Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 2129677, 14 pages
https://doi.org/10.1155/2022/2129677

*Research Article*

# Statistical Inference of Stress-Strength Reliability of Gompertz Distribution under Type II Censoring

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Received 23 February 2022; Revised 4 August 2022; Accepted 24 August 2022; Published 3 December 2022

Academic Editor: Zengtao Chen

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This paper develops the problem of estimating stress-strength reliability for Gompertz lifetime distribution. First, the maximum likelihood estimation (MLE) and exact and asymptotic confidence intervals for stress-strength reliability are obtained. Then, Bayes estimators under informative and noninformative prior distributions are obtained by using Lindley approximation, Monte Carlo integration, and MCMC. Bayesian credible intervals are constructed under these prior distributions. Also, simulation studies are used to illustrate these inference methods. Finally, a real dataset is analyzed to show the implementation of the proposed methodologies.

## 1. Introduction

The stress-strength reliability $R = P(X > Y)$ is an assessment of the reliability of a component based on its strength $X$ and its stress $Y$. The idea of a stress-strength reliability was introduced by Birnbaum [1] and spread two years later by Birnbaum and McCarty [2].

Recently, the study on reliability has been considered by the authors, which we refer to some recent studies. Qixuan and Wenhao [3] worked on the Bayesian and classical estimation of stress-strength reliability for inverse Weibull lifetime models. Abravesh et al. [4] obtained classical and Bayesian estimation of stress-strength reliability in type II censored Pareto distributions. Akgül et al. [5] presented inferences on stress-strength reliability based on ranked set sampling data in the case of Lindley distribution. Byrnes et al. [6] made a Bayesian inference of $R$ for Burr type XII distribution based on progressively first failure-censored samples. Zhang et al. [7] studied the reliability of generalized Rayleigh distribution under progressive type II censoring.

Gompertz distribution which was first proposed by Gompertz [8] is one of the most widely used distributions in the fields of survival, lifetime data, mortality tables, computer, biology, sociology, and marketing [9–12]. Some recent studies on the distribution of Compretz include the following: [13] presented a new and practical generalization of the Compretz distribution. [14] developed acceptance sampling plans for lot sentencing in which the quality characteristic of the products follows the Topp-Leone Gompertz distribution. An application of Gamma-Gompertz distribution was proposed by [15]. [16] estimated the parameters of a new generalization of Gompertz distribution and investigated the features and application of this new model.

The study on reliability by considering Gompertz distribution is one of the most important and interesting issues. Saraçoğlu and Kaya [17] studied MLE and confidence intervals of system reliability for Gompertz distribution in stress-strength models. Kumar and Vaish [18] presented a study of strength reliability for Gompertz distributed stress. Jha et al. [19] obtained reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive type II censoring. Asadi et al. [20] studied inference on adaptive progressive hybrid censored accelerated life test for Gompertz distribution.

A brief explanation of the type II censoring is given. Let $x_1, x_2, \cdots, x_{n_1}$ and $y_1, y_2, \cdots, y_{n_2}$ be independent random samples from $X$ and $Y$ random variable, respectively. Suppose the ordered statistics of these samples are $x_{(1)} < x_{(2)} < \cdots < x_{(n_1)}$

and $y_{(1)} < y_{(2)} < \cdots < y_{(n_2)}$. $x_i$'s and $y_i$'s are collected until $r_1$ failures and $r_2$ failures occur, respectively (where $r_1 \le n_1$ and $r_2 \le n_2$).

The rest of the article is organized as follows. In Section 2, we introduce the Gompertz distribution. In Section 3, we obtain the MLE of stress strength reliability ($R$). In Section 4, we construct the exact and asymptotic confidence intervals for $R$. In Section 5, we calculate the Bayes estimator of $R$ by considering the conjugate informative and Jeffreys noninformative prior distributions. In Section 6, we provide Bayesian credible intervals, including equi-tailed and HPD intervals under the conjugate informative and Jeffreys noninformative prior distributions. In Section 7, the performance of these inference methods is compared by using simulation studies. Finally, Section 8 performs a real data analysis to demonstrate the application of these methods.

## 2. Gompertz Distribution

Let $X \sim \text{Gompertz}(\beta_1, \gamma)$ and $Y \sim \text{Gompertz}(\beta_2, \gamma)$ be two independent random variables. The probability density function (PDF) and cumulative distribution function (CDF) of $X$ and $Y$ are given:

$$\begin{aligned} f_X(x; \beta_1, \gamma) &= \beta_1 e^{\gamma x} e^{-\beta_1/\gamma(e^{\gamma x}-1)}, \; F_X(x; \beta_1, \gamma) = 1 - e^{-\beta_1/\gamma(e^{\gamma x}-1)}, \; x > 0, \beta_1, \gamma > 0, \\ f_Y(y; \beta_2, \gamma) &= \beta_2 e^{\gamma y} e^{-\beta_2/\gamma(e^{\gamma y}-1)}, \; F_Y(y; \beta_2, \gamma) = 1 - e^{-\beta_2/\gamma(e^{\gamma y}-1)}, \; y > 0, \beta_2, \gamma > 0. \end{aligned} \tag{1}$$

The reliability function is calculated as follows:

$$\begin{aligned} R &= P(X > Y) \\ &= \int_{-\infty}^{\infty} F_Y(w) f_X(w) dw \\ &= \int_0^{\infty} \left(1 - e^{-\beta_2/\gamma(e^{\gamma w}-1)}\right) \beta_1 e^{\gamma w} e^{-\beta_1/\gamma(e^{\gamma w}-1)} dw \\ &= \frac{\beta_2}{\beta_1 + \beta_2}. \end{aligned} \tag{2}$$

## 3. MLE of $R$

Let $x_{(1)}, x_{(2)}, \cdots, x_{(r_1)}$ be a type II censored sample from Gompertz$(\beta_1, \gamma)$ and $y_{(1)}, y_{(2)}, \cdots, y_{(r_2)}$ be a type II censored sample from Gompertz$(\beta_2, \gamma)$. Suppose these two samples are independent. The likelihood function is given by

$$\begin{aligned} L(\beta_1, \beta_2, \gamma | \mathbf{x}, \mathbf{y}) &= \frac{n_1! n_2!}{(n_1 - r_1)!(n_2 - r_2)!} \prod_{i=1}^{r_1} f_X\left(x_{(i)}; \beta_1, \gamma\right) \left[S_X\left(x_{(r_1)}\right)\right]^{n_1 - r_1} \\ &\quad \times \prod_{j=1}^{r_2} f_Y\left(y_{(j)}; \beta_2, \gamma\right) \left[S_Y\left(y_{(r_2)}\right)\right]^{n_2 - r_2} \\ &= \frac{n_1! n_2!}{(n_1 - r_1)!(n_2 - r_2)!} \beta_1^{r_1} e^{\gamma \sum_{i=1}^{r_1} x_{(i)}} e^{-\beta_1/\gamma \sum_{i=1}^{r_1} (e^{\gamma x_{(i)}} - 1)} e^{-\beta_1 (n_1 - r_1)/\gamma \left(e^{\gamma x_{(r_1)}} - 1\right)} \\ &\quad \times \beta_2^{r_2} e^{\gamma \sum_{j=1}^{r_2} y_{(j)}} e^{-\beta_2/\gamma \sum_{j=1}^{r_2} (e^{\gamma y_{(j)}} - 1)} e^{-\beta_2 (n_2 - r_2)/\gamma \left(e^{\gamma y_{(r_2)}} - 1\right)} \\ &= \frac{n_1! n_2!}{(n_1 - r_1)!(n_2 - r_2)!} \beta_1^{r_1} \beta_2^{r_2} e^{\gamma \left[\sum_{i=1}^{r_1} x_{(i)} + \sum_{j=1}^{r_2} y_{(j)}\right]} e^{-\beta_1 b_1'} e^{-\beta_2 b_2'}, \end{aligned} \tag{3}$$

where

$$b_1' = \frac{1}{\gamma} \left[\sum_{i=1}^{r_1} e^{\gamma x_{(i)}} + (n_1 - r_1) e^{\gamma x_{(r_1)}} - n_1\right], \tag{4}$$

$$b_2' = \frac{1}{\gamma} \left[\sum_{j=1}^{r_2} e^{\gamma y_{(j)}} + (n_2 - r_2) e^{\gamma y_{(r_2)}} - n_2\right]. \tag{5}$$

Then, the log-likelihood function is

$$\begin{aligned} l(\beta_1, \beta_2, \gamma | \mathbf{x}, \mathbf{y}) &= \log (n_1! n_2!) - \log \left[(n_1 - r_1)!(n_2 - r_2)!\right] \\ &\quad + \gamma \left[\sum_{i=1}^{r_1} x_{(i)} + \sum_{j=1}^{r_2} y_{(j)}\right] + r_1 \log \beta_1 + r_2 \log \beta_2 \\ &\quad - \frac{\beta_1}{\gamma} \left[\sum_{i=1}^{r_1} e^{\gamma x_{(i)}} + (n_1 - r_1) e^{\gamma x_{(r_1)}} - n_1\right] \\ &\quad - \frac{\beta_2}{\gamma} \left[\sum_{j=1}^{r_2} e^{\gamma y_{(j)}} + (n_2 - r_2) e^{\gamma y_{(r_2)}} - n_2\right]. \end{aligned} \tag{6}$$

To obtain the MLE of parameters $\beta_1, \beta_2$ and $\gamma$, it is sufficient to derive the log-likelihood function with respect to parameters $\beta_1, \beta_2$ and $\gamma$ and equal them to zero:

$$\frac{\partial l}{\partial \beta_1} = \frac{r_1}{\beta_1} - \frac{1}{\gamma} \left[\sum_{i=1}^{r_1} e^{\gamma x_{(i)}} + (n_1 - r_1) e^{\gamma x_{(r_1)}} - n_1\right] = 0, \tag{7}$$

$$\frac{\partial l}{\partial \beta_2} = \frac{r_2}{\beta_2} - \frac{1}{\gamma} \left[\sum_{j=1}^{r_2} e^{\gamma y_{(j)}} + (n_2 - r_2) e^{\gamma y_{(r_2)}} - n_2\right] = 0, \tag{8}$$

$$\begin{aligned} \frac{\partial l}{\partial \gamma} &= \frac{r_1}{\gamma} + \frac{r_2}{\gamma} + \left[\sum_{i=1}^{r_1} x_{(i)} + \sum_{j=1}^{r_2} y_{(j)}\right] \\ &\quad - \frac{\beta_1}{\gamma} \left[\sum_{i=1}^{r_1} x_{(i)} e^{\gamma x_{(i)}} + (n_1 - r_1) x_{(r_1)} e^{\gamma x_{(r_1)}}\right] \\ &\quad - \frac{\beta_2}{\gamma} \left[\sum_{j=1}^{r_2} y_{(j)} e^{\gamma y_{(j)}} + (n_2 - r_2) y_{(r_2)} e^{\gamma y_{(r_2)}}\right]. \end{aligned} \tag{9}$$

From Equations (7) and (8), we get

$$\widehat{\beta}_1 = \frac{\gamma r_1}{\left[\sum_{i=1}^{r_1} e^{\gamma x_{(i)}} + (n_1 - r_1) e^{\gamma x_{(r_1)}} - n_1\right]} = \frac{r_1}{b_1'}, \tag{10}$$

$$\widehat{\beta}_2 = \frac{\gamma r_2}{\left[\sum_{j=1}^{r_2} e^{\gamma y_{(j)}} + (n_2 - r_2) e^{\gamma y_{(r_2)}} - n_2\right]} = \frac{r_2}{b_2'}. \tag{11}$$

Now, by substituting (10) and (11) into (9), the MLE of parameter $\gamma(\widehat{\gamma})$ is obtained. Then, to get $\widehat{\beta}_1$ and $\widehat{\beta}_2$, we substitute $\widehat{\gamma}$ into Equations (10) and (11). Therefore, the MLE of $R$ is

$$\widehat{R} = \frac{\widehat{\beta}_2}{\widehat{\beta}_1 + \widehat{\beta}_2}. \tag{12}$$

## 4. Confidence Interval of $R$

In this section, the exact and asymptotic confidence intervals for $R$ are calculated.

*4.1. Exact Confidence Interval.* Let $x_{(1)}, x_{(2)}, \cdots, x_{(r_1)}$ be a type II censored sample from Gompertz$(\beta_1, \gamma)$. Consider $W_i = \beta_1/\gamma(e^{\gamma x_{(i)}} - 1), i = 1, 2, \cdots, r_1$, where $w_1 \le w_2 \le \cdots \le w_{r_1}$ is a type II censored dependent sample from the standard exponential distribution (SED). Now, apply the following conversion:

$$W_i' = (n_1 - i + 1)(W_i - W_{i-1}). \tag{13}$$

It can be concluded that $W_1', W_2', \cdots, W_{r_1}' \sim^{\text{ind}}$ SED. Therefore,

$$2\sum_{i=2}^{r_1} W_i' \sim \chi^2_{(2(r_1-1))}. \tag{14}$$

Similarly, suppose $y_{(1)}, y_{(2)}, \cdots, y_{(r_2)}$ be a type II censored sample from Gompertz$(\beta_2, \gamma)$. Define $M_j = \beta_2/\gamma(e^{\gamma y_{(j)}} - 1)$, $j = 1, 2, \cdots, r_2$, where $m_1 \le m_2 \le \cdots \le m_{r_1}$ is a type II censored dependent sample from the SED. Now, apply the following transformation:

$$M_j' = (n_2 - j + 1)\left(M_j - M_{j-1}\right). \tag{15}$$

It results that $M_1', M_2', \cdots, M_{r_2}' \sim^{\text{ind}}$ SED. So,

$$2\sum_{j=2}^{r_2} M_j' \sim \chi^2_{(2(r_2-1))}. \tag{16}$$

Based on the independence of $\sum_{i=2}^{r_1} W_i'$ and $\sum_{j=2}^{r_2} M_j'$ can be written

$$F_0 = \frac{(r_2 - 1)\sum_{i=2}^{r_1} W_i'}{(r_1 - 1)\sum_{j=2}^{r_2} M_j'} \sim F_{(2(r_1-1), 2(r_2-1))}. \tag{17}$$

Then, confidence interval for $R$ is

$$\left(\frac{1}{1 + Q_2/Q_1 F_{1-\xi/2}(2(r_1 - 1), 2(r_2 - 1))}, \frac{1}{1 + Q_2/Q_1 F_{\xi/2}(2(r_1 - 1), 2(r_2 - 1))}\right), \tag{18}$$

where

$$\begin{aligned} Q_1 &= \frac{r_2 - 1}{\gamma}\sum_{i=2}^{r_1}(n_1 - i + 1)\left(e^{\gamma x_{(i)}} - e^{\gamma x_{(i-1)}}\right), \\ Q_2 &= \frac{r_1 - 1}{\gamma}\sum_{j=2}^{r_2}(n_2 - j + 1)\left(e^{\gamma y_{(j)}} - e^{\gamma y_{(j-1)}}\right). \end{aligned} \tag{19}$$

*4.2. Asymptotic Confidence Interval.* In this section, the asymptotic confidence interval for $R$ is calculated using Wald statistics. Based on Wald statistics, we have

$$W = \frac{(1/r_1 + 1/r_2)^{-1/2}\left(\widehat{R} - R\right)}{\widehat{\eta}} \xrightarrow{D} N(0, 1), \tag{20}$$

where $\widehat{\eta} = \text{Var}(\widehat{R}) = \widehat{\beta}_1\widehat{\beta}_2/\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2$.

**Theorem 1.** *Let $r_1 \longrightarrow \infty$ and $r_2 \longrightarrow \infty$, then*

$$\left[2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right]^{-1/2}\left(\widehat{R} - R\right) \xrightarrow{D} N\left(0, \kappa^2\right), \tag{21}$$

*where $\kappa^2 = 2(\beta_1\beta_2)^2/(\beta_1 + \beta_2)^2$.*

*Proof.* Given that the maximum likelihood estimator is asymptotically normal [21], when $r_1 \longrightarrow \infty$ and $r_2 \longrightarrow \infty$, then

$$\left(\sqrt{r_1}\left(\widehat{\beta}_1 - \beta_1\right), \sqrt{r_2}\left(\widehat{\beta}_2 - \beta_2\right)\right)^T \xrightarrow{D} N(0, \varpi), \tag{22}$$

where

$$\varpi = \begin{bmatrix} \beta_1^2 & 0 \\ 0 & \beta_2^2 \end{bmatrix}. \tag{23}$$

Define $h(\beta_1, \beta_2) = \beta_2/\beta_1 + \beta_2$. According to Taylor expansion $h(\widehat{\beta}_1, \widehat{\beta}_2)$ around $\beta_1$ and $\beta_2$, we have

$$\begin{aligned} \widehat{R} = h\left(\widehat{\beta}_1, \widehat{\beta}_2\right) &= h(\beta_1, \beta_2) + \nabla h(\beta_1, \beta_2)^T \begin{bmatrix} \widehat{\beta}_1 & -\beta_1 \\ \widehat{\beta}_2 & -\beta_2 \end{bmatrix} + \varsigma_1 \\ &= R + \begin{bmatrix} \dfrac{-\beta_2}{(\beta_1 + \beta_2)^2} & \dfrac{\beta_1}{(\beta_1 + \beta_2)^2} \end{bmatrix} \cdot \begin{bmatrix} \widehat{\beta}_1 & -\beta_1 \\ \widehat{\beta}_2 & -\beta_2 \end{bmatrix} + \varsigma_1, \text{ a.s.}, \end{aligned} \tag{24}$$

where the remaining sentences in the following relation apply:

$$\varsigma_1 = O_P\left[\left(\left(\widehat{\beta}_1 - \beta_1\right)^2 + \left(\widehat{\beta}_2 - \beta_2\right)\right)^2\right]. \tag{25}$$

Based on (22) and (24), when $r_1 \longrightarrow \infty$ and $r_2 \longrightarrow \infty$, then

$$E\left[\left(2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right)^{-1/2}\left(\widehat{R}-R\right)\right]=0, \tag{26}$$

$$\begin{aligned}\operatorname{Var}\left[\left(2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right)^{-1/2}\left(\widehat{R}-R\right)\right] &= 2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^{-1}\operatorname{Var}\left(\widehat{R}-R\right)\\ &\coloneqq 2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^{-1}S^2,\end{aligned} \tag{27}$$

where

$$\begin{aligned}S^2 &= \operatorname{Var}\left(\begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2} & \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix}\cdot\begin{bmatrix}\widehat{\beta}_1-\beta_1\\ \widehat{\beta}_2-\beta_2\end{bmatrix}\right)\\ &= \begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2} & \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix}\cdot\operatorname{Var}\left(\begin{bmatrix}\widehat{\beta}_1-\beta_1\\ \widehat{\beta}_2-\beta_2\end{bmatrix}\right)\\ &\quad\cdot\begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2}\\ \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix} = \begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2} & \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix}\\ &\quad\cdot\begin{bmatrix}\frac{\operatorname{Var}\left[\sqrt{r_1}\left(\widehat{\beta}_1-\beta_1\right)\right]}{r_1} & 0\\ 0 & \frac{\operatorname{Var}\left[\sqrt{r_2}\left(\widehat{\beta}_2-\beta_2\right)\right]}{r_2}\end{bmatrix}\\ &\quad\cdot\begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2}\\ \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix} = \begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2} & \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix}\cdot\begin{bmatrix}\frac{\beta_1^2}{r_1} & 0\\ 0 & \frac{\beta_2^2}{r_2}\end{bmatrix}\\ &\quad\cdot\begin{bmatrix}\frac{-\beta_2}{(\beta_1+\beta_2)^2}\\ \frac{\beta_1}{(\beta_1+\beta_2)^2}\end{bmatrix} = \frac{\kappa^2}{r_1}+\frac{\kappa^2}{r_2} = \kappa^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right).\end{aligned} \tag{28}$$

Therefore,

$$\operatorname{Var}\left[\left(2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right)^{-1/2}\left(\widehat{R}-R\right)\right]=\kappa^2. \tag{29}$$

Equation (26) holds because according to the property of MLE $\widehat{R}\xrightarrow{P}R$. □

**Corollary 2.** *A $100(1-\xi)\%$ asymptotic confidence interval for $R$ is*

$$\left[\widehat{R}-\widehat{\eta}z_{\xi/2}\sqrt{\frac{1}{r_1}+\frac{1}{r_2}},\widehat{R}+\widehat{\eta}z_{\xi/2}\sqrt{\frac{1}{r_1}+\frac{1}{r_2}}\right], \tag{30}$$

*where* $\widehat{\eta}=\widehat{\beta}_1\widehat{\beta}_2/(\widehat{\beta}_1+\widehat{\beta}_2)^2$.

*Proof.* Define $\eta^2=(\beta_1\beta_2)^2/(\beta_1+\beta_2)^4$ and $\widehat{\eta}^2=(\widehat{\beta}_1\widehat{\beta}_2)^2/(\widehat{\beta}_1+\widehat{\beta}_2)^4$. According to the asymptotic property of the MLE, $\widehat{\eta}/\eta$ tends to 1 in probability. On the other hand, according to Theorem 1,

$$\frac{(1/r_1+1/r_2)^{-1/2}\left(\widehat{R}-R\right)}{\eta}\xrightarrow{D}N(0,1). \tag{31}$$

Therefore, according to the Slutsky theorem, we have

$$\frac{(1/r_1+1/r_2)^{-1/2}\left(\widehat{R}-R\right)}{\widehat{\eta}}=\frac{(1/r_1+1/r_2)^{-1/2}\left(\widehat{R}-R\right)/\eta}{\widehat{\eta}/\eta}\xrightarrow{D}N(0,1). \tag{32}$$

Thus,

$$P\left(\widehat{R}-\widehat{\eta}z_{\xi/2}\sqrt{\frac{1}{r_1}+\frac{1}{r_2}}\le R\le\widehat{R}+\widehat{\eta}z_{\xi/2}\sqrt{\frac{1}{r_1}+\frac{1}{r_2}}\right)=1-\xi. \tag{33}$$

□

## 5. Bayesian Estimation of $R$

Bayes [22] and Laplace [23] found that the uncertainty about the parameters of a model, which we represent with $\boldsymbol{\theta}$, could be modeled on $\Theta$ through a probability distribution such as $\pi(\boldsymbol{\theta})$, called the prior distribution. With this approach, the inference is based on the conditional distribution $\boldsymbol{\theta}$ on $\mathbf{x}$, $\pi(\boldsymbol{\theta}|\mathbf{x})$. This conditional distribution is called the posterior distribution. In this section, Bayesian estimation is obtained by using the conjugate informative and Jeffreys noninformative prior distributions.

*5.1. Conjugate Informative Prior.* Let $\beta_1\sim\text{Gamma}(\alpha_1,\eta_1)$ and $\beta_2\sim\text{Gamma}(\alpha_2,\eta_2)$ and are independent. The PDF of these priors is as follows:

$$\begin{aligned}\pi(\beta_1)&=\frac{\eta_1^{\alpha_1}}{\Gamma(\alpha_1)}\beta_1^{\alpha_1-1}e^{-\eta_1\beta_1},\ \alpha_1,\eta_1>0,\\ \pi(\beta_2)&=\frac{\eta_2^{\alpha_2}}{\Gamma(\alpha_2)}\beta_2^{\alpha_2-1}e^{-\eta_2\beta_2},\ \alpha_2,\eta_2>0,\end{aligned} \tag{34}$$

where $\alpha_1, \eta_1, \alpha_2$ and $\eta_2$ are the hyperparameters. Then, the joint prior possibility distribution is

$$\pi(\beta_1, \beta_2) = \pi(\beta_1)\pi(\beta_2) = \frac{\eta_1^{\alpha_1}}{\Gamma(\alpha_1)} \beta_1^{\alpha_1 - 1} e^{-\eta_1 \beta_1} \frac{\eta_2^{\alpha_2}}{\Gamma(\alpha_2)} \beta_2^{\alpha_2 - 1} e^{-\eta_2 \beta_2}. \tag{35}$$

The posterior probability distribution is calculated as follows:

$$\pi(\beta_1, \beta_2, \gamma | \mathbf{x}, \mathbf{y}) = \frac{L(\beta_1, \beta_2 ; \mathbf{x}, \mathbf{y})\pi(\beta_1, \beta_2)}{\int_0^\infty \int_0^\infty L(\beta_1, \beta_2 ; \mathbf{x}, \mathbf{y})\pi(\beta_1, \beta_2) d\beta_1 d\beta_2} \propto \beta_1^{\alpha_1 + r_1 - 1} e^{-\beta_1 \left(\eta_1 + b_1{}'\right)} \beta_2^{\alpha_2 + r_2 - 1} e^{-\beta_2 \left(\eta_2 + b_2{}'\right)}, \tag{36}$$

where $b_1'$ and $b_2'$ were given in (4) and (5), respectively. For $i = 1, 2$, we can write

$$\beta_i | \mathbf{x}, \mathbf{y} \sim \text{Gamma}(\alpha_i^*, \eta_i^*), \tag{37}$$

where $\alpha_i^* = \alpha_i + r_i$ and $\eta_i^* = \eta_i + b_i'$. [24] proposed $\alpha_1 = \eta_1 = \alpha_2 = \eta_2 = 0.001$, and Robert [25] suggested an empirical Bayesian approach to determining the values of hyperparameters. According to the approach presented by Robert, to get $\alpha_1$ and $\eta_1$, we maximize the following function:

$$\begin{aligned} m(\mathbf{x} | \alpha_1, \eta_1) &= \int_0^\infty f(\mathbf{x} | \alpha_1, \eta_1) \pi(\beta_1 | \alpha_1, \eta_1) d\beta_1 \\ &= \frac{n_1! \eta_1^{\alpha_1} e^{\gamma \sum_{i=1}^{r_1} x_{(i)}}}{\Gamma(\alpha_1)(n_1 - r_1)!} \int_0^\infty \beta_1^{\alpha_1 + r_1 - 1} e^{-\beta_1 \left(\eta_1 + b_1{}'\right)} d\beta_1 \\ &= \frac{n_1! \eta_1^{\alpha_1 \Gamma(\alpha_1 + r_1)} e^{\gamma \sum_{i=1}^{r_1} x_{(i)}}}{\Gamma(\alpha_1)(n_1 - r_1)! \left(\eta_1 + b_1'\right)^{\alpha_1 + r_1}}. \end{aligned} \tag{38}$$

We have

$$\begin{aligned} M(\mathbf{x} | \alpha_1, \eta_1) &= \log \left[ m(\mathbf{x} | \alpha_1, \eta_1) \right] \\ &= \log \left[ \frac{n_1!}{(n_1 - r_1)!} \right] + \alpha_1 \log \eta_1 + \log \Gamma(\alpha_1 + r_1) \\ &\quad - \log \Gamma(\alpha_1) - (\alpha_1 + r_1) \log \left( \eta_1 + b_1' \right) \\ &\quad + \gamma \sum_{i=1}^{r_1} x_{(i)}. \end{aligned} \tag{39}$$

$\alpha_1$ and $\eta_1$ are obtained by solving the following equations:

$$\frac{\partial M}{\partial \alpha_1} = \log \eta_1 + \psi(r_1 + \alpha_1) - \psi(\alpha_1) - \log \left( \eta_1 + b_1' \right) = 0, \tag{40}$$

$$\frac{\partial M}{\partial \eta_1} = \frac{\alpha_1}{\eta_1} - \frac{\alpha_1 + r_1}{\eta_1 + b_1'} = 0, \tag{41}$$

where $\psi$ shows the digamma function. By solving Equation (41), $\eta_1$ is

$$\eta_1 = \frac{\alpha_1 b_1'}{r_1}. \tag{42}$$

By substituting (42) into (40), we get

$$\begin{aligned} 0 &= \log \left( \frac{\eta_1}{\eta_1 + b_1'} \right) + \psi(r_1 + \alpha_1) - \psi(\alpha_1) \\ &= \log \left( \frac{\alpha_1}{r_1 + \alpha_1} \right) + \psi(r_1 + \alpha_1) - \psi(\alpha_1) \\ &= \log \left( \frac{\alpha_1}{r_1 + \alpha_1} \right) + \sum_{k=0}^{r_1 - 1} \frac{1}{\alpha_1 + k}. \end{aligned} \tag{43}$$

Similarly, this method can be repeated to calculate $\alpha_2$ and $\eta_2$. So, $\eta_2$ is as follows:

$$\eta_2 = \frac{\alpha_2 b_2'}{r_2}. \tag{44}$$

Also, $\alpha_2$ is obtained by solving the following equation:

$$0 = \log \left( \frac{\alpha_2}{r_2 + \alpha_2} \right) + \sum_{k=0}^{r_2 - 1} \frac{1}{\alpha_2 + k}. \tag{45}$$

Abravesh et al. [4] showed that Equations (43) and (45) have no answer and set $\alpha_1 = \alpha_2 = 1$ to solve this problem. Then, $\eta_1 = b_1'/r_1$ and $\eta_2 = b_2'/r_2$.

*5.2. Posterior Distribution of R.* To calculate the posterior distribution of $R$, we have the following transformations:

$$R = \frac{\beta_2}{\beta_1 + \beta_2}, \tag{46}$$

$$V = \beta_2. \tag{47}$$

Transformations (46) and (47) are equivalent to $\beta_1 = V(1-R/R)$ and $\beta_2 = V$. The posterior distribution of $R$ and $V$ can be calculated by the following formula:

$$\pi(r, v|\mathbf{x}, \mathbf{y}) = |J| \cdot \pi\left(v\left(\frac{1-r}{r}\right), v|\mathbf{x}, \mathbf{y}\right). \tag{48}$$

In the above formula, $J$ is called Jacobin and is calculated as follows:

$$|J| = \left|\det\begin{bmatrix} \dfrac{\partial\beta_1}{\partial v} & \dfrac{\partial\beta_2}{\partial v} \\ \dfrac{\partial\beta_1}{\partial r} & \dfrac{\partial\beta_2}{\partial r}\end{bmatrix}\right| = \left|\det\begin{bmatrix} \dfrac{1-r}{r} & 1 \\ -\dfrac{v}{r^2} & 0\end{bmatrix}\right| = \frac{v}{r^2}. \tag{49}$$

The marginal distribution of $R$ is calculated from the joint distribution in (48) as

$$\begin{aligned}\pi(r|\mathbf{x}, \mathbf{y}) &= \int_0^\infty \pi_I(r, v|\mathbf{x}, \mathbf{y})du \\ &= \int_0^\infty \frac{v}{r^2}\cdot\frac{\eta_1^{*\alpha_1^*}\eta_2^{*\alpha_2^*}}{\Gamma(\alpha_1^*)\Gamma(\alpha_2^*)}\left[v\left(\frac{1-r}{r}\right)\right]^{\alpha_1^*-1} e^{-\eta_1^*[v(1-r/r)]}v^{\alpha_2^*-1}e^{-\eta_2^* v}du \\ &= \frac{\eta_1^{*\alpha_1^*}\eta_2^{*\alpha_2^*}\Gamma(\alpha_1^*+\alpha_2^*)}{\Gamma(\alpha_1^*)\Gamma(\alpha_2^*)}\cdot\frac{(1/r-1)^{\alpha_1^*-1}}{r^2[\eta_1^*(1/r-1)+\eta_2^*]^{\alpha_1^*+\alpha_2^*}}, 0 \\ &\le r \le 1.\end{aligned} \tag{50}$$

*5.3. Jeffreys Noninformative Prior.* In this section, by using Jeffreys noninformative prior [26], the Bayesian estimation of $R$ is obtained. The Jeffreys prior is as follows:

$$\pi_J(\beta_1, \beta_2) \propto \sqrt{\det\,[I(\beta_1, \beta_2)]}, \tag{51}$$

where

$$I(\beta_1, \beta_2) = -\begin{bmatrix} E\left(\dfrac{\partial^2 l}{\partial\beta_1^2}\right) & E\left(\dfrac{\partial^2 l}{\partial\beta_1\partial\beta_2}\right) \\ E\left(\dfrac{\partial^2 l}{\partial\beta_2\partial\beta_1}\right) & E\left(\dfrac{\partial^2 l}{\partial\beta_2^2}\right)\end{bmatrix}. \tag{52}$$

Considering Jeffreys prior $\pi_J \propto 1/\beta_1\beta_2$, the marginal posterior distribution is given by

$$\beta_i|\mathbf{x}, \mathbf{y} \sim \text{Gamma}\left(r_i, b_i'\right), i = 1, 2. \tag{53}$$

Similar to the process in Subsection 5.2, the marginal posterior distribution can be obtained:

$$\pi_J(r|\mathbf{x}, \mathbf{y}) = \frac{b_1'^{r_1} b_2'^{r_2}\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)}\cdot\frac{(1/r-1)^{r_1-1}}{r^2\left[b_1'(1/r-1)+b_2'\right]^{r_1+r_2}}, 0 \le r \le 1. \tag{54}$$

*5.4. Lindley Approximation.* Lindley [27] proposed a method for approximating the ratio of integrals. The Lindley approximation of $R$ can be calculated using the following formula:

$$\begin{aligned}E(R(\boldsymbol{\tau})|\text{data}) &= \frac{\int R(\boldsymbol{\tau})e^{l(\boldsymbol{\tau})+\Pi(\boldsymbol{\tau})}d\boldsymbol{\tau}}{\int e^{l(\boldsymbol{\tau})+\Pi(\boldsymbol{\tau})}d\boldsymbol{\tau}} \\ &\approx R(\widehat{\boldsymbol{\tau}}) + \frac{1}{2}\left(\sum_{i,j}\left(\rho_{ij}+2\rho_i\Pi_i\right)\sigma_{ij} + \sum_{i,j,k,l} l_{ijk}\rho_l\sigma_{ij}\sigma_{kl}\right),\end{aligned} \tag{55}$$

where $\Sigma = [\sigma_{ij}(\widehat{\boldsymbol{\tau}})] = [-l_{ij}(\widehat{\boldsymbol{\tau}})]^{-1}$, $\widehat{\boldsymbol{\tau}}$ is MLE of $\boldsymbol{\tau}$ and

$$\rho_i = \frac{\partial R}{\partial\tau_i}\bigg|_{\boldsymbol{\tau}=\widehat{\boldsymbol{\tau}}}, \rho_{ij} = \frac{\partial^2 R}{\partial\tau_i\partial\tau_j}\bigg|_{\boldsymbol{\tau}=\widehat{\boldsymbol{\tau}}}, \Pi_i = \frac{\partial\Pi}{\partial\tau_i}\bigg|_{\boldsymbol{\tau}=\widehat{\boldsymbol{\tau}}}, l_{ijk} = \frac{\partial^3 l}{\partial\tau_i\partial\tau_j\partial\tau_k}\bigg|_{\boldsymbol{\tau}=\widehat{\boldsymbol{\tau}}}. \tag{56}$$

Here, $l$ is log-likelihood function and $\Pi$ is log-prior distribution.

*5.4.1. Informative Prior.* Based on the prior distribution (35) and $R = \beta_2/\beta_1+\beta_2$, we obtain

$$\begin{aligned}\Pi = \log\pi(\beta_1, \beta_2) &= \alpha_1\log\eta_1 - \log\Gamma(\alpha_1) + (\alpha_1-1)\log\beta_1 \\ &\quad - \eta_1\beta_1 + \alpha_2\log\eta_2 - \log\Gamma(\alpha_2) + (\alpha_2-1)\log\beta_2 - \eta_2\beta_2.\end{aligned} \tag{57}$$

So,

$$\begin{aligned}\Pi_1 &= \frac{\partial\Pi}{\partial\beta_1} = \frac{\alpha_1-1}{\beta_1} - \eta_1, \\ \Pi_2 &= \frac{\partial\Pi}{\partial\beta_2} = \frac{\alpha_2-1}{\beta_2} - \eta_2, \\ \rho_1 &= \frac{\partial R}{\partial\beta_1} = \frac{-\beta_2}{(\beta_1+\beta_2)^2}, \\ \rho_2 &= \frac{\partial R}{\partial\beta_2} = \frac{\beta_1}{(\beta_1+\beta_2)^2}, \\ \rho_{11} &= \frac{\partial^2 R}{\partial\beta_1^2} = \frac{2\beta_2}{(\beta_1+\beta_2)^3}, \\ \rho_{22} &= \frac{\partial^2 R}{\partial\beta_2^2} = \frac{-2\beta_1}{(\beta_1+\beta_2)^3}, \\ \rho_{12} &= \frac{\partial^2 R}{\partial\beta_2\partial\beta_1} = \frac{\beta_2-\beta_1}{(\beta_1+\beta_2)^3}.\end{aligned} \tag{58}$$

Inverse of the Hessian matrix is given by

$$\Sigma = \begin{bmatrix} \dfrac{\beta_1^2}{r_1} & 0 \\ 0 & \dfrac{\beta_2^2}{r_2}\end{bmatrix}. \tag{59}$$

1: Generate a MCMC sample $\{R_i, i = 1, 2, \cdots, k\}$ from $\pi(R|\mathbf{x}, \mathbf{y})$;
2: Sort $\{R_i, i = 1, 2, \cdots, l\}$ and suppose $R_{(1)} \le R_{(2)} \le \cdots \le R_{(l)}$;
3: Consider $C_j = (R_j, R_{j+[(1-\xi)l]}), j = 1, 2, \cdots, l - [(1-\xi)l]$;
4: Consider $W_j = R_{j+[(1-\xi)l]} - R_j$;
5: Select $j'$ such that $W_{j'} = \min\{W_j, j = 1, 2, \cdots, l - [(1-\xi)l]\}$;
6: Introduce $C_{j'}$ as a $100(1-\xi)\%$ HPD interval for $R$.

Algorithm 1: Chen-Shao algorithm for $R$.

So, $\sigma_{11} = \beta_1^2/r_1, \sigma_{22} = \beta_2^2/r_2, \sigma_{12} = \sigma_{21} = 0$ and

$$l_{iii} = \frac{\partial^3 l}{\partial \beta_i^3} = \frac{2r_i}{\beta_i^3}, i = 1, 2. \tag{60}$$

Finally, the Lindley approximation of the Bayes estimator of $R$ is

$$\begin{aligned} R_{\text{Bayes}} &\approx R(\boldsymbol{\tau}) + \frac{1}{2}\left(\sum_{i,j}\left(\rho_{ij} + 2\rho_i \Pi_i\right)\sigma_{ij} + \sum_{i,j,k,l} l_{ijk}\rho_l \sigma_{ij}\sigma_{kl}\right) \\ &= R\left(\widehat{\beta}_1, \widehat{\beta}_2\right) + \frac{1}{2}\sum_{i=1}^{2}(\rho_{ii} + 2\rho_i \Pi_i)\sigma_{ii} + \frac{1}{2}\sum_{i=1}^{2} l_{iii}\rho_i \sigma_{ii}^2 \\ &= R\left(\widehat{\beta}_1, \widehat{\beta}_2\right) + \frac{\widehat{\beta}_1^2}{r_1}\left[\frac{\widehat{\beta}_2}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^3} + \frac{-\widehat{\beta}_2}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2}\left(\frac{\alpha_1 - 1}{\widehat{\beta}_1} - \eta_1\right)\right] \\ &\quad + \frac{\widehat{\beta}_2^2}{r_2}\left[\frac{-\widehat{\beta}_1}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^3} + \frac{\widehat{\beta}_1}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2}\left(\frac{\alpha_2 - 1}{\widehat{\beta}_2} - \eta_2\right)\right] \\ &\quad + \frac{\widehat{\beta}_1\widehat{\beta}_2}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \end{aligned} \tag{61}$$

*5.4.2. Noninformative Prior.* Under Jeffreys prior ($\pi_J(\beta_1, \beta_2) = 1/\beta_1\beta_2$), we have

$$\begin{aligned} \Pi &= -\log \beta_1 - \log \beta_2, \\ \Pi_1 &= -\frac{1}{\beta_1}, \Pi_2 = -\frac{1}{\beta_2}. \end{aligned} \tag{62}$$

Therefore, the Bayes estimator of $R$ using the Lindley approximation is

$$\begin{aligned} R_{\text{Bayes}} &\approx R(\boldsymbol{\tau}) + \frac{1}{2}\left(\sum_{i,j}\left(\rho_{ij} + 2\rho_i \Pi_i\right)\sigma_{ij} + \sum_{i,j,k,l} l_{ijk}\rho_l \sigma_{ij}\sigma_{kl}\right) \\ &= R\left(\widehat{\beta}_1, \widehat{\beta}_2\right) + \frac{1}{2}\sum_{i=1}^{2}(\rho_{ii} + 2\rho_i \Pi_i)\sigma_{ii} + \frac{1}{2}\sum_{i=1}^{2} l_{iii}\rho_i \sigma_{ii}^2 \\ &= R\left(\widehat{\beta}_1, \widehat{\beta}_2\right) + \frac{\widehat{\beta}_1^2}{r_1}\left[\frac{\widehat{\beta}_2}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^3} + \frac{\widehat{\beta}_2}{\widehat{\beta}_1\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2}\right] \\ &\quad + \frac{\widehat{\beta}_2^2}{r_2}\left[\frac{-\widehat{\beta}_1}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^3} + \frac{\widehat{\beta}_1}{\widehat{\beta}_2\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2}\right] \\ &\quad + \frac{\widehat{\beta}_1\widehat{\beta}_2}{\left(\widehat{\beta}_1 + \widehat{\beta}_2\right)^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \end{aligned} \tag{63}$$

*5.5. Monte Carlo Integration.* The Monte Carlo integration method was introduced by Metropolis and Ulam [28] and Neumann [29]. Let $\theta_1, \theta_2, \cdots, \theta_k$ be a random sample from posterior density $\pi(\boldsymbol{\theta}|\text{observations})$. In this case, according to the strong law of large numbers, for large $k$, an approximation for the expected values of posterior is equal to

$$E(\zeta(\boldsymbol{\theta})|\text{observation}) \simeq \frac{1}{k}\sum_{i=1}^{k}\zeta(\boldsymbol{\theta}_i). \tag{64}$$

This method was very simple and does not involve complicated calculations. The only problem this method may have is generating a random sample of posterior density.

Now, the Bayes estimator of $R$ is obtained using this method. Let $\{\beta_i^{(1)}, \beta_i^{(2)}, \cdots, \beta_i^{(N)}\}$ be the random sample from $\pi(\beta_i|\mathbf{x}, \mathbf{y}), i = 1, 2$, then for large $N$,

$$E(R|\mathbf{x}, \mathbf{y}) \approx \frac{1}{N}\sum_{j=1}^{N} R\left(\beta_1^{(j)}, \beta_2^{(j)}\right) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{\beta_2^{(j)}}{\beta_1^{(j)} + \beta_2^{(j)}}\right). \tag{65}$$

*5.5.1. Informative Prior.* To calculate the Bayes estimator of $R$ under the conjugate prior (35), we assume $\beta_i^{(1)}, \beta_i^{(2)}, \cdots, \beta_i^{(N)} \sim^{iid} \text{Gamma}(\alpha_i + r_i, \eta_i + b_i'), i = 1, 2$. So,

$$\widehat{R}_{\text{MC}}^{I} = \frac{1}{N}\sum_{j=1}^{N} R\left(\beta_1^{(j)}, \beta_2^{(j)}\right) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{\beta_2^{(j)}}{\beta_1^{(j)} + \beta_2^{(j)}}\right). \tag{66}$$

*5.5.2. Noninformative Prior.* Under the Jeffreys prior, we consider $\beta_i^{(1)}, \beta_i^{(2)}, \cdots, \beta_i^{(N)} \sim^{iid} \text{Gamma}(r_i, b_i'), i = 1, 2$. Then,

$$\widehat{R}_{\text{MC}}^{J} = \frac{1}{N}\sum_{j=1}^{N} R\left(\beta_1^{(j)}, \beta_2^{(j)}\right) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{\beta_2^{(j)}}{\beta_1^{(j)} + \beta_2^{(j)}}\right). \tag{67}$$

*5.6. MCMC.* To solve the stated problem of the Monte Carlo integration method, a more general method is used to generate approximate random variables from the posterior distribution, called the Markov chain Monte Carlo (MCMC) method [30]. The Metropolis-Hastings algorithm is used to create Markov chains with a given distribution. The application of

Table 1: The bias and MSE values (MSE in parentheses) of estimators for $\beta_1 = 1, \beta_2 = 1, \gamma = 1, R = 1/2$.

| $n_1$ | $n_2$ | $r_1$ | $r_2$ | Prior | MLE | Lindley | MC | MCMC |
|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | Conjugate | 0.011339 (0.015528) | 0.010107 (0.012152) | 0.010295 (0.012573) | 0.006669 (0.006495) |
| | | | | Jeffreys | — | 0.010822 (0.014271) | 0.011231 (0.014295) | 0.007375 (0.007236) |
| | | 9 | 8 | Conjugate | 0.009474 (0.021751) | 0.003927 (0.016272) | 0.007728 (0.016904) | 0.005395 (0.008246) |
| | | | | Jeffreys | — | 0.004289 (0.019732) | 0.007793 (0.019736) | 0.004971 (0.009175) |
| | | 8 | 8 | Conjugate | -0.000753 (0.021541) | -0.000852 (0.016715) | -0.000681 (0.017157) | -0.000596 (0.008737) |
| | | | | Jeffreys | — | -0.000815 (0.019670) | -0.000749 (0.019676) | -0.000502 (0.008889) |
| 10 | 30 | 10 | 30 | Conjugate | -0.015306 (0.010153) | 0.010855 (0.008091) | -0.006733 (0.008151) | -0.005904 (0.004985) |
| | | | | Jeffreys | — | 0.009125 (0.009500) | -0.006959 (0.009400) | -0.006247 (0.005666) |
| | | 9 | 28 | Conjugate | -0.003033 (0.009196) | 0.024440 (0.007826) | 0.004501 (0.007476) | 0.002187 (0.004340) |
| | | | | Jeffreys | — | 0.024301 (0.009185) | 0.006082 (0.008662) | 0.003254 (0.004896) |
| | | 8 | 27 | Conjugate | -0.017882 (0.016928) | 0.015887 (0.013174) | -0.006873 (0.013339) | -0.005408 (0.007667) |
| | | | | Jeffreys | — | 0.013523 (0.015708) | -0.007212 (0.015539) | -0.006585 (0.008730) |
| 30 | 30 | 30 | 30 | Conjugate | 0.007389 (0.004586) | 0.007087 (0.004188) | 0.007466 (0.004211) | 0.006229 (0.003175) |
| | | | | Jeffreys | — | 0.007269 (0.004444) | 0.007113 (0.004477) | 0.006013 (0.003299) |
| | | 29 | 27 | Conjugate | 0.000869 (0.004186) | -0.000089 (0.003779) | 0.000583 (0.003835) | 0.001015 (0.002807) |
| | | | | Jeffreys | — | -0.000090 (0.004044) | 0.000408 (0.004046) | 0.001044 (0.002948) |
| | | 27 | 27 | Conjugate | -0.002992 (0.005576) | -0.002802 (0.005017) | -0.002798 (0.005059) | -0.002345 (0.003726) |
| | | | | Jeffreys | — | -0.002939 (0.005384) | -0.002854 (0.005423) | -0.002606 (0.003900) |

Table 2: The bias and MSE values (MSE in parentheses) of estimators for $\beta_1 = 1, \beta_2 = 2, \gamma = 1, R = 2/3$.

| $n_1$ | $n_2$ | $r_1$ | $r_2$ | Prior | MLE | Lindley | MC | MCMC |
|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | Conjugate | 0.009500 (0.012304) | -0.015630 (0.010022) | -0.012485 (0.010080) | -0.053973 (0.008464) |
| | | | | Jeffreys | — | 0.002976 (0.011512) | 0.003521 (0.011480) | -0.045037 (0.008308) |
| | | 9 | 8 | Conjugate | 0.047141 (0.019723) | 0.013442 (0.015408) | 0.019974 (0.015494) | -0.030126 (0.010706) |
| | | | | Jeffreys | — | 0.035412 (0.018268) | 0.037776 (0.018229) | -0.019705 (0.011154) |
| | | 8 | 8 | Conjugate | 0.002840 (0.019525) | -0.023443 (0.015882) | -0.020104 (0.016047) | -0.062010 (0.013115) |
| | | | | Jeffreys | — | -0.004222 (0.018351) | -0.004391 (0.018341) | -0.053220 (0.013700) |
| 10 | 30 | 10 | 30 | Conjugate | -0.002736 (0.008137) | 0.008917 (0.006132) | -0.005916 (0.006567) | -0.038466 (0.006042) |
| | | | | Jeffreys | — | 0.014354 (0.007555) | -0.000261 (0.007695) | -0.034790 (0.006342) |
| | | 9 | 28 | Conjugate | -0.000636 (0.011272) | 0.013486 (0.008512) | -0.003448 (0.009072) | -0.037787 (0.007786) |
| | | | | Jeffreys | — | 0.018633 (0.010370) | 0.002493 (0.010620) | -0.036707 (0.008483) |
| | | 8 | 27 | Conjugate | -0.011756 (0.007082) | 0.006556 (0.004932) | -0.013705 (0.005678) | -0.049995 (0.006245) |
| | | | | Jeffreys | — | 0.012011 (0.006199) | -0.007743 (0.006572) | -0.046598 (0.006817) |
| 30 | 30 | 30 | 30 | Conjugate | 0.003700 (0.003215) | -0.005787 (0.002979) | -0.005248 (0.002967) | -0.025586 (0.003080) |
| | | | | Jeffreys | — | 0.001298 (0.003137) | 0.001252 (0.003141) | -0.019025 (0.002924) |
| | | 29 | 27 | Conjugate | 0.009701 (0.009701) | -0.001228 (0.004512) | -0.000352 (0.004518) | -0.021084 (0.004090) |
| | | | | Jeffreys | — | 0.006303 (0.004813) | 0.006694 (0.004831) | -0.014817 (0.004119) |
| | | 27 | 27 | Conjugate | 0.013939 (0.004330) | 0.003420 (0.003420) | 0.004086 (0.003858) | -0.018673 (0.003403) |
| | | | | Jeffreys | — | 0.011203 (0.004182) | 0.011117 (0.004171) | -0.013123 (0.003508) |

this algorithm in mechanical physics was first developed by Metropolis et al. [30]. A few years later, Hastings and Keith generalized the algorithm in more statistical detail [31]. Using the MCMC method, the following integral is approximated:

$$E(R|\mathbf{x},\mathbf{y}) = \int_0^1 r\pi(r|\mathbf{x},\mathbf{y})dr. \quad (68)$$

Let $r_1, r_2, \cdots, r_l$ be an ergodic MCMC sample from $\pi(r|\mathbf{x},\mathbf{y})$, we have

$$E(R|\mathbf{x},\mathbf{y}) \approx \frac{1}{l}\sum_{j=1}^{l} r_j. \quad (69)$$

*5.6.1. Informative Prior.* Considering the conjugate prior

Table 3: The bias and MSE values (MSE in parentheses) of estimators for $\beta_1 = 2, \beta_2 = 1, \gamma = 1, R = 1/3$.

| $n_1$ | $n_2$ | $r_1$ | $r_2$ | Prior | MLE | Lindley | MC | MCMC |
|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | Conjugate | -0.006236 (0.013546) | 0.017747 (0.011277) | 0.014489 (0.011353) | 0.055189 (0.009427) |
| | | | | Jeffreys | — | 0.000054 (0.012776) | -0.000131 (0.012800) | 0.046938 (0.009554) |
| | | 9 | 8 | Conjugate | -0.020955 (0.020210) | 0.001998 (0.015037) | 0.002221 (0.015536) | 0.049048 (0.010972) |
| | | | | Jeffreys | — | -0.017516 (0.018642) | -0.014752 (0.018694) | 0.039356 (0.011794) |
| | | 8 | 8 | Conjugate | -0.036934 (0.018266) | -0.006422 (0.014205) | -0.010386 (0.014360) | 0.039886 (0.010667) |
| | | | | Jeffreys | — | -0.028748 (0.016873) | -0.028721 (0.016784) | 0.027892 (0.011565) |
| 10 | 30 | 10 | 30 | Conjugate | -0.019006 (0.006476) | 0.029157 (0.006079) | 0.010961 (0.005102) | 0.044632 (0.005154) |
| | | | | Jeffreys | — | 0.006823 (0.006389) | -0.007084 (0.006079) | 0.031843 (0.004776) |
| | | 9 | 28 | Conjugate | -0.022646 (0.008856) | 0.029192 (0.008100) | 0.008836 (0.006968) | 0.044723 (0.006398) |
| | | | | Jeffreys | — | 0.005889 (0.008797) | -0.009568 (0.008326) | 0.031616 (0.006042) |
| | | 8 | 27 | Conjugate | -0.020403 (0.009430) | 0.040819 (0.009588) | 0.016503 (0.007684) | 0.053626 (0.007476) |
| | | | | Jeffreys | — | 0.012477 (0.009715) | -0.005752 (0.008934) | 0.035300 (0.007476) |
| 30 | 30 | 30 | 30 | Conjugate | -0.005453 (0.004098) | 0.003841 (0.003746) | 0.003399 (0.003739) | 0.023368 (0.003607) |
| | | | | Jeffreys | — | -0.003058 (0.003991) | -0.002791 (0.004009) | 0.018217 (0.003467) |
| | | 29 | 27 | Conjugate | -0.003748 (0.004346) | 0.005040 (0.004015) | 0.005143 (0.004039) | 0.026241 (0.003894) |
| | | | | Jeffreys | — | -0.002042 (0.004232) | -0.001604 (0.004234) | 0.019994 (0.003906) |
| | | 27 | 27 | Conjugate | -0.007624 (0.003878) | 0.002911 (0.003491) | 0.002431 (0.003503) | 0.023633 (0.003301) |
| | | | | Jeffreys | — | -0.004931 (0.003755) | -0.005036 (0.003742) | 0.017996 (0.003314) |

1: Select $n_1, n_2, \beta_1, \beta_2, r_1, r_2$, and $\gamma$ values;
2: Generate $(\tau_1, \tau_2, \cdots, \tau_{n_1})$ and $(\delta_1, \delta_2, \cdots, \delta_{n_2})$ from the SED;
3: Consider $x_i = (1/\gamma) \log ((\gamma/\beta_1)\tau_i + 1), i = 1, 2, \cdots, n_1$ and $y_j = (1/\gamma) \log ((\gamma/\beta_2)\delta_j + 1), i = 1, 2, \cdots, n_2$;
4: Sort $x_i s$ and $y_j s$ and suppose $x_{(1)} < x_{(2)} < \cdots < x_{(n_1)}$ and $y_{(1)} < y_{(2)} < \cdots < y_{(n_2)}$;
5: Report $(x_{(1)}, x_{(2)}, \cdots, x_{(r_1)})$ and $(y_{(1)}, y_{(2)}, \cdots, y_{(r_2)})$ as the type II censored samples from Gompertz$(\beta_1, \gamma)$ and Gompertz$(\beta_2, \gamma)$, respectively.

Algorithm 2: The type II censored sample generation algorithm.

distribution (35) for $\beta_1$ and $\beta_2$, we obtain the posterior distribution (50). By generating an ergodic sample $r_1^I, r_2^I, \cdots, r_l^I$ from $\pi_I$ using the Metropolis-Hastings algorithm, the $R$ Bayesian estimator is as follows:

$$E(R|\mathbf{x}, \mathbf{y}) \approx \frac{1}{l}\sum_{j=1}^{l} r_j^I. \tag{70}$$

*5.6.2. Noninformative Prior.* Similarly, using the Jeffreys prior and the posterior distribution (54) for $\beta_1$ and $\beta_2$, we generate an ergodic sample $r_1^J, r_2^J, \cdots, r_l^J$ from $\pi_J$ using the Metropolis-Hastings algorithm; the $R$ Bayesian estimator is given by

$$E(R|\mathbf{x}, \mathbf{y}) \approx \frac{1}{l}\sum_{j=1}^{l} r_j^J. \tag{71}$$

## 6. Bayesian Credible Interval

In fact, the Bayesian view offers confidence interval that are more realistic than its classic counterpart. We start this section with two definitions.

*Definition 3.* Set $\mathbf{C}_x$ is called a $\xi$-credible region whenever

$$P_\pi(\theta \in \mathbf{C}_\mathbf{x}|\mathbf{x}) \geq 1 - \xi, \tag{72}$$

where $P_\pi$ is the $\theta$ posterior probability function of condition $\mathbf{x}$.

*Definition 4.* The $\xi$ -credible region $\mathbf{C}_\mathbf{x}$ is called a region with the highest posterior density (HPD) whenever it can be written as follows:

$$\mathbf{C}_\mathbf{x}(\nu) = \{\theta : \pi(\theta|\mathbf{x}) \geq \nu\}, \tag{73}$$

where $\nu$ is the largest fixed number that applies to

$$P_\pi(\theta \in \mathbf{C}_\mathbf{x}(\nu)) \geq 1 - \xi. \tag{74}$$

Although the $100(1-\xi)\%$ HPD interval is an optimal answer among the $\xi$-credible intervals, in some cases, it is not easy to calculate directly, and approximate methods must be used to obtain it [32]. It is usually easier to calculate approximate intervals with equal tails than HPD interval

Table 4: $L$ and CP of confidence interval for $R = 1/2$, $\beta_1 = 1$, $\beta_2 = 1$ and $\gamma = 1$.

| $n_1$ | $n_2$ | $r_1$ | $r_2$ | | Asymptotic | Exact | Credible equi-tailed Conjugate | Credible equi-tailed Jeffreys | HPD Conjugate | HPD Jeffreys |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | $L$ | 0.410623 | 0.417916 | 0.328047 | 0.337885 | 0.385356 | 0.400056 |
| | | | | CP | 0.995 | 0.900 | 0.936 | 0.939 | 0.910 | 0.900 |
| | | 9 | 8 | $L$ | 0.435411 | 0.447157 | 0.342757 | 0.354050 | 0.406417 | 0.423652 |
| | | | | CP | 0.987 | 0.875 | 0.922 | 0.920 | 0.886 | 0.870 |
| | | 8 | 8 | $L$ | 0.443006 | 0.454225 | 0.347341 | 0.358796 | 0.412829 | 0.431075 |
| | | | | CP | 0.725 | 0.857 | 0.922 | 0.916 | 0.869 | 0.855 |
| 10 | 30 | 10 | 30 | $L$ | 0.344899 | 0.352722 | 0.287869 | 0.295543 | 0.327492 | 0.338830 |
| | | | | CP | 0.242 | 0.929 | 0.954 | 0.953 | 0.932 | 0.922 |
| | | 9 | 28 | $L$ | 0.359745 | 0.369685 | 0.297746 | 0.306382 | 0.340650 | 0.353603 |
| | | | | CP | 1.000 | 0.932 | 0.965 | 0.955 | 0.944 | 0.931 |
| | | 8 | 27 | $L$ | 0.373987 | 0.386498 | 0.306679 | 0.316532 | 0.353156 | 0.367765 |
| | | | | CP | 1.000 | 0.912 | 0.950 | 0.942 | 0.926 | 0.910 |
| 30 | 30 | 30 | 30 | $L$ | 0.248407 | 0.249667 | 0.223203 | 0.226709 | 0.242077 | 0.245737 |
| | | | | CP | 1.000 | 0.934 | 0.946 | 0.943 | 0.943 | 0.940 |
| | | 29 | 27 | $L$ | 0.257244 | 0.258682 | 0.230090 | 0.233250 | 0.250227 | 0.254282 |
| | | | | CP | 1.000 | 0.941 | 0.949 | 0.949 | 0.942 | 0.938 |
| | | 27 | 27 | $L$ | 0.261087 | 0.262638 | 0.233356 | 0.236327 | 0.253800 | 0.258014 |
| | | | | CP | 1.000 | 0.928 | 0.937 | 0.935 | 0.929 | 0.927 |

Table 5: $L$ and CP of confidence interval for $R = 2/3$, $\beta_1 = 1$, $\beta_2 = 2$ and $\gamma = 1$.

| $n_1$ | $n_2$ | $r_1$ | $r_2$ | | Asymptotic | Exact | Credible equi-tailed Conjugate | Credible equi-tailed Jeffreys | HPD Conjugate | HPD Jeffreys |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | $L$ | 0.358133 | 0.371438 | 0.311195 | 0.317441 | 0.349515 | 0.355055 |
| | | | | CP | 0.987 | 0.867 | 0.907 | 0.908 | 0.882 | 0.869 |
| | | 9 | 8 | $L$ | 0.370774 | 0.390539 | 0.324441 | 0.332062 | 0.365106 | 0.370252 |
| | | | | CP | 0.925 | 0.850 | 0.902 | 0.902 | 0.869 | 0.844 |
| | | 8 | 8 | $L$ | 0.383027 | 0.403275 | 0.329960 | 0.338316 | 0.374478 | 0.380763 |
| | | | | CP | 0.724 | 0.850 | 0.894 | 0.900 | 0.874 | 0.849 |
| 10 | 30 | 10 | 30 | $L$ | 0.304199 | 0.307665 | 0.260316 | 0.264883 | 0.290042 | 0.296947 |
| | | | | CP | 0.996 | 0.886 | 0.907 | 0.897 | 0.905 | 0.883 |
| | | 9 | 28 | $L$ | 0.322409 | 0.326611 | 0.271776 | 0.277204 | 0.305627 | 0.313888 |
| | | | | CP | 1.000 | 0.909 | 0.908 | 0.906 | 0.922 | 0.911 |
| | | 8 | 27 | $L$ | 0.333331 | 0.338090 | 0.277764 | 0.283595 | 0.314420 | 0.323653 |
| | | | | CP | 0.759 | 0.877 | 0.886 | 0.879 | 0.899 | 0.880 |
| 30 | 30 | 30 | 30 | $L$ | 0.219360 | 0.222087 | 0.204815 | 0.206564 | 0.217282 | 0.218471 |
| | | | | CP | 1.000 | 0.930 | 0.942 | 0.938 | 0.938 | 0.927 |
| | | 29 | 27 | $L$ | 0.225229 | 0.228234 | 0.210635 | 0.211962 | 0.223388 | 0.224514 |
| | | | | CP | 1.000 | 0.903 | 0.915 | 0.918 | 0.923 | 0.906 |
| | | 27 | 27 | $L$ | 0.229774 | 0.232950 | 0.214164 | 0.215451 | 0.227483 | 0.228800 |
| | | | | CP | 1.000 | 0.915 | 0.920 | 0.919 | 0.928 | 0.920 |

[33]. Chen and Shao [34] have proposed an algorithm to construct an approximate HPD interval. We obtain confidence intervals of equal tails and HPD.

*6.1. An Equi-Tailed Bayesian Credible Interval.* In this subsection, confidence intervals with equal tails are calculated under the conjugate and Jeffreys prior distributions.

Table 6: $L$ and CP of confidence interval for $R = 1/3, \beta_1 = 2, \beta_2 = 1$ and $\gamma = 1$.

| $n_1$ | $n_2$ | $r_1$ | $r_2$ | | Asymptotic | Exact | Credible equi-tailed Conjugate | Credible equi-tailed Jeffreys | HPD Conjugate | HPD Jeffreys |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | $L$ | 0.361056 | 0.375095 | 0.312100 | 0.318856 | 0.352094 | 0.357747 |
| | | | | CP | 0.956 | 0.916 | 0.930 | 0.936 | 0.925 | 0.900 |
| | | 9 | 8 | $L$ | 0.381043 | 0.396260 | 0.324565 | 0.332243 | 0.369455 | 0.376389 |
| | | | | CP | 0.939 | 0.857 | 0.906 | 0.891 | 0.880 | 0.858 |
| | | 8 | 8 | $L$ | 0.381905 | 0.400602 | 0.329760 | 0.337189 | 0.373895 | 0.379686 |
| | | | | CP | 0.922 | 0.845 | 0.899 | 0.901 | 0.876 | 0.847 |
| 10 | 30 | 10 | 30 | $L$ | 0.301283 | 0.320492 | 0.280337 | 0.286296 | 0.302704 | 0.306193 |
| | | | | CP | 0.823 | 0.927 | 0.932 | 0.948 | 0.948 | 0.930 |
| | | 9 | 28 | $L$ | 0.315321 | 0.337685 | 0.290839 | 0.297912 | 0.316907 | 0.321067 |
| | | | | CP | 0.095 | 0.933 | 0.934 | 0.938 | 0.936 | 0.919 |
| | | 8 | 27 | $L$ | 0.326139 | 0.353599 | 0.301778 | 0.309364 | 0.329406 | 0.334021 |
| | | | | CP | 1.000 | 0.923 | 0.942 | 0.951 | 0.939 | 0.929 |
| 30 | 30 | 30 | 30 | $L$ | 0.219806 | 0.222444 | 0.205892 | 0.206672 | 0.217676 | 0.218890 |
| | | | | CP | 1.000 | 0.931 | 0.925 | 0.926 | 0.935 | 0.925 |
| | | 29 | 27 | $L$ | 0.227859 | 0.230704 | 0.211658 | 0.213410 | 0.225177 | 0.226650 |
| | | | | CP | 1.000 | 0.924 | 0.936 | 0.939 | 0.932 | 0.921 |
| | | 27 | 27 | $L$ | 0.229601 | 0.232700 | 0.214087 | 0.215231 | 0.227314 | 0.228632 |
| | | | | CP | 1.000 | 0.909 | 0.919 | 0.919 | 0.918 | 0.906 |

Table 7: Real data.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 156 | 442 | 285 | 247 | 173 | 168 | 253 | 112 | 125 | 286 |
| | 166 | 202 | 852 | 261 | 133 | 365 | 559 | 227 | 309 | 702 |
| $Y$ | 230 | 568 | 1101 | 218 | 169 | 115 | 285 | 342 | 178 | 280 |
| | 734 | 431 | 271 | 305 | 177 | 143 | 129 | 326 | 493 | 381 |

Table 8: Estimators of $R$ for real data.

| Prior | MLE | Lindley | MC | MCMC |
|---|---|---|---|---|
| Conjugate | 0.4169946 | 0.4192417 | 0.4202460 | 0.4294883 |
| Jeffreys | — | 0.4192367 | 0.4158864 | 0.4422381 |

*6.1.1. Informative Prior.* We use the prior distribution (35). The posterior distributions of $\beta_1$ and $\beta_2$ are as follows:

$$\beta_i|\mathbf{x}, \mathbf{y} \overset{\text{ind}}{\sim} \text{Gamma}(\alpha_i^*, \eta_i^*), i = 1, 2. \tag{75}$$

Therefore, one can conclude

$$2\eta_i^* \beta_i|\mathbf{x}, \mathbf{y} \sim \chi^2(2\alpha_i^*), \forall i = 1, 2. \tag{76}$$

Given that the posterior distributions of $\beta_1$ and $\beta_2$ are independent, we get

$$\frac{\eta_1^* \alpha_2^*}{\alpha_1^* \eta_2^*} \cdot \frac{\beta_1|\mathbf{x}, \mathbf{y}}{\beta_2|\mathbf{x}, \mathbf{y}} \sim F(2\alpha_1^*, 2\alpha_2^*). \tag{77}$$

Thus, a $100(1-\xi)\%$ Bayesian credible interval with equal tails for $R$ under the conjugate prior is

$$P\left[\frac{1}{1 + \alpha_1^* \eta_2^* / \eta_1^* \alpha_2^* F_{1-\xi/2}(2\alpha_1^*, 2\alpha_2^*)} < R < \frac{1}{1 + \alpha_1^* \eta_2^* / \eta_1^* \alpha_2^* F_{\xi/2}(2\alpha_1^*, 2\alpha_2^*)}\right] = 1 - \xi. \tag{78}$$

*6.1.2. Noninformative Prior.* Similarly, considering the Jeffreys prior, the posterior distributions of $\beta_1$ and $\beta_2$ are $\beta_1|\mathbf{x}, \mathbf{y} \sim^{\text{ind}} \text{Gamma}(r_1, b_1')$ and $\beta_2|\mathbf{x}, \mathbf{y} \sim^{\text{ind}} \text{Gamma}(r_2, b_2')$, respectively. So, we have

$$2b_i' \beta_i|\mathbf{x}, \mathbf{y} \sim \chi^2(2r_i), i = 1, 2. \tag{79}$$

Table 9: Confidence intervals of $R$ for real data.

| | Asymptotic | Exact | Credible equi-tailed Conjugate | Credible equi-tailed Jeffreys | HPD Conjugate | HPD Jeffreys |
|---|---|---|---|---|---|---|
| Lower | 0.2581656 | 0.2414667 | 0.2727656 | 0.2691575 | 0.3078994 | 0.2989704 |
| Upper | 0.5758236 | 0.5554383 | 0.5769903 | 0.5814321 | 0.5804470 | 0.5704925 |
| $L$ | 0.317658 | 0.313972 | 0.272548 | 0.271522 | 0.304225 | 0.312275 |

Hence,

$$\frac{b_1' r_2}{r_1 b_2'} \cdot \frac{\beta_1 | \mathbf{x}, \mathbf{y}}{\beta_2 | \mathbf{x}, \mathbf{y}} \sim F(2r_1, 2r_2). \tag{80}$$

A $100(1-\xi)\%$ Bayesian credible interval with equal tails for $R$ under the Jeffreys prior is

$$P\left[\frac{1}{1 + r_1 b_2' / b_1' r_2 F_{1-\xi/2}(2r_1, 2r_2)} < R < \frac{1}{1 + r_1 b_2' / b_1' r_2 F_{\xi/2}(2r_1, 2r_2)}\right] = 1 - \xi. \tag{81}$$

*6.2. HPD Interval.* As mentioned earlier, it is difficult to obtain the HPD interval directly. Therefore, in this subsection, the Chen-Shao algorithm [34] is used to calculate the approximate HPD interval for $R$. This algorithm is expressed as follows.

In Step 3, $C_j$'s are $100(1-\xi)\%$ credible intervals for $R$. To obtain the HPD interval under the conjugate and Jeffreys priors, it is enough to substitute the posterior distributions (50) and (54) in Step 1 of Algorithm 1.

## 7. Simulation Study

In this section, we use simulation to compare estimators and confidence intervals of R. Therefore, for different sample sizes, different numbers of type II censorship, and different $R$ values with 1000 repetitions, bias and mean square error (MSE) values of $R$ estimators are calculated. For the conjugate informative prior distribution, hyperparameters $\alpha_1 = \alpha_2 = 1, \eta_1 = b_1'/r_1$ and $\eta_2 = b_2'/r_2$ are considered. Tables 1–3 show biases and MSEs of point estimators with $R = 1/2$, $R = 2/3$, and $R = 1/3$, respectively. Using Algorithm 2, type II censored samples are generated from two independent Gompertz distributions.

The results of the proposed methods for point estimation are summarized in Tables 1–3. Based on these tables, the following results can be achieved:

(i) The MCMC method has the lowest MSE

(ii) For small sample size, the Bayesian method performs better than the MLE method

(iii) The MSE of Bayesian estimators under conjugate and Jeffreys priors is not significantly different

(iv) The MSE of all estimators decreases significantly with increasing sample size

Tables 4–6 compare the proposed confidence intervals using the interval lengths ($L$) and coverage probabilities (CPs). From these tables, the following results are obtained:

(i) The Bayesian credible intervals equi-tailed under the conjugate priors have the shortest, and the exact interval has the longest interval length

(ii) For classical and Bayesian methods, the $L$'s and the CPs have been improved by increasing the sample size.

Among these interval estimators, the CPs of the exact intervals are close to nominal level 95%.

(iii) The CPs of the Bayesian credible intervals equi-tailed are almost the same under the conjugate and Jeffreys priors

(iv) When $R$ is close to 1/2, the HPD intervals have an overestimate in estimating the CP

(v) When $R$ is far from 1/2, the HPD intervals have an underestimate in estimating the CP

## 8. Application

The sample lifetime of a steel particular type under two different pressures of 35.5 ($X$) and 35 ($Y$) is reported in Table 7. This data contains 20 observations in each sample. This data has been studied by Kimber [35]. To verify that the data have a Gompertz distribution, we perform the Kolmogorov-Smirnov test. Based on the test statistics and $P$ value, it is concluded that $X$ has a Gompertz distribution with parameters $\beta_1 = 0.0013$ and $\gamma = 0.0027 (D = 0.2, P\text{-value} = 0.832)$ and $Y$ has a Gompertz distribution with parameters $\beta_2 = 0.00093$ and $\gamma = 0.0027 (D = 0.35, P\text{-value} = 0.1745)$. We consider $r_1 = r_2 = 2$. In Bayesian estimation, the values of the hyperparameters are $\alpha_1 = \alpha_2 = 1, \eta_1 = b_1'/r_1$ and $\eta_2 = b_2'/r_2$. Now, we apply the proposed methods for point estimation and confidence interval estimation to this data. The results are summarized in Tables 8 and 9. Based on the value of $R = P(X > Y) = 0.417$, it can be concluded that the lifetime of steel under pressure 35 is greater than the lifetime of steel under pressure 35.5.

## 9. Conclusion

This paper proposed a classical and Bayesian inference for stress-strength reliability of Gompertz distribution under type II censoring. First, the MLE of $R$ was obtained. Then, exact and asymptotic confidence intervals for $R$ were presented. In addition, Bayesian estimators of $R$ obtained using Lindley approximation, Monte Carlo, and MCMC under conjugate informative and Jeffreys noninformative priors were discussed. Also, Bayesian credible intervals with equi-tailed and HPD intervals under conjugate and Jeffreys prior distributions were obtained. The proposed methods were compared with simulation studies. Finally, the application of these methods was examined with a real data.

## Data Availability

This paper has no associated data.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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