Action of Finite Group Presentations on Signal Space

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

Original Research Article

Abstract

The use of finite group presentations in signal processing has not been exploit in the current literature. Based on the existing signal processing algorithms (not necessarily group theoretic approach), various signal processing transforms have unique decomposition capabilities, that is, different types of signal has different transformation combination. This paper aimed at studying representation of finite groups via their actions on Signal space and to use more than one transformation to process a signal within the context of group theory. The objective is achieved by using group generators as actions on Signal space which produced output signal for every corresponding input signal. It is proved that the subgroup presentations act on signal space by conjugation. Hence, a different approach to signal processing using group of transformations and presentations is established.

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1 Introduction

Representation theory served as a tool to obtain information on finite groups with applications to many areas of sciences such as signal processing, cryptography, sound compression which is based on the Fast Fourier Transform (FFT) for finite groups (see Knapp, [1]; Hamermesh, [2]). Its emergence was also to serve as a tool for obtaining the information through the methods of linear algebra. Representation theory is also used as a tool in signal processing. Generally, Signal is the output of an array of sensor configured in time and (or) space. It is a conveyor of information and whenever a signal is projected or registered, it is then processed so as to extract all information coded in it. More generally, signals are regarded as physical processes which spread over time. One can observe signal directly at the moment (and in the form) in which it appears and only within a finite time interval. This is irrespective of its duration and parameters. Signals are also functions from groups into fields.

Also as discussed by Riley, et al. [3], Knapp, [1], Duzhin et al. [4], Hamermesh, [2] and Mallat, [5], group theory gives the most crucial part of the puzzle which is the dimension of the vector space where the signal exists. It is also observed that group theory provides the process for choosing any suitable transformation that is required for a specific application customized for a specific characteristic. Among many groups, it is found that finite non-abelian groups have more and useful applications in areas of sciences than other groups. Hence, the groups S_n and D_n are chosen for demonstration purposes.

1.1 Preliminaries

An action of a group G on any set X is the same as homomorphism $\phi: G \to S_X$, where S_X is the symmetric group on X (see Ngulde et al. [6], Peter, [7]). This motivates the following:

If K is a field and G is a group, then we define a representation of G as the pair (ρ, V) where V is a vector space over K and ρ is a homomorphism of G, given by ρ:G→GL$_K$(V). We also defined K-algebra as a ring whose underlying Abelian group is a K-vector space such that the multiplication map R×R→R is K-bilinear.

Again let R be a K-algebra. An R-module is a pair (ρ, V) where V is a vector space over K such that $\tilde{\rho}: R \to End_K V$ and a group algebra K[G] is a vector space with basis $\{1_g : g \in G\}$ and multiplication $1_g 1_h = 1_{gh}$.

Equivalently, the set K[G] represent the space of functions from G to K, f:G→K with binary operation as multiplication defined as follows:

$$f_1 * f_2(g) = \sum_{xy=g} f_1(x) f_2(y).$$

Hence, we have the following definitions (see Anupam, [8]).

Definition 1.1.1: Let r be an arbitrary element of R and define a function $\tilde{L}(r): R \to R$ from R to R by $x \mapsto r \cdot x$. Then the pair $(\tilde{L}, R)$ is called left regular module and if $R = K[G]$, then the representation (L, K[G]) is called the left regular representation of g∈G defined by the relation $L(g)1_x = 1_{gx}$.

In a similar manner, the right regular representation is defined by the relation $R(G)1_x = 1_{xg^{-1}}$.

Definition 1.1.2: (Equivalence representations): Two representations defined by φ:G→GL(V) and ψ:G→GL(W) are said to be equivalent if there exists an isomorphism between them. i.e. T:V→W such that the relation $\psi_g = T\varphi_g T^{-1}$ hold for all elements g of G and we write $\varphi \sim \psi$.

Definition 1.1.3: (Irreducible representation): A representation $\phi : G \rightarrow GL(V)$ is irreducible if and only if the only G-invariant subspace of V are {0} and V.

Definition 1.1.4: (Completely reducible): A representation $\phi : G \rightarrow GL(V)$ is completely reducible if and only if V can be expressed as a direct sum $V_1 \oplus V_2 \oplus \ldots \oplus V_n$ where all the V_i are non-zero G-invariant subspaces and $\phi | V_i$ is irreducible for all $i = 1, 2,\ldots,n$.

Definition 1.1.5: (Decomposable): The space V is decomposable if and only if $V = V_1 \oplus V_2$ where V_1 and V_2 are non-zero G-invariant subspaces, otherwise V is indecomposable.

Definition 1.1.6: Let (ρ_1, V_1) and (ρ_2, V_2) be representations. The linear map $T:V_1 \rightarrow V_2$ is called an intertwiner if

$T(\rho_1(g)v) = \rho_2(g)(T(v))$ or $T \circ \rho_1(g) = \rho_2(g) \circ T$ for all $g \in G$ (Ngulde, 2018).

Note:

The conjugate of an element x by y in a group G is define by $y^{-1}xy$. This convention is well defined with right conjugation action so that $x^y = y^{-1}xy$ or equivalently, $x_y = y^{-1}xy$ and $x^e = x_e = x$ where e is the identity element of G and $(x^y)^z = (x_y)_z = x_{yz}$ for all $x, y, z \in G$.

Definition 1.1.7: Signal Space [9]: If signal can be represented by n-tuple, then it can be treated in much the same way as n-dimensional vector space. Hence, the n-dimensional Euclidean space is called Signal space.

2 Review of Relevant Work

Let V be an inner product space and define a representation $\phi: G \rightarrow GL(V)$ on G. Then the representation is called *unitary* if and only if ϕ_g is unitary for all elements $g \in G$, i.e., $\langle \phi_g(v), \phi_g(w) \rangle = \langle v,w \rangle$, $v,w \in W$. Equivalently, identifying $GL_1(C)$ with the space C, then a complex number z is unitary if and only if $\bar{z}z = 1$. But this means that $|z| = 1$ so that $U_1(C)$ is the unit circle S^1 in the space C. Hence, a one-dimensional unitary representation is a homomorphism $\phi: G \rightarrow S^1$.
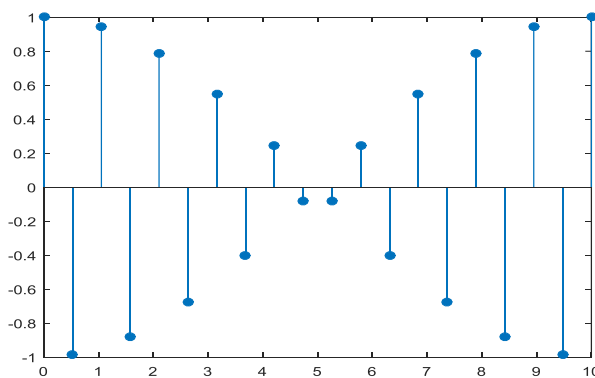

Fig. 1. Symmetric signal produced by a Unitary representation

A simple example is a representation $\varphi: R \rightarrow S^1$ defined by $\varphi(x) = e^{2\pi ix}$. Then φ is a unitary representation of R since $\varphi(x + s) = e^{2\pi i(x+y)} = e^{2\pi ix}e^{2\pi iy} = \varphi(x)\varphi(y)$ and produced a symmetric signal as follows:

>> x = linspace(0,10,20);

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>> Y = (exp(2*pi*i*x));
>> stem(x,Y,'filled')
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It can be deduce from the above statement that every representation of a finite group is equivalent to some unitary representations. Similar result is also established by Benjamin, [10].

2.1 Permutation representations

In permutation representation, every permutation group S_n has at least, a one-dimensional representation which is called the trivial representation. Another irreducible representation of degree 1 exist for $n \geq 2$, called the sign representation that maps a permutation to 1 by 1 matrix whose entry is ±1. These are the only one-dimensional representations of the symmetric group S_n and for all *n*, there is a natural permutation representation which is *n*-dimensional of order *n!* consisting of *n* coordinates. The trivial sub representation in this case consist of vectors with equal coordinates and for $n \geq 2$, the representation is called Standard Representation which is an $(n-1)$-dimensional irreducible representation which can be found and evaluated by the sign representation. Standard representation in terms of matrices of permutation representation was also presented by [11]. The Symmetric group S_n was defined by A^k where $k = 1, 2, \dots n!$ such that

$$A^k = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ a_1^k & a_1^k & a_1^k & . & . & . & a_1^k \end{pmatrix}$$

where $1 \leq a_i^k \leq n$ and all a_i^k are distinct. In the permutation representation, the n × n matrices are given through their elements by $(A^k)_{ij} = \delta_{a_i^k j}$, $1 \leq i, j \leq n$ which is a unitary representation. The Permutation representations can be defined in terms of left and right transversal of a subgroup in a group as follows: Suppose $(\alpha_1, \alpha_2, \dots, \alpha_r)$ is a left transversal of H in G such that [G:H] = r and $\tau \in G$ is fixed, then there is a unique permutation $\sigma_\tau \in S_n$ of left cosets of H in G by left multiplication such that $\tau \alpha_i H = \alpha_{\sigma_\tau(i)} H$ for all $1 \leq i \leq r$ and if $(\beta_1, \beta_2, \dots, \beta_r)$ is a right transversal of H in G, then there is a unique permutation $w_\tau \in S_n$ of right cosets of H in G by right multiplication such that $H\beta_i \tau = H\beta_{\sigma_\tau(i)}$ for all $1 \leq i \leq r$. The above representations are used to generate a function called transfer function which gives output for every corresponding input.

2.2 Group Action and Arbitrary Function

Representation of a group is given as a function ρ from a group *G* to the general linear group of some vector space V such that ρ is a homomorphism. Some measures on how closed an arbitrary function defined between non trivial groups is to being a homomorphism were also presented by Hawthorn et al. [12]. It was shown that the functions exhibit some properties which are similar to that of conjugates and commutators. The authors also showed that there exists a theory based on the given structures and the same theory was used in unifying some approach like group cohomology and transfer. Instead of trying to prove that some homomorphism exist especially by orbit counting, they tried to build using direct approach which means to begin with any given arbitrary function, then average out the same function action. For example, given a finite group G and H≤G such that [G:H] = r. Define a homomorphism ρ:H→K from H to an Abelian group K and a collection of all cosets representatives {$\alpha_i : 1 \leq i \leq r$} of H in G for the cosets {α_iH}. Then a function σ was defined by σ:G→K such that σ(hα_i) = ρ(h). It was also shown that if h$_o \in$H, then

$$\sigma^{h_0}(h\alpha_i) = \sigma(h_0)^{-1}\sigma(h_0 h\alpha_i) = \rho(h_0)^{-1}\rho(h_0 h) = \rho(h) = \sigma(h\alpha_i)$$

so that H stabilized σ under the function action. Now, since the transfer function in this case is regarded as a power of the average function, it was concluded that it is a homomorphism. Finally, the function action serves as

a tool to measure the failure of an arbitrary function from being a homomorphism, the same as the way conjugation action measures the failure to commute. Another similar construction which is analogous to that of commutators, called distributor measures the extent for an arbitrary function to preserve group structure was presented by Hawthorn et al, [12].

3 Methodology

Given a finite set X with n elements and a bijection $\zeta:X{\to}X$ on X, then a collection of all such bijections on X formed a group called the symmetric group on X denoted by S_n of order $n!$ (see Ngulde et al. [6], Derek et al. [13]). Thus if $\sigma{\in}S_n$, then σ is considered as a transformation on S_n. In this section, a method for constructing new homomorphism called transfer function on S_n is presented based on the idea of group representation as a homomorphism and the concept of arbitrary functions in group theory [12]. It is defined as a mathematical function that accept information, process it and gives possible output values. The idea of "Transversal" of a subgroup is first introduced using group of transformations as permutation representations as follows:

Suppose G is a group and $H{\leq}G$ is a non-trivial subgroup of G such that $[G{:}H] = r$ for some $r \geq 0$, $r{\leq}n$. Then any finite collection T containing exactly one representative of each coset (left or right) of the subgroup H in G is referred to as the transversal of H in G, given by $T = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ where each $\alpha_i \in \alpha_i H$. In this case,

$$G = \bigcup_{i=1}^{r} \alpha_i H .$$

Also, $\alpha_i H \bigcap \alpha_j H = \Phi$ for $i \neq j$ and whenever $\alpha_i \in \alpha_j H$, then $\alpha_i H = \alpha_j H$. Again, if $\{\alpha_i \mid 1 \leq i \leq r\}$ is defined as a left transversal of the subgroup H in G and an element $\tau \in G$ is fix, then there is always a unique permutation $\sigma_\tau \in G$ of left cosets such that $\tau\alpha_i H = \alpha_{\sigma_\tau(i)} H$ which implies that $\tau\alpha_i \in \alpha_{\sigma_\tau(i)} H$. Similarly, if $\{\beta_i \mid 1 \leq i \leq r\}$ is a right transversal of H in G such that $\tau \in G$ is fix, then there exists $\eta_\tau \in G$ such that $H\beta_i\tau = H\beta_{\eta_\tau(i)}$ which implies that $\beta_i\tau \in H\beta_{\eta_\tau(i)}$. From the above constructions, the mappings from G to S_n, $G \to S_n$, defined by $\tau \to \sigma_\tau$ and $\tau \to \eta_\tau$ are called the permutation representations of the group G in S_n with respect to the left and right transversals. Thus, if H is a non-trivial Abelian subgroup of a group G such that $[G{:}H] = r$ with left and right transversals as above and $g{\in}G$, then there exists $h_i \in H$, $\delta \in G$ depending on g such that $g\alpha_i = \alpha_{i\delta} h_i$ and if there exists another $k_i \in H$, $\delta \in G$, then

$$\prod_{i=1}^{r} h_r = \prod_{i=1}^{r} k_i .$$

Alternatively, let $q = \delta \in G$ and $g = e$. Then there exists $p_i \in H$ and $\xi \in G$ satisfying $\beta_i = \alpha_{i\xi} p_i$ for $1 \leq i \leq r$ and using the identity $\alpha_{i\xi\delta} = \alpha_{i\xi\delta\xi\xi^{-1}}$, we have

$$g\beta_i = g(\alpha_{i\xi} p_i) = (\alpha_{i\xi\delta} h_{i\xi}) p_i = \beta_{i\xi\delta\xi^{-1}} (p_{i\xi\delta\xi^{-1}})^{-1} h_{i\xi} p_i .$$

Then since $\xi\delta\xi^{-1} \in G$ and $(p_{i\xi\delta\xi^{-1}})^{-1} h_{i\xi} p_i \in H$, set $k = \xi\delta\xi^{-1}$ and $f_i = (p_{i\xi\delta\xi^{-1}})^{-1} h_{i\xi} p_i$. Thus, $g\beta_i = \beta_{ik} f_i$ and since H is Abelian,

$$\prod_i f_i = \prod_i (p_{i\xi\delta\xi^{-1}})^{-1} h_{i\xi} p_i = \prod_i p_i^{-1} \prod_i h_i \prod_i p_i = \prod_i h_i$$

The above construction defined a homomorphism called transfer function as follows:

Giving a finite group G (Abelian or non-Abelian) and a subgroup $H \leq G$ such that $[G:H] = r$. Then the notations set up above defined a function $\xi : G \rightarrow H$ by $g\xi = \prod_{i=1}^{r} h_i$ for all elements $g \in G$ called the transfer of $H \leq G$ independent of the cosets representatives h_i of H in G.

3.1 Action of the group S_n on finite sets

The concept of group presentation and group action on finite sets gives the basic idea behind group actions in which every elements of the given group can be viewed as transformations on a set in such a way that product of any corresponding transformations equals multiplication in the original group which is also a homomorphism [7]. The function $\sigma : G \rightarrow K$ between finite groups G and K defined a new function such that $\sigma^{\tau}(\alpha) = \sigma(\tau)^{-1} \sigma(\tau\alpha)$ [12]. Supposed G act on a finite set X, then the action $\sigma \cdot x$ can be viewed as a function for which σ is fixed. This means that for each $\sigma \in G$, any function $\rho_{\sigma} : X \rightarrow X$ on X defined as $\rho_{\sigma}(x) = \sigma \cdot x$ most satisfy the following conditions: $\sigma_1 \cdot (\sigma_2 \cdot x) = (\sigma_1 \sigma_2) \cdot x$ so that $\rho_{\sigma 1} \circ \rho_{\sigma 2} = \rho_{\sigma 1 \sigma 2}$ and the composition of functions on the set X is equivalent to multiplication in the group G. Note that if the group G is the group of linear transformations GL$_n$(R), then the group acts on the vectors in $\mathfrak{R}^n$ such that the product of the matrix A with a column vector v is given by $A \cdot v = Av$. If the set X is the set of polynomials and G is the symmetric group S$_n$, then G acts on X by permuting the variables as follows: For $\sigma \in G$,

$$(\sigma \cdot \rho)(y_1, y_2, ..., y_n) = \rho(y_{\sigma(1)}, y_{\sigma(2)}, ..., y_{\sigma(n)})$$

where $y_1, y_2, ..., y_n \in X$ are polynomials and ρ is the representative function defined on X.

Supposed the group G acts on the n-dimensional vector space $\mathfrak{R}^n$ by permuting the n-coordinates, then for any $\sigma \in G$, choose $\pi = (x_1, x_2, ..., x_n) \in \mathfrak{R}^n$ such that

$$\rho_{\sigma}(\pi) = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) \in \mathfrak{R}^n.$$

Then for all $\sigma_1, \sigma_2 \in G$ and $(x_1, x_2, ..., x_n) \in \mathfrak{R}^n$, we have

$$\rho_{(\sigma 1)}(\rho_{(\sigma 2)}(\pi)) = \rho_{(\sigma 1)}(x_{(\sigma 2)(1)}, x_{(\sigma 2)(2)}, ..., x_{(\sigma 2)(n)}) = \rho_{\sigma 1}(w_1, w_2, ..., w_n) \text{ where } w_i = x_{\sigma 2(i)}$$
$$= (w_{\sigma 1(1)}, w_{\sigma 1(2)}, ..., w_{\sigma 1(n)})$$
$$= (x_{\sigma 2(\sigma 1(1))}, x_{\sigma 2(\sigma 1(2))}, ..., x_{\sigma 2(\sigma 1(n))})$$
$$= (x_{(\sigma 2 \sigma 1)(1)}, x_{(\sigma 2 \sigma 1)(2)}, ..., x_{(\sigma 2 \sigma 1)(n)}) = \rho_{\sigma 1 \sigma 2}(\pi).$$

These methods of group actions are used and redefined on the n-dimensional space $\mathfrak{R}^n$ called signal space. The effect of the group S$_n$ on the signal space under group action manipulates signal by given different output for every corresponding input signal.

The method of distributors is also introduced in this section since it measures and gives the condition and extent to which the transfer function ξ defined above distributes over the elements of G with respect to multiplication

and preserves group structure. The idea is very useful most especially in this electronic era where the concept is used in recycling. The construction is also analogous to that of commutators as follows:

Theorem 3.1: [12]: Suppose $\xi : G \to H$ is a transfer function from G to H and let $\sigma, \rho, \omega \in G$. Then the ξ-distributor satisfy

$$[\rho, \omega : \xi][\sigma, \rho\omega : \xi] = [\sigma, \rho : \xi]^{\xi(\omega)}[\sigma\rho, \omega : \xi].$$

Theorem 3.2: [12]: Suppose $\xi : G \to H$ is a transfer function from G to H and let $\sigma, \rho, \omega \in G$. Then the ξ-distributor satisfy

$$[\sigma\rho, \omega : \xi] = [\sigma, \omega : \xi][\rho, \omega : \xi^{\sigma}].$$

4 Result and Discussion

From the construction in Section 3 above, the following are some facts defined on transversals of subgroups of a finite group. Let G and H be defined as in Section 3 above such that $[G:H] = r$ and define a function $\xi : G \to H$ by $g\xi = \prod_{i=1}^{r} h_i$ for all $g \in G$. Then ξ is called the transfer of H in the group G. The next result shows that the function $\xi : G \to H$ from the group G to H is a homomorphism.

Theorem 4.1: Let G and H be defined as above such that $[G:H] = r$. Then the transfer function $\xi : G \to H$ is also a homomorphism.

Proof: Suppose $g, q \in G$ are elements of G depending on δ, σ respectively such that $g\alpha_i = (\alpha_i)_\delta p_i$, $q\alpha_i = (\alpha_i)_\sigma k_i$, $p_i, k_i \in H$ and let $\{\alpha_1, \alpha_2, ..., \alpha_r\}$ be a set of transversal of $H \leq G$. With $\delta, \sigma \in G$,
$$gq\alpha_i = g(\alpha_i)_\sigma k_i = (\alpha_i)_{\sigma\delta} (p_i)_\sigma k_i.$$

Thus, from the identities in Section 3.2.1,

$$gq\xi = \prod_i (p_i)_\sigma k_i = \prod_i p_i \cdot \prod_i k_i = g\xi \cdot q\xi.$$

Hence, ξ is a homomorphism.

4.1 Action of subgroup presentations on signal space

Let $(X, \pi) = \{\phi_1, \phi_2, ..., \phi_n\}$ be a signal space where π is the representative of the functions over X and let $\sigma \in S_n$ with $n \geq 2$. Then from Section 3, the group $G = S_n$ acts on the signal space X and permute its elements as follows:

$$(\sigma \cdot \pi)(\phi_1, \phi_2, ..., \phi_n) = \pi(\phi_{\sigma(1)}, \phi_{\sigma(2)}, ..., \phi_{\sigma(n)}) \qquad 4.1.1$$

In this case, every element $\sigma \in G$ satisfy $\phi_i \mapsto \phi_{\sigma(i)}$ in (X, π). Hence, we have the following results.

Lemma 4.1.1: The function defined in Equation 4.1.1 also defined a group action of the group G on (X, π).

Proof: Obviously, $i \cdot \pi = \pi$. Next, we show that $\sigma \cdot (\delta \cdot \pi) = (\sigma\delta) \cdot \pi$ for all $\sigma, \delta \in G$. Now,

$$(\sigma \cdot (\delta \cdot \pi))(\phi_1, \phi_2, ..., \phi_n) = (\delta \cdot \pi)(\phi_{\sigma(1)}, \phi_{\sigma(2)}, ..., \phi_{\sigma(n)})$$
$$= \pi(\phi_{\sigma(\delta(1))}, \phi_{\sigma(\delta(2))}, ..., \phi_{\sigma(\delta(n))})$$
$$= \pi(\phi_{(\sigma\delta)(1)}, \phi_{(\sigma\delta)(2)}, ..., \phi_{(\sigma\delta)(n)})$$
$$= ((\sigma\delta) \cdot \pi)(\phi_{(1)}, \phi_{(2)}, ..., \phi_{(n)})$$

as required.

Lemma 4.1.2: Let $\sigma, \delta \in G$ and $(X, \pi) = \{\phi_1, \phi_2, ..., \phi_n\} \in \Re^n$ be a signal space. Then $\pi_\sigma \circ \pi_\delta = \pi_{\delta\sigma}$.

Proof: Supposed $\sigma, \delta \in G$ and $w = (\varphi_1, \varphi_2, ..., \varphi_n) \in (X, \pi)$ be arbitrary.

Then

$$\pi_\sigma \circ \pi_\delta(w) = \pi_\sigma(\pi_\delta(w))$$
$$= \pi_\sigma(\pi_\delta(\varphi_1, \varphi_2, ..., \varphi_n)$$
$$= \pi_\sigma(\varphi_{\delta(1)}, \varphi_{\delta(2)}, ..., \varphi_{\delta(n)})$$
$$= \pi_\sigma(\lambda_1, \lambda_2, ..., \lambda_n) \text{ where } \lambda = \varphi_{\delta(i)}$$
$$= (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, ..., \lambda_{\sigma(n)})$$
$$= (\varphi_{\delta(\sigma(1))}, \varphi_{\delta(\sigma(2))}, ..., \varphi_{\delta(\sigma(n))})$$
$$= (\varphi_{(\delta\sigma)(1)}, \varphi_{(\delta\sigma)(2)}, ..., \varphi_{(\delta\sigma)(n)})$$
$$= \pi_{\delta\sigma}(\varphi_1, \varphi_2, ..., \varphi_n)$$
$$= \pi_{\delta\sigma}(w).$$

Since w is arbitrary, the result is true for all $w \in X$.

Lemma 4.1.3: With $G = S_n$, let X be defined as above and Y be a signal space not necessarily isomorphic to X. Let $\xi(X, Y)$ be the collection of all maps from X to Y, i.e. $\xi : X \to Y$. Then the action of G on $\xi(X, Y)$ is given by the rule

$$(\pi_\sigma \xi)(w) = \xi(\sigma w).$$

4.1.2

Proof: Obviously, σw is the action of the element $\sigma \in G$ on $w \in X$ and $\pi_\sigma \xi$ is also a function from X to Y. To find a group action of G on $\xi(X, Y)$, note that G acts on X from the left. Now, replace σ with σ^{-1} in Equation 4.3.2 so that $(\sigma \cdot \xi)(w) = \xi(\sigma^{-1}w)$, then we have

$$(\sigma_1 \cdot (\sigma_2 \cdot \xi))(w) = (\sigma_2 \cdot \xi)(\sigma_1^{-1}w)$$
$$= \xi(\sigma_2^{-1}(\sigma_1^{-1}w))$$
$$= \xi((\sigma_2^{-1}\sigma_1^{-1})w)$$
$$= \xi((\sigma_1\sigma_2)^{-1}w)$$
$$= ((\sigma_1\sigma_2) \cdot \xi)(w).$$

Hence, $\sigma_1 \cdot (\sigma_2 \cdot \xi) = (\sigma_1\sigma_2) \cdot \xi$ so that $(\pi_\sigma \xi)(w) = \xi(\sigma w)$ defined a group action on $\xi(X, Y)$.

Example 4.1.4: Let $G = S_n$ and choose $n = 5$. Then $G = S_5$ and $X = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\} \in \Re^5$.

Now, from

$$(\sigma \cdot (\delta \cdot \pi))(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\delta \cdot \pi)(\phi_{\sigma(1)}, \phi_{\sigma(2)}, \phi_{\sigma(3)}, \phi_{\sigma(4)}, \phi_{\sigma(5)})$$

and Lemma 4.1.1,

$$(\sigma_3 \cdot (\delta_1 \cdot \pi))(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\delta_1 \cdot \pi)(\phi_{(4)}, \phi_{(1)}, \phi_{(3)}, \phi_{(2)}, \phi_{(5)})$$
$$= (\phi_{(3)}, \phi_{(4)}, \phi_{(1)}, \phi_{(5)}, \phi_{(2)}).$$

But $\sigma_3 \cdot \delta_1 = \delta_{18}$ in G and

$$(\delta_{18} \cdot \pi)(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\phi_{(3)}, \phi_{(4)}, \phi_{(1)}, \phi_{(5)}, \phi_{(2)}).$$

Hence, $(\sigma_3 \cdot (\delta_1 \cdot \pi))(w) = ((\sigma_3 \delta_1) \cdot \pi))(w) = (\delta_{18} \cdot \pi)(w)$.

Example 4.1.5: Again, let $w = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \Re^5$. Then $\pi_{t_2} \circ \pi_{\gamma_{15}}(w) = \pi_{\gamma_{15}\tau_2}(w)$ (Lemma 4.1.2), where $\tau_2, \gamma_{15} \in G$.

To see this, we compute

$$\pi_{\tau_2} \circ \pi_{\gamma_{15}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = \pi_{\gamma_{15}}(\varphi_1, \varphi_3, \varphi_4, \varphi_5, \varphi_2)$$
$$= (\varphi_3, \varphi_1, \varphi_4, \varphi_2, \varphi_5).$$

Now, $\gamma_{15} \circ \tau_2 = \tau_{11}$ in G and

$$\pi_{\tau_{11}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = (\varphi_3, \varphi_1, \varphi_4, \varphi_2, \varphi_5).$$

Thus, $\pi_{\tau_2} \circ \pi_{\gamma_{15}}(w) = \pi_{\tau_{11}}(w)$. This is true for all elements of G.

Now, the elements of G act on the signal space X as functions. Hence for any signal space X, if G is a subgroup of S_X then X is called a G-set with respect to the group action

$$(\sigma, \phi(t)) \mapsto \sigma(\phi(t))$$

for all $\sigma \in$ G, $\phi(t) \in$ X.

Lemma 4.1.6: Let $|G| = n$ such that G≅X. Then G acts on X by the left regular representation given b

$$(\sigma, \phi(t)) \mapsto \pi_\sigma(\phi(t)) = \sigma\phi(t).$$

Proof: Since π_σ is a left multiplication,

$$i \cdot \phi(t) = \pi_i \phi(t) = i\phi(t) = \phi(t)$$

where *i* is the identity element of G. Also,

$$(\sigma\tau)\cdot\phi(t) = \pi_{(\sigma\tau)}\phi(t) = \pi_\sigma\pi_\tau\phi(t) = \pi_\sigma(\tau\phi(t)) = \sigma\cdot(\tau\cdot\phi(t))$$

and this established the result.

Lemma 4.1.7: Let $|G| = n$ such that $G\cong X$ and let K be a subgroup of G. Then X is a K-set under conjugation. i.e., an action $K\times X \to X$ of K on X defined by

$$(\delta,\phi(t)) \mapsto \delta(\phi(t))\delta^{-1}$$

for all $\delta\in K$ and $\phi(t)\in X$.

Proof: Clearly, the first axiom for group action is satisfied. Next, observed that

$$\begin{aligned}
(\sigma\tau,\phi(t)) &= \sigma\tau(\phi(t))(\sigma\tau)^{-1} \\
&= \sigma\tau(\phi(t))(\tau^{-1}\sigma^{-1}) \\
&= \sigma(\tau(\phi(t))\tau^{-1})\sigma^{-1} \\
&= (\sigma,(\tau,\phi(t)))
\end{aligned}$$

which shows that condition two is satisfied as required.

Example 4.1.8: We defined Signals as functions on some discrete groups which is identified with the group Z of integers or integers modulo p, Z_p, i.e. $\varphi : Z_p \to Z_p$. Let $p = 2$. Then Z_p is isomorphic to S_2. Now, consider a binary symmetric channel described as model consisting of transmitter that is capable of transmitting binary signal together with a receiver. One of the possible coding schemes can be achieved by sending a signal many times so as to compare any output signals with another. Supposed the signal which is to be encoded is (1 1 0 1 0 0) into a binary $4n$-tuple, let $\sigma\in S_n$. Then $\sigma : Z_2 \to Z_2$ encode (1 1 0 1 0 0) into a binary $4n$-tuple as

$$(110100) \mapsto (110100110100110100110100).$$

The decoded signal depends on the function $\sigma\in S_n$. The function σ is also required to be one-to-one in order that two signals will not be encoded into the same image.

NOTE: Given any transformation $T : G \to G$ defined by $\omega \mapsto T(\omega)$, $\omega \in G$, then $T_\tau(\omega) = (\omega)^\tau = \tau(\omega) = (\omega)\tau^{-1}$ for all elements $\tau \in G$. The defined operation can be used to modify any given signal such that the signal is well-defined on the domain of the transformation $\tau \in G$, called a mapping on G. Hence, a discrete signal $x(n)$ can be transformed as $\omega^\tau(x(n)) = \omega\tau^{-1}(x(n))$.

In the next result, it is assumed that the group G is isomorphic to the geometric group D_n.

Lemma 4.1.9: Let G be a finite 2-generator group and $\zeta : G \to G$ be a mapping on G such that $\zeta_{\phi\eta\gamma} = \zeta_\tau$ satisfying $\eta\tau = e$, the identity element of G, $\phi,\eta,\gamma,\tau \in G$. Then any discrete signal $x(n)$ can be transformed by the relation $\phi(\eta(\gamma(x(n)))) = \tau(x(n))$.

Proof: Set $\zeta = \gamma\tau^n\gamma^{-1}$ such that τ,γ represent a rotation and reflection respectively. But ζ is a homomorphism and taken H = G, we have for any given signal $x(n)$,

$$\begin{aligned}
\zeta(x(n)) &= \gamma\tau^n\gamma^{-1}(x(n)) = \gamma(x(n))\tau^n\gamma^{-1}(x(n)) \\
&= \gamma(x(n))\tau^n(x(n))\gamma^{-1}(x(n))
\end{aligned}$$

$$= \gamma(x(n))(\tau^{-n})^{-1}(x(n))\gamma^{-1}(x(n))$$
$$= \gamma(x(n))[(\tau^{-n})^{-1}\gamma^{-1}](x(n))$$
$$= \gamma(x(n))[(\gamma\tau^{-n})^{-1}](x(n))$$
$$= \gamma(x(n))[\gamma(\tau^{-n})](x(n))$$
$$= \gamma(x(n))\gamma(x(n))(\tau^{-n})(x(n))$$
$$= (\tau^{-n})(x(n))$$

i.e. $\zeta(x(n)) = \tau^{-n}(x(n))$.

But $\tau^{-n}\tau^{n} = i$. Hence, the result follows.

Corollary 4.1.10: Suppose G is finite and a 2-generator group. Let $\omega, \tau \in G$ be rotation and reflection respectively. Then there exists an element $\eta \in G$ in G such that $\eta = \tau\omega\tau^{-1}$ with $\eta\omega = e$. This means that the element $\eta \in G$ is decomposed into product of transformations.

Remark 4.1.11: Suppose $H \subseteq G$ such that $H = \{\eta^{n} : \eta \in G\}$ and the index $[G{:}H] = 2$. If $\eta, \tau \in G$ are arbitrary transformations in G, then $H \lhd G$ and there exist a relation $(\circ) : G \to G$ defined on G such that for any given signal $x(n)$, the following relations holds:

    i. $\eta^{r} \circ \tau^{s}(x(n)) = \eta^{(r+s)\bmod n}(x(n))$ iff $\tau \in H$;
    ii. $\eta^{r} \circ \tau^{s}(x(n)) = \tau^{(r+s)\bmod n}(x(n))$ iff $\tau \notin H$;
    iii. $\tau^{r} \circ \eta^{s}(x(n)) = \eta^{(r-s)\bmod n}(x(n))$ iff $\tau \notin H$;
    iv. $\mu^{r} \circ \tau^{s}(x(n)) = \eta^{(r-s)\bmod n}(x(n))$ iff $\mu, \tau \notin H$.

In general, since H is an Abelian subgroup, the elements of H satisfy $(\tau \circ \mu)(x(n)) = \tau(x(n)) \circ \mu(x(n))$ for all $\tau, \mu \in H$.

The transfer function $\xi : G \to H$ also distributes over the elements of G as follows: For any $\sigma, \rho \in G$, the ξ-distributor $[\sigma, \rho : \xi]$ is given by

$$[\sigma, \rho : \xi] = \xi(\rho)^{-1}\xi(\sigma)^{-1}\xi(\sigma\rho) = \xi(\rho)^{-1}\xi^{\sigma}(\rho).$$

This set of distributors measures the extent to which the ξ-distributor fails to be a homomorphism and if $\sigma, \rho, w \in G$, then from Theorem 3.1 and Theorem 3.2 (Section 3),

$$[\sigma, \omega : \xi][\rho, \omega : \xi^{\sigma}] = \xi(\omega)^{-1}\xi(\sigma\rho)^{-1}\xi(\sigma\rho\omega).$$

Note that if $\tau \in G$, then the distributor of τ defines a new operator ξ_{τ} on the function $\xi : G \to H$ such that $\xi_{\tau}(x) = [x, \tau : \xi]$. This implies that $(\xi_{\tau}(x))^{g} = \xi_{\tau}^{g}(x)$ for all $g \in G$. Hence, the distributors operators on G commute with action of conjugation on functions.

5 Conclusion

The result in this paper shows that any 2-generator group of order $2n$ with presentation

$$G = \langle \alpha, \beta : \alpha^n = \beta^2 = e, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle$$

can be partition into two disjoint subsets H and S such that

$$H = \langle \alpha^i : 1 \le i \le n \rangle, \quad S = \langle \alpha^i \beta : 1 \le i \le n \rangle \text{ and } |H| = |S| = n.$$

In this case, the action of the subgroup H on the set S is given by $H \bullet S = S$. This means that each element of the subgroup H described the action of H on S. The results described how the representations of S_n act on the signal space (X, π) by transforming $\phi_i \rightarrow \phi_{\sigma(i)}$ for all $\phi \in (X, \pi)$ and $\sigma \in S_n$. It was proved that each $\sigma \in S_n$ can be used to transform signal from one form to another in a manner that each representation act on a signal as input, processed it using permutation and then produced the result as output. The result also shows that the function defined by $(\sigma \cdot \pi)(\phi_1, \phi_2, ..., \phi_n) = \pi(\phi_{\sigma(1)}, \phi_{\sigma(2)}, ..., \phi_{\sigma(n)})$ is a group action of S_n on the signal space (X, π) and that if $\xi : X \rightarrow Y$ is a transfer function from a signal space X to a signal space Y, then $(\pi_g \xi)(w) = \xi(gw)$ is the action of $G (= S_n)$ on $\xi(X, Y)$. The functions $(\sigma, \phi(t)) \mapsto \pi_\sigma(\phi(t)) = \sigma\phi(t)$ and $(\delta, \phi(t)) \mapsto \delta(\phi(t))\delta^{-1}$ also act on the signal space X (when $G \cong X$) and $K \times X \rightarrow X$ of K on X respectively. The collection of these results proves that if G is a finite 2-generator group with representation elements $\omega, \tau \in G$ where ω and τ are rotations and reflections respectively, then an element $\eta \in G$ can be decomposed into product of transformations as $\eta = \tau\omega\tau^{-1}$ such that $\eta\omega = e$. These results are used to manipulate signals which produced output for every corresponding input signal.

The function $\zeta : G \rightarrow G$ in this case is used to decompose single transformation into product of two or more transformations such that $\zeta(\omega) = \prod_{r \in N} \eta_r$ for some $\omega, \eta \in G$.

We therefore conclude that a different approach and presentation to signal processing using group of transformations is presented. The derived relations shows that whenever a change occur in any given signal at a particular point, the derived relations will propagates this same change across different conjugacy classes.

Competing Interests

Authors have declared that no competing interests exist.

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