-m^2}sc(x+y-\chi t)\right)\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\cdot\left(\frac{1}{dc(x+y-\chi t)\pm\sqrt{1-m^2}sc(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{87}$$

or

$$V_{342}(x+y-\chi t)=\Phi_{342}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)},$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}\left(-dc(x+y-\chi t)\mp\sqrt{1-m^2}sc(x+y-\chi t)\right)\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\cdot\left(\frac{1}{-dc(x+y-\chi t)\mp\sqrt{1-m^2}sc(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{88}$$

and $((\beta_1+\beta_2+\beta_3)(1-2m^2/2)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=9/4(\beta_1+\beta_2+\beta_3)$.

Besides, if $m\longrightarrow 0$ and $m\longrightarrow 1$, then the Jacobi elliptic functions become vestigial trigonometric and hyperbolic functions ([31]). Now we give specific examples as follows:

Case 3 If $m\longrightarrow 0$ then $cs\xi\longrightarrow\cot\xi$, then

$$\Phi_{35}(x+y-\chi t)=\pm\sqrt{\frac{2}{\beta_4}}\cot(x+y-\chi t)\pm\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\tan(x+y-\chi t);\tag{89}$$

$$V_{35}(x+y-\chi t)=\Phi_{35}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}=\left(\pm\sqrt{\frac{2}{\beta_4}}\cot(x+y-\chi t)\pm\sqrt{\frac{2(\beta_1+\beta_2+\beta_3)}{\beta_4}}\tan(x+y-\chi t)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)},\tag{90}$$

and $(2(\beta_1+\beta_2+\beta_3)+(\beta_3kl-\omega-\beta_1l^2-\beta_2k^2))^2=36(\beta_1+\beta_2+\beta_3)$.

If $m\longrightarrow 1$ then $cs\xi\longrightarrow csch\xi$, and

$$\Phi_{35}(x+y-\chi t)=\pm\sqrt{\frac{2}{\beta_4}}\operatorname{csch}(x+y-\chi t);$$

$$V_{35}(x+y-\chi t)=\Phi_{35}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\pm\sqrt{\frac{2}{\beta_4}}\,\mathrm{csch}\,(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{91}$$

and $((\beta_1+\beta_2+\beta_3)+(\beta_3 kl-\omega-\beta_1 l^2-\beta_2 k^2))^2=0$.

Case15. If $m\longrightarrow 0$, then $ds\xi\pm cs\xi\longrightarrow \csc\xi\pm\cot\xi$ or $-ds\xi\mp cs\xi\longrightarrow -\csc\xi\mp\cot\xi$,

$$\Phi_{317}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(\csc\,(x+y-\chi t)\pm\cot\,(x+y-\chi t))$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\frac{1}{\csc\,(x+y-\chi t)\pm\cot\,(x+y-\chi t)}, \tag{92}$$

or

$$\Phi_{318}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}(-\csc\,(x+y-\chi t)\mp\cot\,(x+y-\chi t))$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\frac{1}{(-\csc\,(x+y-\chi t)\mp\cot\,(x+y-\chi t))}; \tag{93}$$

$$V_{317}(x+y-\chi t)=\Phi_{317}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}(\csc\,(x+y-\chi t)\pm\cot\,(x+y-\chi t))\right.$$
$$\left.\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)(1-m^2)^2}{2\beta_4}}\frac{1}{(\csc\,(x+y-\chi t)\pm\cot\,(x+y-\chi t))}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{94}$$

or

$$V_{318}(x+y-\chi t)=\Phi_{318}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}=\left(\pm\sqrt{\frac{1}{2\beta_4}}(-\csc\,(x+y-\chi t)\mp\cot\,(x+y-\chi t))\right.$$
$$\left.\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)(1-m^2)^2}{2\beta_4}}\frac{1}{(-\csc\,(x+y-\chi t)\mp\cot\,(x+y-\chi t))}\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{95}$$

and $((\beta_1+\beta_2+\beta_3)((1+m^2)/2)+(\beta_3 kl-\omega-\beta_1 l^2-\beta_2 k^2))^2$ $=(9(1-m^2)^2/4)(\beta_1+\beta_2+\beta_3)$.

Case23. If $m\longrightarrow 1$, then $(sn\xi/1\pm dn\xi)\longrightarrow(\tanh\xi/1\pm sech\xi)$ and

$$\Phi_{335}(x+y-\chi t)=\pm\sqrt{\frac{1}{2\beta_4}}\left(\frac{\tanh\,(x+y-\chi t)}{1\pm\mathrm{sech}\,(x+y-\chi t)}\right)$$
$$\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1\pm\mathrm{sech}\,(x+y-\chi t)}{\tanh\,(x+y-\chi t)}\right);$$

$$V_{335}(x+y-\chi t)=\Phi_{335}(x+y-\chi t)e^{i(-lx+ky+\omega t+\varepsilon_2)}$$
$$=\left(\pm\sqrt{\frac{1}{2\beta_4}}\left(\frac{\tanh\,(x+y-\chi t)}{1\pm\mathrm{sech}\,(x+y-\chi t)}\right)\right.$$
$$\left.\pm\sqrt{\frac{(\beta_1+\beta_2+\beta_3)}{2\beta_4}}\left(\frac{1\pm\mathrm{sech}\,(x+y-\chi t)}{\tanh\,(x+y-\chi t)}\right)\right)e^{i(-lx+ky+\omega t+\varepsilon_2)}, \tag{96}$$

and $((\beta_1+\beta_2+\beta_3)(-1/2)+(\beta_3 kl-\omega-\beta_1 l^2-\beta_2 k^2))^2=9/4$ $(\beta_1+\beta_2+\beta_3)$.

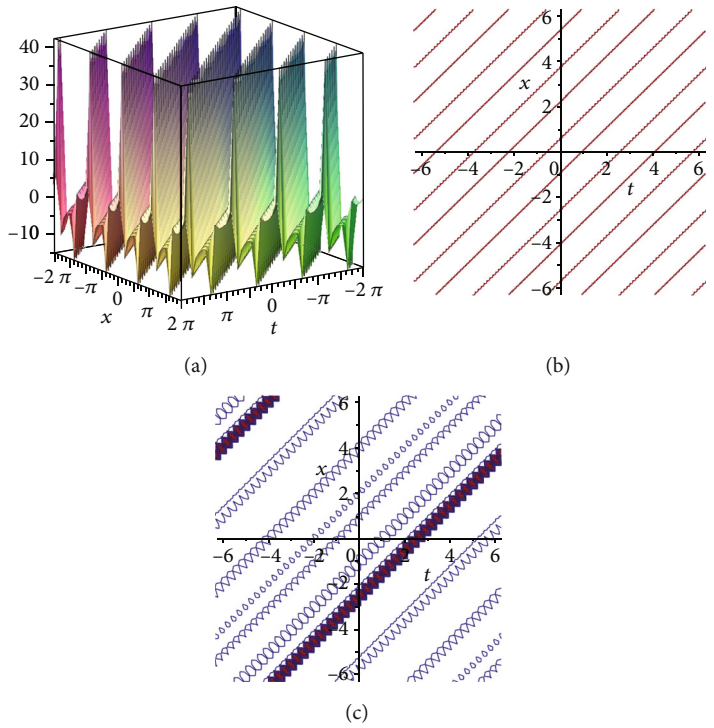

Figure 1: The 3D, 2D, and contour images of $\Phi_{35}(\xi.)$.

4. Comparison

In this paper, employing the EMVJEEM to the HFSCE, we found that various forms exact solutions for HFSCE. In 2021, Ünal et al. [30] investigated the exact solutions of space-time fractional symmetric regularized long wave equation using ideas of the Jacobi elliptic functions. In our paper, Equation (10) is $\Phi(\xi) = \sum_{j=-N}^{N} a_j K^j(\xi)$. Here, j starts from the negative number to the corresponding positive number. However, in [30], j only takes $0, 1, \cdots, N$, this is the biggest difference of [30]. From there, j can take the negative number; our form is more extensive than [30].

Comparing with [29, 32], our article still varies greatly. In this paper, Equation (11) is $(K')^2(\xi) = r + qK^2(\xi) + pK^4(\xi)$. However, in [29, 32], what is similar to Equation (11) is $(K')^2(\xi) = a_0 + a_1K + a_2K^2 + a_3K^3 + a_4K^4$. Due to the importance of Formula (11), this is obviously a big difference between our paper and [29].

From the perspective of the complex calculation process, our paper is more complex than [27, 30]. At the same time, the results in our paper are more concise and clearer than [29, 32].

Recently, the complex method [33, 34] is used to study the exact solutions of nonlinear evolution equations and found the Weierstrass elliptic functions (WEFs) solution, hyperbolic function or trigonometric meromorphic solutions, and rational solutions. In our paper, we also can find more solutions by the EMVJEEM, including hyperbolic function solutions or trigonometric solutions. The WEFs solutions $\wp(\xi)$ satisfy the equation $(K')^2(\xi) = 4K^3(\xi) - g_2K(\xi) - g_3$. In this paper, Equation (11) is $(K')^2(\xi) = pK^4(\xi) + qK^2(\xi) + r$. In the above two equations, K of 3 power varies greatly from K of 4 power. At the same time, the relationship between the JEFs and WEFs is

$$\wp(\xi, g_2, g_3) = k_2 - (k_2 - k_3)cn^2\left(\sqrt{k_1 - k_3}\,; m\right), \tag{97}$$

here $m^2 = (k_2 - k_3)/(k_1 - k_3)$ is the modulus number of JEFs and $k_i (i = 1, 2, 3, k_1 \geq k_2 \geq k_3)$ are the roots of equation $4\xi^2 - g_2\xi - g_3 = 0$.

It is well know that if $m \longrightarrow 1$ and $e_2 \longrightarrow e_1$, then $cn(\xi\,; m) \longrightarrow \operatorname{sech}(\xi)$, or if $m \longrightarrow 0$ then $cn(\xi\,; m) \longrightarrow \cos(\xi)$. Equation (97) is the bridge linking among the WEFs, hyperbolic function, trigonometric function, and JEFs.

5. Computer Simulations

In this section, we are trying to explain the results through computer simulation images and further analyze the nature of the $\Phi_{35}(\xi)$, $\Phi_{317}(\xi)$, $\Phi_{318}(\xi)$, and $\Phi_{335}(\xi)$ in the equations.

Figure 1 The 3D, 2D, and contour images of $\Phi_{35}(\xi)$, when $m \longrightarrow 0$, we take the values of $\beta_1 = 1$, $\beta_2 = 4$, $\beta_3 = 3$, $\beta_4 = 4$, $y = 1$, and $q = 1$; the graphs demonstrate the localized interactions of diverse solitons of $\Phi_{35}(\xi)$ on the domain. We also find the waves train in Figure 1(a).

Figure 2: The 3D, 2D, and contour images of $\Phi_{35}(\xi)$, when $m \longrightarrow 1$, we take the values of $\beta_4 = 2$, $y = 1$, and $q = 1$; the graphs depict the single kink solitary wave solution $\Phi_{35}(\xi)$ on the domain.

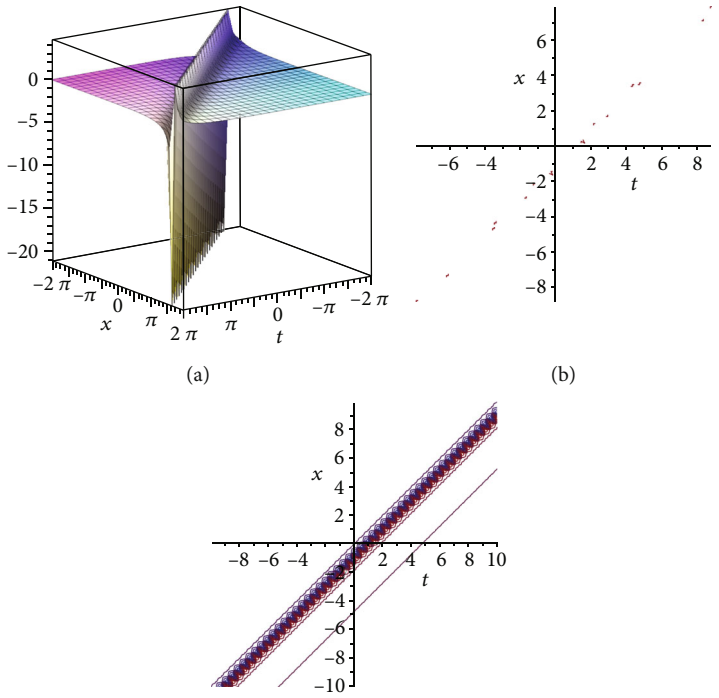

Figure 2: The 3D, 2D, and contour images of $\Phi_{35}(\xi)$.

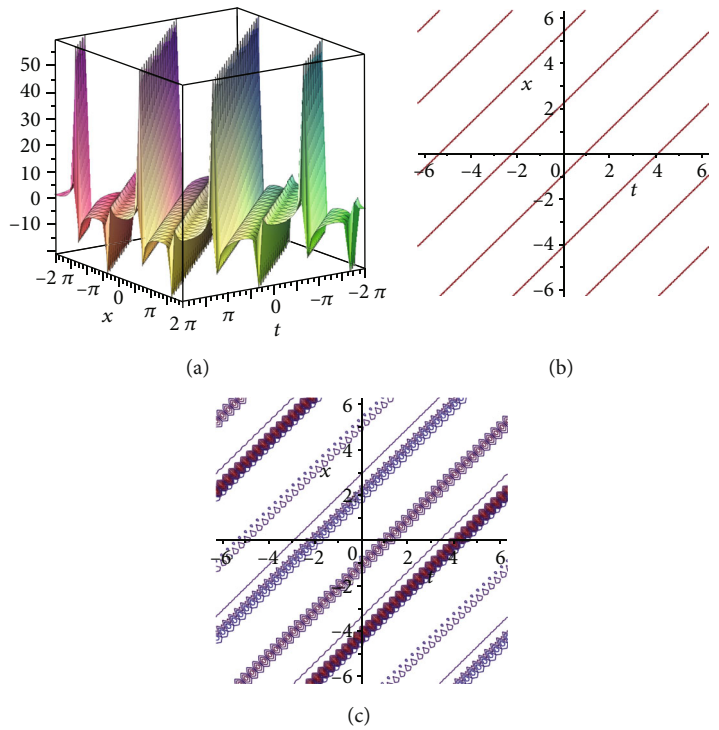

Figure 3: The 3D, 2D, and contour images of $\Phi_{317}(\xi)$.

Figure 3: The 3D, 2D, and contour images of $\Phi_{317}(\xi)$, we take the values of $\beta_1 = 1$, $\beta_2 = 4$, $\beta_3 = 3$, $\beta_4 = 2$, $y = 1$, and $q = 1$; the graphs clearly demonstrate the multiperiodic characteristic of $\Phi_{317}(\xi)$ on the domain.

Figure 4: The 3D, 2D, and contour images of $\Phi_{318}(\xi)$, we take the values of $\beta_1 = 1$, $\beta_2 = 4$, $\beta_3 = 3$, $\beta_4 = 2$, $y = 1$, and $q = 1$; the graphs show the periodic curved parabolic structure of $\Phi_{318}(\xi)$ on the domain.

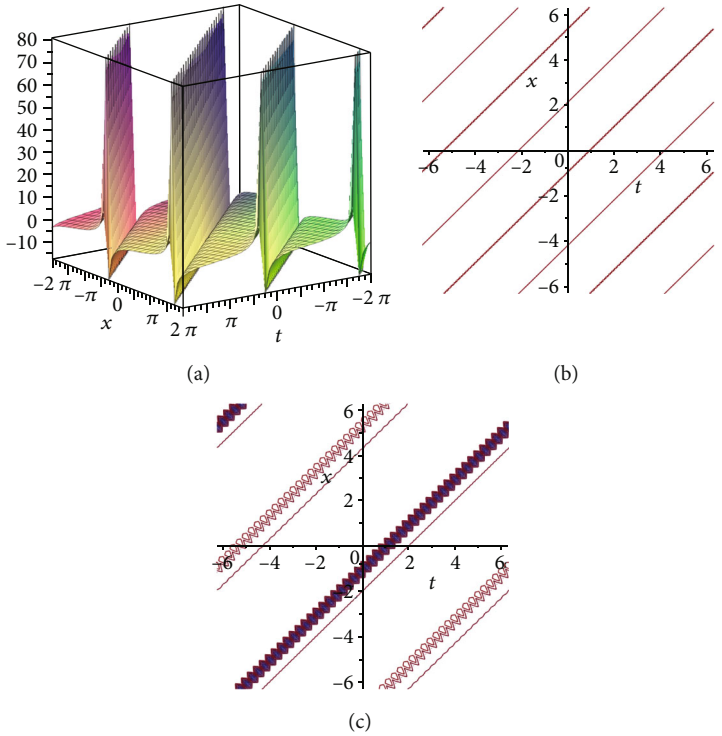

FIGURE 4: The 3D, 2D, and contour images of $\Phi_{318}(\xi)$.

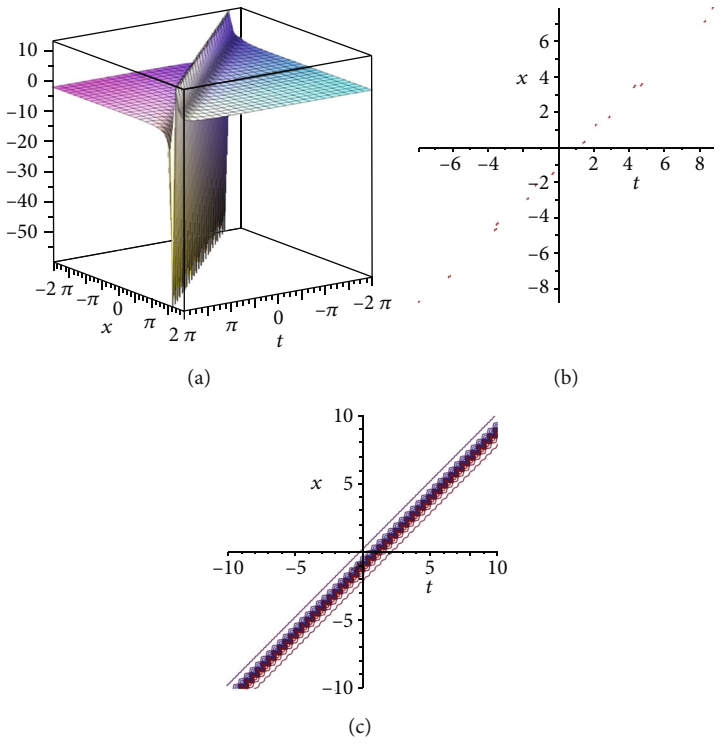

FIGURE 5: The 3D, 2D, and contour images of $\Phi_{335}(\xi)$.

Figure 5: The 3D, 2D, and contour images of $\Phi_{335}(\xi)$ by considering the values $\beta_1 = 1$, $\beta_2 = 4$, $\beta_3 = 3$, $\beta_4 = 2$, $y = 1$, and $q = 1$; the graphs demonstrate the interactions between one soliton with kink waves of $\Phi_{335}(\xi)$ on the domain.

The all above wave profiles solutions of HFSCE keep their velocities, shapes, and amplitudes invariant during the appropriate value of some specific parameters. In Figures 1, 3,and 4(a), we can see M-shaped waves train. In

Figures 2 and 5(a), we also see the multiple bright and dark peak solitons and also seen their behavior in contour (Figures 1–5(c)) and 2D (Figures 1–5(b)) structures, respectively.

6. Conclusion and Future Study

The EMVJEEM was implemented perfectly for the first time in the framework of these techniques to design these new types of soliton solutions for this model. The achieved soliton solutions show the potentially traveling wave solutions to the (2 + 1)-dimensional HFSCE. The resulting sample of the achieved solutions offers a rich podium to study the nonlinear spin dynamics in magnetic materials. The derived solutions having abundant applications to handle spin dynamics in magnetic materials and transmission of high-frequency waves in tranquil medium. We have successfully used EMVJEEM to construct a rich variety of exact solutions of HFSCE under various family cases in this paper. By choosing suitable best values for the constant parameters, these new complex soliton solutions established the dynamical behaviors through 3D and 2D wave profiles via simulation. When compared with [27, 35], our paper has the largest number solutions. Because there are 42 types of linear independent solutions for 23 different cases in the presented method. Hence, these new exact solutions will play an important role in understanding and investigating the HFSCE.

In summary, the results of the full text eloquently prove that the above EMVJEEM is very efficient and powerful in solving the exact solutions of nonlinear evolution equations now and in the future. We can apply EMVJEEM of this research to other nonlinear evolution equations.

In the meanwhile, we notice if there is additional noise, the equation becomes a stochastic differential equation (SDE),

$$(\beta_1 + \beta_2 + \beta_3)\Phi'' - (\omega + \beta_1 l^2 + \beta_2 k^2 - \beta_3 kl)\Phi - \beta_4 \Phi^3 - \xi = 0. \tag{98}$$

This model, compared to the previous model (8), is more pervasive in scientific computing experiments, for the reason that such noise exists in both complicated nature phenomenon and artificial model errors. Usually, the noise is Gaussian white noise. According to the theory of existence and uniqueness of solution to SDE, one can find solution to the SDE problem (98) under certain conditions [36].

Numerical solutions can also be found by various methods, for example, the Runge-Kutta method [37], the Euler method [38, 39], and the Milstein method [40, 41]. Since the Runge-Kutta method is far more complicated than the Euler method and the Milstein method, one usually performs the Euler method and the Milstein method in real applications. One can perform numerical experiments to the SDE and compare the strong and weak convergence, numerical errors, and stability of the numerical methods [42].

For the noise perturbed system (8), one can observe the property of energy landscape and the phenomenon of exit from basin of attraction under the framework of the large deviation theory [43]. We will discuss about that in later work. From the data assimilation point of view, when Equation (8) has unobservable state Φ and noisy observations of Φ, one can perform filtering strategy to recover the state Φ, based on a the Bayesian framework [44]. We will also explore this issue in future work.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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