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A Constructive Method for Generating Short Presentations for the Symmetric Groups S_{m+n}, S_{2m} and S_{mn}

D. Samaila[1*], G. N. Shu'aibu[2] and B. A. Modu[2]

[1]Department of Mathematics, Adamawa State University, Mubi, Nigeria.
[2]Mathematical Sciences Department, University of Maiduguri, Borno State, Nigeria.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

A long-standing problem is how to create a short-length presentation for finite groups of degree n. This paper aimed at presenting a concrete method for generating presentations for the groups S_{m+n}, S_{2m} and S_{mn} for all $m,n \in Z^+$ with fewer relations than the existing literature from the presentations of S_m and S_n. The aim is achieved by considering finite groups acting on sets and Cartesian product of groups which lead to the construction of multiple transformations as representatives of some finite groups.

Keywords: Cartesian product; group action; representation; symmetric group; permutation.

1 Introduction

The idea of Group arises in mathematics as "sets of symmetries (of an object), which are closed under composition and inverses". A concrete example is the Symmetric group S_n whose elements consists of all possible permutations of n - objects; the group of even permutations in S_n called Alternating group A_n; the

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**Corresponding author: Email: danjuma1981@gmail.com;*

Dihedral group D_{2n} (also called geometric group) which is the group of symmetries of regular *n*-sided polygon; the Orthogonal group O(3) also known as the group of distance-preserving transformations in the Euclidean plane that fixes the origin. From geometric point of view, questions such as "Given a geometric object *X*, what is its group of symmetries?" aroused while the same question is reversed in Representation theory such as "Given a group *G*, what objects *X* does it act on?". The attempt to answer such question leads to the classification of *X* up to isomorphism.

In group theory, a presentation of a group G is described as a homomorphism from the group into another group, say K. It is considered as a compact way of describing the structure of any group. A representation of a group is also a presentation such that the target group is given by the group of automorphisms of a vector space. In this case, every element of the group is mapped to an invertible linear transformation in the space. Group representation theory also serves as a tool to study the structure of groups via their actions on vector spaces. Such result can be achieved by considering groups acting on sets such as the Sylow theorems. Also, more detail information about group can be obtained when the group act on vector space. This is the basic idea behind representation theory. It also served as a powerful tool to obtain information about finite groups with applications to many areas of sciences such as signal processing, cryptography, sound compression using Fast Fourier Transform (FFT) for finite groups [1,2]. It also provide information about finite groups through the methods of linear algebra.

This paper aimed at addressing a long-standing problem for creating short-length presentation for finite groups of degree *n*. An attempt by Bray *et al* 2007, paved a way for such construction for which some short presentations for finite groups were derived. But these presentations can be made shorter with fewer relations which leads to the novelty of this paper.

1.1 Preliminaries

Let K be a field, V be a vector space over K and G be a group. Then a representation of G can be define as the pair (ρ, V) where ρ is a homomorphism of G defined by $\rho:G\to GL_K(V)$. Again, a K-algebra can be defined as a ring for which underlying Abelian group is a K-vector space with multiplication map R×R→R. We shall now define the following terms (see [3]).

Definition 1.1.1: (Equivalence): The representations $\varphi:G\to GL(V)$ and $\psi:G\to GL(W)$ are said to be equivalent if there is an isomorphism $T:V\to W$ between the two representations such that $\psi_g = T\varphi_g T^{-1}$ for all elements g∈G, i.e. $\psi_g T = T\varphi_g$, g∈G. Hence, we write φ~ψ.

Definition 1.1.2: (Irreducible representation): Let $\varphi : G \to GL(V)$ be a representation. Then φ is irreducible if the only G-invariant subspace of V are {0} and V.

Definition 1.1.3: (Completely reducible): A representation $\varphi : G \to GL(V)$ is completely reducible if and only if $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ such that the V_i are non-zero G-invariant subspaces and each $\varphi|V_i$ is irreducible for all *i* = 1, 2,…,n. Equivalently, if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus \ldots \oplus \varphi^{(n)}$ where $\varphi^{(i)}$ are irreducible representations, then φ is completely reducible.

Definition 1.1.4: (Decomposable): The space V is decomposable if and only if $V = V_1 \oplus V_2$ where V_1 and V_2 are non-zero G-invariant subspaces. Otherwise, V is indecomposable.

Definition 1.1.5: If (ρ_1, V_1) and (ρ_2, V_2) are representations, then the linear map $T:V_1\to V_2$ from V_1 to V_2 is called an intertwiner if it satisfies

$$T(\rho_1(g)v) = \rho_2(g)(T(v)) \text{ or } T\circ\rho_1(g) = \rho_2(g)\circ T \text{ for all } g\in G \text{ [4].}$$

Lemma 1.1.6: (Shur's Lemma 1): Supposed K is algebra closed and V is finite dimensional simple representation of G, then every self-intertwiner $T:V\to V$ is a scalar multiple of id_V.

Note: Two distinct spaces V_1 and V_2 are said to be isomorphic if there exists a bijective intertwiner $T:V_1 \rightarrow V_2$ between them denoted by $V_1 \cong V_2$.

Lemma 1.1.7: (Shur's Lemma 2): Let V_1 and V_2 be simple. Then every non-zero intertwiner of V_1 and V_2 is an isomorphism. Consequently, either $V_1 \cong V_2$ or $Hom_G(V_1, V_2) = 0$.

We shall now write ϕ_g for $\phi(g)$ and the action of ϕ_g on $v \in V$ by $\phi_g(v)$ or $\phi_g v$.

Note: We shall now define a Coxeter group W as a group with the following presentations:

$$\left\langle x_1, x_2, \ldots, x_m \mid (x_i x_j)^{n_{ij}} = e \right\rangle$$

where $n_{ij} = e$ and $n_{ij} \geq 2$ for $i \neq j$ and the condition that $n_{ij} = \infty$ means there is no any relation of the form $(x_i x_j)^n$. The pair (W,S) with set of generators $S = \{x_1, \ldots, x_n\}$ is called a Coxeter system. Hence, we have the following Coxeter relations:

i. The relation $n_{ii} = e$ means that $(x_i x_i)^1 = (x_i)^2 = e$ for all i;

ii. If $n_{ij} = 2$, then the generators x_i and x_j commute since $aa = bb = e$ with $abab = e$ implies that $ab = a(abab)b = (aa)ba(bb) = ba$. Alternatively, the generators are involutions so that $x_l = x_i^{-1}$ and thus,

$$(x_i x_j)^2 = x_i x_j x_i x_j = x_i x_j x_i^{-1} x_j^{-1} = [x_i, x_j],$$

equal to the commutator.

iii. If redundancy among relations most be avoided, then it is necessary to assume that $n_{ij} = n_{ji}$ by observing that $xx = e$ and $(xy)^n = e$ implies that

$$(xy)^n = (yx)^n xx = x(xy)^n x.$$

Alternatively, using conjugate elements, we have the relation

$$y(xy)^m y^{-1} = (yx)^m yy^{-1} = (yx)^m$$

2 Review of Relevant Work

If $\phi : Z_n \rightarrow C$ and $\varphi : Z_n \rightarrow C$ are representations on Z_n defined by $\phi_m = e^{\frac{2\pi i m}{n}}$ and $\varphi_m = e^{-\frac{2\pi i m}{n}}$ respectively, then the sum $\phi \oplus \varphi$ can be define by

$$(\phi \oplus \varphi)_m = \begin{pmatrix} e^{\frac{2\pi m i}{n}} & 0 \\ 0 & e^{\frac{-2\pi m i}{n}} \end{pmatrix}.$$

Now, since representations are considered as special homomorphism, suppose a set X generate the group G. Then any representation ϕ of G is uniquely determined by its values on X; [5]. Again if ϕ: $G \rightarrow GL(V)$ is any representation and $W \leq V$ is G-invariant subspace, then the representation ϕ can be restricted so as to obtain a new representation $\phi|_W : G \rightarrow GL(W)$ by setting $(\phi|_W)_g(w) = \phi_g(w)$, $w \in W$. Thus, since W is G-invariant, then the element $\phi_g(w) \in W$ and $\phi|_W$ is called a *sub-representation* of ϕ. Also, any degree one representation ϕ: $G \rightarrow C$ is irreducible where $G = \{1\}$ and if ϕ: $G \rightarrow GL(V)$ is a representation, then $\phi = e$ and if ϕ: $G \rightarrow GL(V)$ is another representation of degree 2, then we say that ϕ is irreducible if and only if there is no common eigenvector v to all ϕ_g with $g \in G$ [5].

Despite the fact that numerous properties of group representations are presented in various literature, no attempt for generating and producing shorter length presentations for finite groups. In the quest to generate short presentations for finite groups, [6] derived new families of presentations for finite groups which is based on generators and relations from the presentations for the symmetric group S_n and the group of even permutations in S_n. The literature also includes presentations with length linear in log$n$ and 2-generator presentations with a bounded number of relations independent of n. The authors were able to derived the presentations for finite groups S_{m+n} with $|M| + |N| + 12$ relations, S_{2m} with $|M| + 6$ relations and S_{mn} with $|M| + |N| + 20$ relations based on the presentation of S_n as follows:

Theorem 2.1: Let $P = \{A \mid M\}$ and $Q = \{B \mid N\}$ be presentations for the finite groups S_m and S_n with m,n≥3 respectively and let the generating set A for S_m contains r and v representing transposition (1 2) and the m-cycle (1 2 … m) respectively and the generating set B for S_n contains elements s and w standing for the transposition (1 2) and the n-cycle (1 2 … n) respectively. Then

$$\{A,B,t,y \mid M,N,t^2,(rt)^3,(ts)^3,y^{-1}wtv,[r,s],[r,w],[v,s],[v,w],[rv,t],[vrv^{-1},t],[t,ws][t,w^{-1}sw]\}$$

is a presentation for S_{m+n} on a generating set that includes the elements y standing for the (m + n)-cycle (1 2 … m + n) and t standing for a transposition fo the form (i, i+1). This presentation has |A| + |B| + 2 generators and |R| + |S| + 12 relations, and presentation length of at most l(P) + l(Q) + 64 where l(P) and l(Q) are the lengths of the presentations P and Q [6].

Theorem 2.2: Let $P = \{A \mid M\}$ be a presentation for the symmetric group S_n of degree n ≥ 3, such that the generating set A contains x and w standing for the transposition (1 2) and the n-cycle (1 2 … n) respectively.

Then

$$\{A,y \mid M,y^{2n},(xy)^{2n-1},[x,wy^{-1}],[w^2xw^{-1},wy^{-1}],[x,y^n]^2,[x,y^{n-1}]^2\}$$

is a presentation for S_{2n} on a generating set that includes the elements y standing for the 2n-cycle (1 2 … 2n) and x standing for a transposition fo the form (i, i+1). This presentation has |A| + 1 generators and |R| + 6 relations [6].

Theorem 2.3: Let $P = \{A \mid M\}$ and $Q = \{B \mid N\}$ be presentations for the finite groups S_m and S_n with m,n≥3 respectively and let the generating set A for S_m contains r and v representing transposition (1 2) and the m-cycle (1 2 … m) respectively and the generating set B for S_n contains elements s and w standing for the transposition (1 2) and the n-cycle (1 2 … n) respectively. Then

$$\{A,B,t,y \mid M,N,t^2,s^{-1}(v^{-1}tw^{-1}v^{-1}w)^m,w^{-1}y^m,y^{-1}wv(wtv)^{n-1},y^{-1}vyrv^{-1}t,(v^2rv^{-2}t)^3,(tw^{-1}rw)^3,$$
$$[r,t],[v^2rv^{-1},t],[r,vy^{-1}],yty^{-1}v^2rv^{-2},y^{-1}tyw^{-1}rw,[r,w^{-1}rw],[r,w^{-1}vw],[v,w^{-1}rw],[v,w^{-1}vw],$$
$$[r,ws],[r,w^{-1}sw],[v,ws],[v,w^{-1}sw]\}$$

gives a presentation for the group S_{mn} on a generating set which includes the elements y representing the mn-cycle (1, 2, …, mn) and t representing a transposition of the form (i, i+1). This presentation has |A| + |B| + 2 generators and |R| + |S| + 20 relations [6]

It is observed that the generated presentations in this literature can be obtained with fewer relations. This work therefore, presents a concrete technique for generating shorter presentations for finite groups with few relations.

3 Methodology

In this section, the method of constructing presentations for the finite group S_n of length linear in n is presented as discussed by Bray et al. [6]. But we shall first present the Cartesian product of non-empty sets $S_1, S_2, \ldots, S_n$ called the set of all ordered n-tuples $\{x_1, x_2, \ldots, x_n \mid x_i \in S_i\}$. The Cartesian product of these sets is denoted by either

$$S_1 \otimes S_2 \otimes \ldots \otimes S_n \text{ or by } \prod_{i=1}^{n} S_i .$$

Now, let the binary operations on the groups $G_1, G_2, \ldots, G_n$ be multiplication. Regarding the G_i as sets, we can form the Cartesian product $\prod_{i=1}^{n} G_i$ of the groups $G_1, G_2, \ldots, G_n$. It is also easy to make $\prod_{i=1}^{n} G_i$ into a group by means of a binary operation of multiplication by components. Hence, new groups can be formed from Cartesian product of known groups as presented by the following theorems:

Theorem 3.1: (see [7]): Let $G_1, G_2, \ldots, G_n$ be groups. For $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\prod_{i=1}^{n} G_i$, define $(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$. Then $\prod_{i=1}^{n} G_i$ is a group called the External Direct Product of the groups $G_1, G_2, \ldots, G_n$ under this binary operation.

Remark 3.2: It can be deduced from the above theorem that for the groups $G_1, G_2, \ldots, G_n$ with orders $r_1, r_2, \ldots, r_n$ respectively, we have

$|G_1 \otimes G_2 \otimes \ldots \otimes G_n| = |G_1|\,|G_2| \ldots |G_n| = r_1 r_2 \ldots r_n$ where the product $G_1 \otimes G_2 \otimes \ldots \otimes G_n$ is a new group which may or may not be isomorphic to the group $G_{r_1 r_2 \ldots r_n}$.

Theorem 3.3: (see [7]): The isomorphism $Z_m \otimes Z_n \cong Z_{mn}$ is possible if and only if $(m, n) = 1$.

Now, let G = S_n whose elements are bijections on the set S. Then to obtain a presentation for G, we introduced an n-cycle $\xi = (1, 2, \ldots, n)$ as a new generator which is used from the fact that $\xi^{-j}\alpha_i\xi^j = (j, j+1) = \alpha_{j+1}$ to eliminate the generator α_i for $1 \le j < n$. If we take an arbitrary generator α and then eliminate further redundancy from the relations under conjugation by ξ, then the presentation is given by

$$\{\alpha, \xi \mid \alpha^2 = \xi^n = (\alpha\xi)^2 = e, (\alpha\xi^{-1}\alpha\xi)^3 = e, (\alpha\xi^{-j}\alpha\xi^j)^2 = e; j = 2, \ldots, n/2\}.$$

However, if we define $\xi_j = \xi^j$, then $(\alpha\xi^{-j}\alpha\xi^j)^2 = e$ is replaced by $(\alpha\xi_j^{-1}\alpha\xi_j)^2 = e$. Hence, we have the following:

Theorem 3.4: [6]: For all $n \ge 3$, the finite group S_n has the following presentation:

$$\{\alpha, \xi_1, \ldots, \xi_{n/2} \mid \alpha^2 = \xi_1^n = (\alpha\xi_1)^{n-1} = e, (\alpha\xi_1^{-1}\alpha\xi_1)^3 = e, \xi_{j-1}\xi_1\xi_j^{-1} = e, (\alpha\xi_j^{-1}\alpha\xi_j)^2 = e\}$$

with $1 + n/2$ generators and $n + 2$ relations.

Again, let $S_n = \langle \sigma_i \mid 0 \le i \le n-1 \rangle$ where σ is any bijection from 1 to n such that $\sigma_0 = \sigma_n = e$, the identity element of G. Then we shall have the following relations:

If $\sigma_i = (i, i+1)$, then

Relation 1: $(\sigma_i)^2 = e$;

Relation 2: for all i, $(\sigma_i\sigma_{i+1})^3 = e$; $(\sigma_i\sigma_{i+1} = (i, i+2, i+1), (\sigma_i\sigma_{i+1})^{-1} = (i, i+1, i+2))$;

Relation 3: for all i, j, $|i-j| \geq 2$, $(\sigma_i\sigma_j)^2 = e$;

So that if $P = \langle \sigma_i^2 \mid 0 \leq i \leq n-1 \rangle$, $Q = \langle (\sigma_i\sigma_j)^2, i < j, | j-i |> i \rangle$ and $R = \langle (\sigma_i\sigma_{i+1})^3 \mid 0 \leq i \leq n-1 \rangle$, taking M as a finite group such that $M = P \cup Q \cup R$, then any finite group $G_n = \langle X \mid M \rangle$, $X = \{\sigma_i \mid 1 \leq i \leq n\}$ is isomorphic to S_n.

From the methods presented above, the presentations for S_{m+n}, S_{2n} and S_{mn} with less relations are obtained in the next Section.

4 Results and Discussion

Following the methodology above, we present in this section the key idea for obtaining short presentations for finite groups S_{m+n} and S_{mn} for all $m,n \in Z^+$ from the presentations of S_m and S_n. When $m = n$, we avoid repetition and this enable us to efficiently construct a shorter presentation for the group S_{2m} from the presentation of S_n. Hence, an inductive process for obtaining a presentation for finite groups is achieved.

Theorem 4.1: Let $S_m = \{X|M\}$ and $S_n = \{Y|N\}$ be presentations for S_m and S_n with generating sets X and Y respectively, where M and N denote the set of relations for S_m and S_n. Let τ, δ be transpositions and ϕ, φ be rotations through $\frac{2\pi}{k}$ rad such that $\tau, \phi \in X$ and $\delta, \varphi \in Y$. Then the presentation for S_r where $r = m+n$, is given by

$$\{X, Y, \nu, \omega | M, N, \nu^2, (\tau\nu)^3, (\delta\nu)^3, [\tau, \varphi], [\tau, \delta], [\phi, \delta], [\phi, \varphi], [\phi\tau\phi^{-1}, \nu], [\nu, \varphi^{-1}\delta\varphi], \omega^{-1}\varphi\nu\phi\}$$

where ω represent the $m+n$ – cycle (1, 2, …, $m+n$) and ν represent a transposition $(i, i+1)$. The given presentation has $|X| + |Y| + 2$ generators and $|M| + |N| + 10$ relations.

Proof: Suppose G is the group described by the given presentation. Define $\tau,\delta,\phi,\varphi \in$ G by

$$\tau = \omega\nu\omega^{-1},\ \phi = (\omega\nu)^i\omega^{-i},\ \delta = \omega^{-1}\nu\omega,\ \varphi = \omega^{-j}(\nu\omega)^j$$

for all i = 1, 2, …, m-1 and j = 1, 2, …, n-1. Then the presentation is transformed into a 2-generator presentation in terms of ν and ω subject to at most $|M| + |N| + 10$ relations. Now, defined a homomorphism $\xi : G \to S_r$ from G to S_r where $r = m + n$ and

$$\xi(\tau) = (i, i+1) \text{ for all transpositions } \tau \in G;\ \ \xi(\phi) = (1,2,..., m);$$
$$\xi(\varphi) = (m+1, m+2,..., m+n);\ \ \xi(\omega) = (1,2,..., m, m+1,..., m+n).$$

Then the permutations satisfy the above relations in G. Again, if we let $\nu_{m-1} = \tau$, $\nu_{m+i-1} = \phi^i\tau\phi^{-i}$, $\nu_{m+1} = \delta$, $\nu_{m+j+1} = \varphi^{-j}\delta\varphi^j$ for all $1 \leq i < m$ and $1 \leq j < m$, then these relations satisfy the Coxeter relations on the group S_r and generate G. Again from the relations 1 to 3 (Section 3), if $\sigma_i\sigma_j = \tau$, then $\sigma_j\sigma_i = \tau^{-1}$. Thus,

$$[\phi\tau\phi^{-1}, v] = (\phi\tau\phi^{-1})^{-1}v^{-1}\phi\tau\phi^{-1}v = \phi\tau^{-1}\phi^{-1}v^{-1}\phi\tau\phi^{-1}v = \phi\tau\phi^{-1}v\phi\tau\phi^{-1}v$$
$$= (\tau\phi^{-1})^{-1}\phi^{-1}v\phi(\tau\phi^{-1})v$$
$$= \alpha^{-1}v\alpha v \text{ where } \alpha = \tau\phi^{-1} \text{ and}$$
$$[\tau\phi, v] = (\tau\phi)^{-1}v^{-1}\tau\phi v = \phi^{-1}\tau^{-1}v^{-1}\tau\phi v = \phi^{-1}\tau v\tau\phi v$$
$$= (\tau\phi)^{-1}v(\tau\phi)v$$
$$= \beta^{-1}v\beta v \text{ where } \beta = \tau\phi.$$

But if H and K are subgroups of G such that $H = \langle\tau\phi^{-1}\rangle$ and $K = \langle\tau\phi\rangle$, then obviously, $H \cong K$.

Similarly, $[v, \varphi^{-1}\delta\varphi] = \eta v\eta^{-1}v$ where $\eta = \varphi^{-1}\delta$ and $[v, \varphi\delta] = \xi v\xi^{-1}v$ where $\xi = \varphi\delta$ so that if $M = \langle\varphi^{-1}\delta\rangle$ and $N = \langle\varphi\delta\rangle$, then $M \cong N$.

Furthermore, by hypothesis, the subgroup $K = \langle v_i\rangle \cong S_m$ and satisfies the Coxeter relation and similarly the subgroup $L = \langle v_{m+i}\rangle$ is isomorphic to S_n and satisfy the Coxeter relation. Thus, the element v_i for $1 \le i < m+n$ satisfies Coxeter relations and since $(\tau v)^3 = (\delta v)^3 = e$, we have $(v_i v_{i+1})^3 = e$ for all $1 \le i \le m+n-1$. The relation $[v_i, v_j] = (v_i, v_j)^2 = e$ holds for $i<j<m$ from the presentation of S_m and holds for $m<i<j$ from the presentation of S_n and if $i<m<j$, then it follows from the relations $[\tau, \delta] = [\tau, \varphi] = [\phi, \delta] = e$. Similarly, since $\tau\phi$ and $\phi\tau\phi^{-1}$ generate a subgroup K of index m in $\langle\tau, \phi\rangle \cong S_m$ which contain the involutions $v_1, v_2, ..., v_m$ and $\varphi\delta$ and $\varphi^{-1}\delta\varphi$ generate a subgroup L of index n in $\langle\delta, \varphi\rangle \cong S_n$ which contain the involutions $v_{m+1}, v_{m+2}, ..., v_{m+n}$, the relation $[\tau\phi, v] = [\phi\tau\phi^{-1}, v] = [v, \varphi\delta] = [v, \varphi^{-1}\delta\varphi] = e$ implies that the element v centralizes $\langle v_1, v_2, ..., v_m\rangle$ and $\langle v_{m+1}, v_{m+2}, ..., v_{m+n}\rangle$ so that $[v_i, v_m] = e$ and $[v_m, v_j] = e$ for all $1 \le i \le m$ and $m \le j \le m+n$.

Hence, the involution $v_1, v_2, ..., v_m, v_{m+1}, ..., v_{m+n}$ generates a subgroup that satisfies the Coxeter relations for S_r. But the relations in S_m (and S_n) implies that each of its elements can be expressed as a word in $v_1, v_2, ..., v_m, v_{m+1}, ..., v_{m+n}$ and the relation $\omega^{-1}\varphi v\phi = e$ imposed the same condition for ω. Thus the same involution generates G. Hence, $G \cong S_r$ and the result follows.

Next, we consider the case $m = n$ such that $S_r = S_{m+n} = S_{2m}$.

Corollary 4.2: Let $S_m = \{X|M\}$ be a presentation for S_m, $m \ge 3$ and let $\tau, \alpha \in X$ such that $\tau = (i, i+1)$ and $\alpha = (1, 2, \ldots, m)$. Then

$$\{X, \omega \mid M, \omega^r, (\tau\omega)^{r-1}, [\tau, \omega\alpha], [\alpha^i\tau\alpha^{-i}, \omega\alpha], [\tau, \omega^m]^2\}$$

is the representation for S_r where $r = 2m$, $1 \le i \le m$ and a generating set that includes $\omega = (1, 2, ..., r)$, $|X| + 1$ generators and $|M| + 5$ relations.

Proof: This follows directly from Theorem 4.2.1 above with $m = n$ and the fact that if $\lambda = \tau w^{-1}$ and $\pi = \tau w$, then

$[\tau, w] = \tau^{-1}w^{-1}\tau w = \tau w^{-1}\tau w = \lambda\pi$, $[\tau, w]^2 = (\tau^{-1}w^{-1}\tau w)(\tau^{-1}w^{-1}\tau w) = \tau w^{-1}\tau w\tau w^{-1}\tau w = (\lambda\pi)^2$ and so on, for all w^i.

The next result is derived from Cartesian product of two groups such that given two groups H and K, then the product HK is given by the set

$$HK = \{x = hk : h \in H, k \in K\}.$$

Theorem 4.3: Let $S_m = \{X|M\}$ and $S_n = \{Y|N\}$ be presentations for the groups S_m and S_n, $m,n \geq 3$ with generating sets X and Y respectively, where M and N denote the set of relations for S_m and S_n. Let τ, δ be transpositions and ϕ, φ be rotations through $\dfrac{2\pi}{k}$ rad such that $\tau, \phi \in X$ and $\delta, \varphi \in Y$. Then the presentation for S_{mn} is given by

$$\{X, Y, v, \omega | M, N, v^2, \delta^{-1}(\phi v \varphi^{-1}\phi^{-1}\varphi)^i, \omega^{-1}\phi\omega\tau\phi^{-1}v, (\phi^2\tau\phi^{-2}v)^3, (v\varphi^{-1}\tau\varphi)^3, [\tau, v], [\phi^2\tau\phi^{-2}, v],$$
$$[\tau, \phi\omega^{-1}], \omega v \omega^{-1}\phi^2\tau\phi^{-2}, \omega^{-1}v\omega\varphi^{-1}\tau\varphi, [\tau, \varphi^{-1}\tau\varphi], [\tau, \varphi^{-1}\phi\varphi], [\phi, \varphi^{-1}\tau\varphi], [\phi, \varphi^{-1}\phi\varphi][\tau, \varphi\delta],$$
$$[\tau, \varphi^{-1}\delta\varphi], [\phi, \varphi\delta], [\phi, \varphi^{-1}\delta\varphi]\}$$

where ω represent the mn – cycle (1, 2, …, mn) and v represents a transposition $(i, i+1)$. The given presentation has $|X| + |Y| + 2$ generators and $|M| + |N| + 18$ relations.

Proof: Supposed G is the finite group defined by the given presentation, define a function $\xi : G \to S_{mn}$ from G to S_r such that $\tau \mapsto (i, i+1)$, $\phi \mapsto (1, 2, ..., m)$, $\delta \mapsto (j, j+1)$, $\varphi \mapsto (1, 1+m, ..., 1+(n-1)m)(2, 2+m, ..., 2+(n-1)m)...(m, 2m, ..., nm)$, $v \mapsto (k, k+1)$, and $\omega \mapsto (1, 2, ..., m, m+1, m+2, ..., 2m, 2m+1, ..., mn)$.

Then ξ is a homomorphism and for some m, n, $w^m = \varphi$ and $w^n = \phi$. In particular, $\xi(\tau)$ and $\xi(\phi)$ generate a subgroup H of S_m such that $H \cong S_m$ and the conjugate of H defined by the multiples of φ generate the direct product of n-copies of S_m. Now, let $v_1 = \tau$, $v_{i+1} = \phi^{-i}\tau\phi^i$, $\lambda_{i+1} = \omega^{-i}\tau\omega^i$ and $v_{im+j} = \varphi^{-i}v_j\varphi^i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ in G. Then we shall show that the $(mn - 1)$ elements $v_1, v_2, ..., v_q$, $q = mn - 1$ satisfies the Coxeter relations on S_{mn} and also generate the group G. To see this, note that $\phi v \varphi^{-1}\phi^{-1}\varphi = \phi v(\phi\varphi)^{-1}\varphi = \phi v \sigma^{-1}\varphi = \phi\xi\varphi$ where $\sigma = \phi\varphi$ is an mn – cycle and $\xi = v\sigma^{-1}$ is an $(mn - 1)$ – cycle, $(\phi v \varphi^{-1}\phi^{-1}\varphi)^2 = (\phi\xi\varphi)^2 = \phi\xi\mu\xi\varphi$ where $\mu = \varphi\phi$ is an mn – cycle. Thus, both the product $w^{-1}\varphi\phi(\varphi v \phi)^2 = w^{-1}\varphi\sigma v \sigma v \phi = w^{-1}\varphi(\xi^{-1})^2\phi = w^{-1}\varphi\pi\phi$, $\pi = (\xi^{-1})^2$ is an $(mn - 1)$ – cycle, and $w^{-1}vw\varphi^{-1}\tau\varphi = w^{-1}\lambda_i\varphi^{-1}\lambda_j$ are mn – cycles.

Now, by the hypothesis on S_m, the elements $v_1, v_2, ..., v_m$ and $v_{im+1}, v_{im+2}, ..., v_{im+q}$ respectively generate the subgroups H and K of S_m such that $H \cong S_m$ and $K \cong S_m$ for $1 \leq i < n$. Again, the commutator relations

$$[\tau, \varphi^{-1}\tau\varphi] = [\tau, \varphi^{-1}\phi\varphi] = [\phi, \varphi^{-1}\tau\varphi] = [\phi, \varphi^{-1}\phi\varphi] = e$$

describe the subgroup H as $H = \langle v_1, v_2, ..., v_m \rangle = \langle \tau, \phi \rangle$ which commute with its conjugate $H_1 = \langle v_{m+1}, v_{m+2}, ..., v_{2m} \rangle$ under φ. The relations

$$[\tau, \varphi\delta] = [\tau, \varphi^{-1}\delta\varphi] = [\phi, \varphi\delta] = [\phi, \varphi^{-1}\delta\varphi] = e$$

implies that the subgroup H is centralized by the set $N_1 = \langle \varphi\delta, \varphi^{-1}\delta\varphi \rangle$ such that

$$[\langle \delta, \varphi \rangle : \langle \varphi\delta, \varphi^{-1}\delta\varphi \rangle] = n \text{ and } N_2 = \langle \delta, \varphi \rangle \cong S_n .$$

Hence, if $N_i = \{H \subseteq S_m$: H is a subgroup$\}$, then N_2 permutes all the subgroups N_i by conjugation which follows from the natural action of the group S_n on the index set $\{1,2,\ldots,n\}$.

Next, the transposition v satisfy the relation $(v_i)^2 = e$ for all i and the relations $(\phi^2\tau\phi^{-2}v)^3 = (v\varphi\tau\varphi^{-1}v)^3 = e$ implies that $(v_i v_{i+1})^3 = e$ for all $1 \le i < m-1$ and then conjugation by multiples of φ gives all the remaining relations. Again, to see that $[v_i, v_j] = (v_i v_j)^2 = e$ for $1 \le i \le j \le mn$, we first consider the presentation for S_m. If $i < j < m$, then the result follows directly from the presentation for S_m and also conjugation by φ^i gives the same result for $km < i < j < (k+1)m$ for some $k \in Z^+$. Also, $[v_i, v_j] = e$ is true if both v_i and v_j lie in different conjugate sets of the subgroup H since the conjugates commutes with each other. And the relations $[\tau, v] = [\phi\tau\phi^{-1}, v] = e$ ensure that v_i commute with all the elements in $\langle \tau, \phi\tau\phi^{-1} \rangle = \langle v_1, v_2, ..., v_m \rangle$. The rest of the relations will follow if $[\tau, \phi\omega^{-1}] = \omega^{-1}\phi\omega\tau\phi^{-1}v = \omega v\omega^{-1}\phi^2\tau\phi^{-2} = \omega^{-1}v\omega\varphi^{-1}\tau\varphi = e$ gives the conjugation of ω on each successive pair of the elements in $\{v_1, v_2, ..., v_{mn}\}$. Thus, since $\phi\omega^{-1}$ centralizes τ, we have

$$\omega v_1 \omega^{-1} = \omega\tau\omega^{-1} = \phi\tau\phi^{-1} = v_2$$

and we find by induction on i, for $1 \le i \le m-2$, that

$$\begin{aligned} \omega v_{i+1}\omega^{-1} &= \omega\phi v_i \phi^{-1}\omega^{-1} = (\omega\phi\omega^{-1})^{-1}(\omega v_i \omega^{-1})(\omega\phi\omega^{-1}) \\ &= (\tau\phi^{-1}v)v_{i+1}(v\phi\tau) = \tau\phi^{-1}v_{i+1}\phi\tau^{-1} = \tau v_{i+2}\tau^{-1} = v_{i+2} \end{aligned}$$

since v commute with each v_i, $\tau = v_1$ commute with v_{i+2}. Also,

$$\omega v_{m-1}\omega^{-1} = \omega\phi^{-2}\tau\phi^2\omega^{-1} = v_m \text{ and } \omega v_m \omega^{-1} = \omega v\omega^{-1} = \varphi\tau\varphi^{-1} = \varphi v_1 \varphi^{-1} = v_{m+1}$$

and since ω centralizes $\varphi = \omega^m$, we find that

$$\omega v_{im+j}\omega^{-1} = \omega\varphi^i v_j \varphi^{-i}\omega^{-1} = \varphi^i \omega v_j \omega^{-1}\varphi^{-i} = \varphi^i v_{j+1}\varphi^{-i} = v_{im+j+1}$$

for $1 \le i \le n$ and $1 \le j \le m$.

Again, conjugation by powers of ω satisfy all the relations of the form $[v_i, v_j] = e$ for $1 \le i < j \le mn$. Thus, the $mn-1$ involutions $\{v_1, v_2, ..., v_{mn-1}\}$ generate subgroups that satisfy the usual Coxeter relations for S_{mn}.

Finally, it can be shown that each generator of G can be expressed as a word in v_i by first considering the relations in S_m. Obviously, the set M satisfy this condition for each element in the set X. In particular, $\tau = v_1$ and

$$\phi = (\phi^{m-2}\tau\phi^{-(m-2)})(\phi^{m-3}\tau\phi^{-(m-3)})...(\phi\tau\phi^{-1})\tau = v_{m-1}v_{m-2}...v_1$$

which follows that

$$\varphi^i\phi\varphi^{-i} = v_{im+m-1}v_{im+m-2}...v_{im+2}v_{im+1}$$

for $1 \le i < n$ and similarly, $v = v_m$ and $\varphi^i v\varphi^{-i} = v_{(i+1)m}$ for $1 \le i \le n-2$. Hence, we deduced from $\omega = \varphi\phi(\varphi v\phi)^{n-1}$ and $\varphi^i\phi\varphi^{-i} = v_{im+m-1}v_{im+m-2}...v_{im+2}v_{im+1}$ that

$$\omega = (v_{mn-1}...v_{m(n-1)+1})v_{m(n-1)}...(v_{2m-1}...v_{m+2}v_{m+1})v_m(v_{m-1}...v_2v_1) = v_{mn-1}v_{mn-2}...v_2v_1$$

and from the relations $\varphi = \omega^m$ and $\delta = (\varphi\phi^{-1}\varphi^{-1}v\phi^{-1})^m$, it shows that both elements δ and φ can be expressed as words in the set v_i. Hence, the involutions $\{v_1, v_2, ..., v_{mn}\}$ generate G so that $G \cong S_{mn}$ and the result follows.

5 Conclusion

This work presented some new families of group presentations by generators and relations. The result gives shorter presentations for the finite groups S_{m+n}, S_{2n} and S_{mn} with |M| + |N| + 10 relations, |M| + 5 relations and |M| + |N| + 18 relations respectively (Theorem 4.1, Corollary 4.2 and Theorem 4.3). For demonstration purpose, with $n \ge 3$, $X = \{\sigma, \tau\}$, $Y = \{\sigma, \tau, \omega\}$ and $Z = \{\sigma, \tau, \omega, \lambda\}$, we have:

$$S_3 \cong \langle X : \sigma^2 = \tau^3 = (\sigma\tau)^2 = e\rangle;$$
$$S_4 \cong \langle X : \sigma^2 = \tau^4 = (\sigma\tau)^3 = e\rangle;$$
$$S_5 \cong \langle Y : \sigma^2 = \tau^5 = \omega^3 = e, (\sigma\omega)^2 = \tau^{-1}\sigma\tau^2\sigma\omega = e\rangle;$$
$$S_6 \cong \langle Z : \sigma^2 = \omega^5 = e, \tau\omega^2\lambda^{-1} = \tau\lambda^{-1}\tau^{-1}\omega\lambda = \sigma\lambda\tau^{-1}\omega^{-1}\lambda = e\rangle;$$
$$S_7 \cong \langle Z : \sigma^2 = e, \tau^{-2}\omega\lambda^2 = \tau^{-1}\omega\lambda^{-1}\omega\lambda = \sigma\tau^{-1}\omega^{-1}\lambda\omega\tau^{-1} = (\sigma\tau^{-1}\lambda^{-1})^2 = e\rangle; \text{ and}$$
$$S_8 \cong \langle Z : \sigma^2 = \omega^2 = (\sigma\omega)^2 = e, \lambda^{-1}\tau^{-1}\sigma\tau\lambda\sigma = \lambda^{-1}\omega\sigma\tau^{-1}\lambda^{-1}\omega = \tau^4\lambda^{-1}\omega\lambda = e\rangle;$$

As group representation theory shows that new representations can be constructed from direct product or tensor product of two or more representations, this work clearly presents a shorter and simpler method for building representations for the finite groups S_{m+n}, S_{2n} and S_{mn} from the representations of S_m and S_n with less number of relations than the existing literature.

Competing Interests

Authors have declared that no competing interests exist.

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