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On the Simultaneous Equations $x - yz = 3w^2, xy = T^3$

Sharadha Kumar, M. A. Gopalan

Abstract*: *The system of double equations given by $x - yz = 3w^2, xy = T^3$ is studied for obtaining its non-zero distinct solutions in integers.

Keywords*: *Double equations, Integer solutions, Pair of equations with 5 unknowns.

I. INTRODUCTION

Systems of indeterminate quadratic equations of the form $ax + c = u^2$, $bx + d = v^2$ where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors [1, 2] and with a few possible exceptions, most of the them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions is a general form. In [3], a general form of the integral solutions to the system of equations $ax + c = u^2$, $bx + d = v^2$ where a, b, c, d are non-zero distinct constants is presented when the product ab is a square free integer whereas the product cd may or may not a square integer. For other forms of system of double diophantine equations, one may refer [4-25].

This communication concerns with yet another interesting system of double Diophantine equations namely $x - yz = 3w^2,\ xy = T^3$ for its infinitely many non-zero distinct integer solutions.

II. METHOD OF ANALYSIS

Consider the pair of equations

$$x - yz = 3w^2 \tag{1}$$

$$xy = T^3 \tag{2}$$

The elimination of y between (1) and (2) gives

$$x^2 - 3w^2 x - zT^3 = 0 \tag{3}$$

Treating (3) as a quadratic in x and solving for x, we have

$$x = \frac{1}{2}\left[3w^2 \pm \sqrt{9w^4 + 4zT^3}\right] \tag{4}$$

The square root on the R.H.S of (4) is eliminated when

(i). $w = T = (k+3)\alpha$, $z = k(k+3)^2\alpha$ (5)

(ii). $w = T = k\alpha$, $z = k^2(k+3)\alpha$ (6)

Now, taking (i)., the corresponding values of x and y are given by

$$x = (k+3)^3\alpha^2 \ , \ y = \alpha \tag{7}$$

and for (ii)

$$x = -k^3\alpha^2 \ , \ y = -\alpha$$

(8)

Note that, the pairs (5), (7) and (6), (8) satisfy (1) and (2) respectively

However, there are other choices of integer solutions to (1) and (2) and they are illustrated as below:

Consider the transformations

$$x = y^2, T = y \tag{9}$$

Note that (2) is satisfied automatically, Substituting (9) in (1), we have

$$y^2 - yz = 3w^2 \tag{10}$$

which is a quadratic in y and solving for y, we have

$$y = \frac{1}{2}\left(z \pm \sqrt{z^2 + 12w^2}\right) \tag{11}$$

which is satisfied by

$$w = 2rs, z = 12r^2 - s^2 \text{ and } y = 12r^2, -s^2 \tag{12}$$

In view of (9), one obtains

$$x = 144r^4, s^4 \text{and } T = 12r^2, -s^2 \tag{13}$$

Note that, (12) and (13) exhibits two sets of integer solutions to (1) and (2)

Also, to eliminate the square root on the R.H.S of (11), assume

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* Correspondence Author

Sharadha Kumar*, Research Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. Email:sharadhak12@gmail.com

M.A.Gopalan, Professor, Department of Mathematics, Shrimathi Indira Gandhi College, Trichy-620002, Tamil Nadu, India. Email:mayilgopalan@gmail.com

$$\alpha^2 = z^2 + 12w^2 \quad (14)$$

which is represented as the system of double equations as shown below in Table1.

Table 1: System of double equations

System	1	2	3	4	5
$\alpha + z$	w^2	$6w$	$12w$	$4w$	$6w^2$
$\alpha - z$	12	$2w$	w	$3w$	2

Solving each of the above systems in Table 1 and performing some algebra, the values of x, y, z, w and T satisfying (1) and (2) are presented below in Table 2.

Table 2: Solutions

System	x	y	z	w	T
1	$4k^4, 36$	$2k^2, -6$	$2k^2 - 6$	$2k$	$2k^2, -6$
2	$9w^2, w^2$	$3w, -w$	$2w$	w	$3w, -w$
3	$144k^2, k^2$	$12k, -k$	$11k$	$2k$	$12k, -k$
4	$16k^2, 9k^2$	$4k, -3k$	k	$2k$	$4k, -3k$
5	$9w^4, 1$	$3w^2, -1$	$3w^2 - 1$	w	$3w^2, -1$

It is to be noted that, one may also write (10) as the system of double equations as in Table 3 below:

Table3: System of double equations

System	1	2	3	4	5	6
y	1	3	w	w^2	$3w$	$3w^2$
$y - z$	$3w^2$	w^2	$3w$	3	w	1

In this case, the corresponding values of x, y, z, w and T are given by the quintuples

$$(x, y, z, w, T) = (1,1,1-3w^2, w,1), (9,3,3-w^2, w,3), (w^2, w, -2w, w, w), (w^4, w^2, w^2-3, w, w^2), (9w^2, 3w, 2w, w, 3w), (9w^4, 3w^2, 3w^2-1, w, 3w^2)$$

Further, write (14) as

$$z^2 + 12w^2 = \alpha^2 = \alpha^2 * 1 \quad (15)$$

Assume

$$\alpha = 4a^2 + 12b^2 \quad (16)$$

Write 1 as

$$1 = \frac{(2 + i\sqrt{12})(2 - i\sqrt{12})}{16} \quad (17)$$

Substituting (16) and (17) in (15) and employing the method of factorization, define

$$z + i\sqrt{12}w = (2a + i\sqrt{12}b)^2 \frac{(2 + i\sqrt{12})}{4}$$

On equating the real and imaginary parts, we have

$$z = 2a^2 - 6b^2 - 12ab, w = a^2 - 3b^2 + 2ab$$

(18)

In view of (11) and (9) we have

$$\left.\begin{array}{l} T = y = 3a^2 + 3b^2 - 6ab, -a^2 - 9b^2 - 6ab \\ x = 9(a-b)^4, (a+3b)^4 \end{array}\right\}$$

(19)

Thus (18) and (19) represent the solutions to (1) and (2).

It is worth mentioning that, in addition to (16), (17), α and 1 may also be written as

$$\alpha = 49(a^2 + 12b^2), 1 = \frac{(1 + i2\sqrt{12})(1 - i2\sqrt{12})}{49}$$

For this choice, the solutions of (1) and (2) are given by

$$x = 784(a - 3b)^4, 441(a + 4b)^4$$

$$T = y = 28(a - 3b)^2, -21(a + 4b)^2$$

$$z = 7(a^2 - 12b^2 - 48ab)$$

$$w = 14(a^2 - 12b^2 + ab)$$

III. CONCLUSION

In this paper, an attempt has been made to obtain many integer solutions to the pair of equations $x - yz = 3w^2, xy = T^3$. The authors wish that the researchers of diophantine equations maybe motivated in solving other choices of double diophantine equations.

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AUTHORS PROFILE

Sharadha Kumar, is M.Phil scholar in Department of Mathematics at Shrimati Indira Gandhi College, Tiruchirappalli. Her area of interest is solving Diophantine equations in various disciplines. She has published 8 papers in National and International journals. She has presented papers in 2 National Conferences and International Conferences.

Dr. M.A. Gopalan, is currently Professor of Mathematics at Shrimati Indira Gandhi College, Tiruchirappalli and has taught Mathematics for nearly three decades. He is interested in problem solving in the area of Diophantine equations and Number patterns. He has published more than 500 papers in National and International journals. He, along with his colleagues, has published 13 books in the area of Diophantine equations and Number patterns. He serves on the editorial boards of IJPMS and IJAR and a life member of Kerala Mathematics Association.