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Stability of a $k-$ Cubic Functional Equation in Quasi - β Normed Spaces: Direct and Fixed Point Methods

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Abstract

In this paper, we introduce and investigate the Hyers - Ulam stability of a $k-$ cubic functional equation of the form

$$kf(x+ky)-f(kx+y)=\frac{k(k^2-1)}{2}\left[f(x+y)+f(x-y)\right]+(k^4-1)f(y)-2k(k^2-1)f(x),$$

for $k\geq 2$ in quasi - β normed spaces using both direct and fixed point methods.

Keywords: Cubic functional equations; Generalized Ulam - Hyers stability; quasi - βnormed spaces; fixed point method.

2010 Mathematics Subject Classification: 39B52; 32B72; 32B82; 47H10

1 Introduction

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

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The first stability problem was raised by [1] during his talk at the University of Wisconsin in 1940. We are given a group $(G_1, \cdot)$ and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

[2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. It was further generalized and excellent results were obtained by a number of authors ([3] - [30]).

The solution and stability of the following cubic functional equations

$$C(x + 2y) + 3C(x) = 3C(x + y) + C(x - y) + 6C(y), \tag{1.1}$$

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.2}$$

$$\begin{aligned} &f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) \\ &\qquad = 2[f(x + y) + 2f(x + z) + 2f(y + z) + 2f(x - z) + 2f(y - z)], \end{aligned} \tag{1.3}$$

$$3f(x + 3y) - f(3x + y) = 12[f(x + y) + f(x - y)] + 80f(y) - 48f(x), \tag{1.4}$$

$$g(2x - y) + g(x - 2y) = 6g(x - y) + 3g(x) - 3g(y), \tag{1.5}$$

$$\begin{aligned} &f\left(ax_1 + b\sum_{i=2}^{n} x_i\right) + f\left(ax_1 - b\sum_{i=2}^{n} x_i\right) + 2a(b^2 - a^2)f(x_1) \\ &\qquad = ab^2\left[f\left(\sum_{i=1}^{n} x_i\right) + f\left(x_1 - \sum_{i=2}^{n} x_i\right)\right], \end{aligned} \tag{1.6}$$

$$\begin{aligned} &f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} f(2x_j) \\ &\qquad = 2f\left(\sum_{j=1}^{n-1} 2x_j\right) + 4\sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right) \end{aligned} \tag{1.7}$$

were investigated by [31, 32, 33, 34, 35, 36, 37, 38].

In this paper, we introduce and investigate the Hyers - Ulam stability of a $k-$ cubic functional equation of the form

$$\begin{aligned} kf(x + ky) - f(kx + y) = &\frac{k(k^2 - 1)}{2}\left[f(x + y) + f(x - y)\right] \\ &+ (k^4 - 1)f(y) - 2k(k^2 - 1)f(x) \end{aligned} \tag{1.8}$$

where $k \geq 2$, in quasi - β normed spaces, by employing direct and fixed point methods.

2 Solution of a $k-$ Cubic Functional Equation

In this section, the general solution of the functional equation (1.8) is given. Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are vector spaces.

Lemma 2.1. *If a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies the functional equation (1.8), then the following properties hold*

(i) $f(0) = 0$,

(ii) $f(kx) = k^3 f(x)$*, for all* $x \in \mathcal{A}$*.*

(iii) $f(-x) = -f(x)$*, for all* $x \in \mathcal{A}$*; that is,* f *is an odd function.*

Proof. Letting (x, y) by $(0, 0)$ in (1.8), we obtain

$$(k-1)f(0) = \left[\frac{2k(k^2-1)}{2} + (k^4-1) - 2k(k^2-1)\right] f(0)$$

and so

$$k^3(k-1)f(0) = 0.$$

Since $k \neq 0, 1$, we find (i).

Replacing (x, y) by $(x, 0)$ in (1.8), we obtain

$$kf(x) - f(kx) = k(k^2-1)f(x) - 2k(k^2-1)f(x), \quad or$$

$$f(kx) = \left[k - k(k^2-1) + 2k(k^2-1)\right] f(x) = k^3 f(x),$$

for all $x \in \mathcal{A}$. Thus, (ii) holds.

Setting x by 0 in (1.8), we get

$$kf(ky) - f(y) = \frac{k(k^2-1)}{2}\left[f(y) + f(-y)\right] + (k^4-1)f(y), \quad or$$

$$k(k^2-1)\left[f(y) + f(-y)\right] = 0,$$

for all $x \in \mathcal{A}$. Finally, (iii) holds, since $k \neq 0, \pm 1$. Thus f is an odd function. Hence the proof is complete □

3 Preliminary Results on Quasi-β Normed Spaces

In this section, we present some preliminary results associated to quasi-β-normed spaces. Let us fix a real number β with $0 < \beta \leq 1$ and denote $\mathbb{K}$ either for $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.1. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\| \cdot \|$ is a real-valued function on X satisfying the following properties:

$(QB1)$ $\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.

$(QB2)$ $\| \lambda x \| = | \lambda |^\beta . \| x \|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

$(QB3)$ There is a constant $K \geq 1$ such that $\| x + y \| \leq K (\| x \| + \| y \|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called quasi-β-normed space if $\| \cdot \|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$.

Definition 3.2. A quasi-β-Banach space is a complete quasi-β-normed space.

Definition 3.3. A quasi-β-norm $\| \cdot \|$ is called a (β, p)-norm $(0 < p \leq 1)$ if

$$\| x + y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

4 Stability Results: Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.8) using the direct method of D.H. Hyers [2].

Throughout this section, let us take $\mathcal{U}$ is a linear space over $\mathbb{K}$ and $\mathcal{V}$ is a (β, p) Banach space with $p-$norm $\|.\|_{\mathcal{V}}$. Let K be the modulus of concavity of $\|.\|_{\mathcal{V}}$. Define a mapping $D_k f : \mathcal{U} \to \mathcal{V}$, by

$$D_k f(x,y) = kf(x+ky) - f(kx+y) - \frac{k(k^2-1)}{2}\left[f(x+y)+f(x-y)\right] - (k^4-1)f(y) + 2k(k^2-1)f(x)$$

for all $x, y \in \mathcal{U}$. Also, hereafter throughout this paper, we use the following notation

$$\zeta(x,y) = \zeta_y^x$$

for all $x, y \in \mathcal{U}$.

Theorem 4.1. *Let $j = \pm 1$. Let $D_k f : \mathcal{U} \to \mathcal{V}$ be a mapping for which there exists a function $\zeta : \mathcal{U}^2 \to [0, \infty)$ with the condition*

$$\lim_{i\to\infty} \frac{1}{k^{3ij}} \zeta_{k^{ij}y}^{k^{ij}x} = 0 \tag{4.1}$$

such that the functional inequality

$$\|D_k f(x,y)\|_{\mathcal{V}} \le \zeta_y^x \tag{4.2}$$

for all $x, y \in \mathcal{U}$. Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ satisfying the functional equation (1.8) and

$$\|f(x) - \mathcal{C}(x)\|_{\mathcal{V}}^p \le \left(\frac{K^{(n-1)}}{k^{3\beta}} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\zeta_0^{k^{ij}x}}{k^{3ij}} \right)^p \tag{4.3}$$

where the mapping $\mathcal{C}(x)$ is defined by

$$\mathcal{C}(x) = \lim_{n\to\infty} \frac{f(k^{nj}x)}{k^{3nj}} \tag{4.4}$$

for all $x \in \mathcal{U}$.

Proof. **Case (i):** Assume $j = 1$.

Replacing (x, y) by $(x, 0)$ in (4.2), we get

$$\left\|f(kx) - k^3 f(x)\right\|_{\mathcal{V}} \le \zeta_0^x \tag{4.5}$$

for all $x \in \mathcal{U}$. Using property $(QB2)$ in (4.5), we obtain

$$\left\| \frac{f(kx)}{k^3} - f(x) \right\|_{\mathcal{V}} \le \frac{\zeta_0^x}{k^{3\beta}} \tag{4.6}$$

for all $x \in \mathcal{U}$. Now replacing x by kx and dividing by k^3 in (4.6), we have

$$\left\| \frac{f(k^2x)}{k^6} - \frac{f(kx)}{k^3} \right\|_{\mathcal{V}} \le \frac{\zeta_0^{kx}}{k^{3\beta}\cdot k^3} \tag{4.7}$$

for all $x \in \mathcal{U}$. From (4.6) and (4.7), we obtain

$$\left\| \frac{f(k^2x)}{k^6} - f(x) \right\|_{\mathcal{V}} \le K\left(\left\| \frac{f(k^2x)}{k^6} - \frac{f(kx)}{k^3} \right\|_{\mathcal{V}} + \left\| \frac{f(kx)}{k^3} - f(x) \right\|_{\mathcal{V}} \right) \le \frac{K}{k^{3\beta}}\left[\zeta_0^x + \frac{\zeta_0^{kx}}{k^3} \right] \tag{4.8}$$

for all $x \in \mathcal{U}$. Generalizing, for a positive integer n, we reach

$$\left\|\frac{f(k^n x)}{k^{3n}} - f(x)\right\|_{\mathcal{V}} \leq \frac{K^{n-1}}{k^{3\beta}} \sum_{i=0}^{n-1} \frac{\zeta_0^{k^i x}}{k^{3n}} \leq \frac{K^{n-1}}{k^{3\beta}} \sum_{i=0}^{\infty} \frac{\zeta_0^{k^i x}}{k^{3n}} \tag{4.9}$$

for all $x \in \mathcal{U}$. To prove the convergence of the sequence

$$\left\{\frac{f(k^n x)}{k^{3n}}\right\},$$

replacing x by $k^m x$ and dividing by k^{3m} in (4.9), for any $m, n > 0$, we get

$$\begin{aligned}\left\|\frac{f(k^{n+m}x)}{k^{3(n+m)}} - \frac{f(k^m x)}{k^{3m}}\right\|_{\mathcal{V}} &= \frac{1}{k^{3m\beta}} \left\|\frac{f(k^n \cdot k^m x)}{k^{3n}} - f(k^m x)\right\|_{\mathcal{V}} \\ &\leq \frac{K^{n-1}}{k^{3\beta}} \sum_{i=0}^{n-1} \frac{\zeta_0^{k^{i+m}x}}{k^{3(n+m)}} \\ &\leq \frac{K^{n-1}}{k^{3\beta}} \sum_{i=0}^{\infty} \frac{\zeta_0^{k^{(i+m)}x}}{k^{3(n+m)}} \\ &\to 0 \quad as \ m \to \infty\end{aligned}$$

for all $x \in \mathcal{U}$. Thus it follows that a sequence $\left\{\frac{f(k^n x)}{k^{3n}}\right\}$ is a Cauchy in $\mathcal{V}$ and so it converges. Therefore, we see that a mapping $\mathcal{C}(x) : \mathcal{U} \to \mathcal{V}$ defined by

$$\mathcal{C}(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}$$

is well defined for all $x \in \mathcal{U}$. In order to show that $\mathcal{C}$ satisfies (1.8), replacing (x, y) by $(k^n x, k^n y)$ and dividing by k^{3n} in (4.2), we have

$$\|\mathcal{C}(x, y)\|_{\mathcal{V}}^p = \lim_{n\to\infty} \frac{1}{k^{3np}} \|D_k f(k^n x, k^n y)\|_{\mathcal{V}}^p \leq \lim_{n\to\infty} \frac{1}{k^{3np}} \left(\zeta_{k^n y}^{k^n x}\right)^p = 0$$

for all $x, y \in \mathcal{U}$ and so the mapping $\mathcal{C}$ is cubic. Taking the limit as n approaches to infinity in (4.9), we find that the mapping $\mathcal{C}$ is a cubic mapping satisfying the inequality (4.3) near the approximate mapping $f : \mathcal{U} \to \mathcal{V}$ of equation (1.8). Hence, $\mathcal{C}$ satisfies (1.8), for all $x, y \in \mathcal{U}$.

To prove that $\mathcal{C}$ is unique, we assume now that there is $\mathcal{C}'$ as another cubic mapping satisfying (1.8) and the inequality (4.3). Then it follows easily that

$$\mathcal{C}(k^n x) = k^{3n}\mathcal{C}(x), \qquad \mathcal{C}'(k^n x) = k^{3n}\mathcal{C}'(x)$$

for all $x \in \mathcal{U}$ and all $n \in \mathbb{N}$. Thus

$$\begin{aligned}\left\|\mathcal{C}(x) - \mathcal{C}'(x)\right\|_{\mathcal{V}}^p &= \frac{1}{k^{3np}} \left\|\mathcal{C}(k^n x) - \mathcal{C}'(k^n x)\right\|_{\mathcal{V}}^p \\ &\leq \frac{K^p}{k^{3np}} \left\{\|\mathcal{C}(k^n x) - f(k^n x)\|_{\mathcal{V}}^p + \left\|f(k^n x) - \mathcal{C}'(k^n x)\right\|_{\mathcal{V}}^p\right\} \\ &\leq \left(\frac{2K^n}{k^{3\beta}} \sum_{i=0}^{\infty} \frac{\zeta_0^{k^{(i+n)}x}}{k^{3(i+n)}}\right)^p\end{aligned}$$

for all $x \in \mathcal{U}$. Therefore, as $n \to \infty$, in the above inequality, one establishes

$$\mathcal{C}(x) - \mathcal{C}'(x) = 0$$

for all $x \in \mathcal{U}$, completing the proof of the claimed uniqueness of $\mathcal{C}$. Hence the theorem holds for $j = 1$.
Case (ii): Assume $j = -1$.

Now replacing x by $\frac{x}{k}$ in (4.5), we get

$$\left\| f(x) - k^3 f\left(\frac{x}{k}\right) \right\|_{\mathcal{V}} \le \zeta_0^{\frac{x}{k}} \tag{4.10}$$

for all $x \in \mathcal{U}$. The rest of the proof is similar to that of case $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. ☐

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.8).

Corollary 4.2. *Let $D_k f : \mathcal{U} \to \mathcal{V}$ be a mapping. If there exist real numbers λ and s such that*

$$\|D_k f(x,y)\|_{\mathcal{V}} \le \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s\right\}, & s \ne 3; \\ \lambda\left\{||x||^s||y||^s + \left\{||x||^{2s} + ||y||^{2s}\right\}\right\}, & s \ne \frac{3}{2}; \end{cases} \tag{4.11}$$

for all $x, y \in \mathcal{U}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that

$$\|f(x) - \mathcal{C}(x)\|_{\mathcal{V}}^p \le \begin{cases} \left(\dfrac{k^3\lambda K^{(n-1)}}{k^{3\beta}|k^3 - 1|}\right)^p, \\ \left(\dfrac{k^3\lambda K^{(n-1)}||x||^s}{k^{3\beta}|k^3 - k^{\beta s}|}\right)^p, \\ \left(\dfrac{k^3\lambda K^{(n-1)}||x||^{2s}}{k^{3\beta}|k^3 - k^{2\beta s}|}\right)^p \end{cases} \tag{4.12}$$

for all $x \in \mathcal{U}$.

The following example is to illustrate that the functional equation (1.8) is not stable for $s = 3$ in condition (ii) of Corollary 4.2.

Example 4.1. Let $\zeta : \mathbb{K} \to \mathbb{K}$ be a function defined by

$$\zeta(x) = \begin{cases} ax^3, & \text{if } |x| < 1 \\ a, & \text{otherwise} \end{cases}$$

where $a > 0$ is a constant, and define a function $f : \mathbb{K} \to \mathbb{K}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\zeta(k^n x)}{k^{3n}} \qquad for\ all \quad x \in \mathbb{K}.$$

Then $D_k f$ satisfies the functional inequality

$$|D_k f(x,y)| \le \frac{ak^9(3k^3 + k^4 - 2k)}{(k^3 - 1)}\left(|x|^3 + |y|^3\right) \tag{4.13}$$

for all $x, y \in \mathbb{K}$. Then there do not exist a cubic mapping $\mathcal{C} : \mathbb{K} \to \mathbb{K}$ and a constant $b > 0$ such that

$$|f(x) - \mathcal{C}(x)| \le b|x|^3 \qquad for\ all \quad x \in \mathbb{K}. \tag{4.14}$$

Proof. Now
$$|f(x)| \le \sum_{n=0}^{\infty} \frac{|\zeta(k^n x)|}{|k^{3n}|} = \sum_{n=0}^{\infty} \frac{a}{k^{3n}} = \frac{k^3\, a}{k^3-1}.$$
Therefore, f is bounded. Now, let us prove that f satisfies (4.13).

If $x = y = 0$ then (4.13) is trivial. If $|x|^3 + |y|^3 \ge \frac{1}{k^3}$ then the left hand side of (4.13) is less than $\frac{a\, k^3(3k^3+k^4-2k)}{(k^3-1)}$. Now suppose that $0 < |x|^3 + |y|^3 < \frac{1}{k^3}$. Then there exists a positive integer ℓ such that
$$\left(\frac{1}{k^3}\right)^{\ell+2} \le |x|^3 + |y|^3 < \left(\frac{1}{k^3}\right)^{\ell+1}, \tag{4.15}$$
so that $k^{3(\ell-1)}|x|^3 < \frac{1}{k^3}$, $k^{3(\ell-1)}|y|^3 < \frac{1}{k^3}$, and consequently
$$k^{3(\ell-1)}(x+ky),\ \ k^{3(\ell-1)}(kx+y),\ \ k^{3(\ell-1)}(x+y),\ \ k^{3(\ell-1)}(x-y),$$
$$k^{3(\ell-1)}(y), k^{3(\ell-1)}(x) \in \left(-\frac{1}{k}, \frac{1}{k}\right).$$
Therefore for each $n = 0, 1, \ldots, \ell-1$, we have
$$k^{3n}(x+ky), k^{3n}(kx+y), k^{3n}(x+y), k^{3n}(x-y), k^{3n}(y), k^{3n}(x) \in \left(-\frac{1}{k}, \frac{1}{k}\right)$$
and
$$k\zeta(k^n(x+ky)) - \zeta(k^n(kx+y)) - \frac{k(k^2-1)}{2}\left[\zeta(k^n(x+y)) + \zeta(k^n(x-y))\right]$$
$$- (k^4-1)\zeta(k^n y) + 2k(k^2-1)\zeta(k^n x) = 0$$
for $n = 0, 1, \ldots, \ell-1$. From the definition of f and (4.15), we obtain that
$$\begin{aligned}
&\Big|kf(x+ky) - f(kx+y) - \frac{k(k^2-1)}{2}\left[f(x+y) + f(x-y)\right] \\
&\qquad - (k^4-1)f(y) + 2k(k^2-1)f(x)\Big| \\
&\le \sum_{n=0}^{\infty} \frac{1}{k^{3n}}\Big|k\zeta(k^n(x+ky)) - \zeta(k^n(kx+y)) - \frac{k(k^2-1)}{2}\left[\zeta(k^n(x+y)) + \zeta(k^n(x-y))\right] \\
&\qquad - (k^4-1)\zeta(k^n y) + 2k(k^2-1)\zeta(k^n x)\Big| \\
&\le \sum_{n=\ell}^{\infty} \frac{1}{k^{3n}}\Big|k\zeta(k^n(x+ky)) - \zeta(k^n(kx+y)) - \frac{k(k^2-1)}{2}\left[\zeta(k^n(x+y)) + \zeta(k^n(x-y))\right] \\
&\qquad - (k^4-1)\zeta(k^n y) + 2k(k^2-1)\zeta(k^n x)\Big| \\
&\le \sum_{n=\ell}^{\infty} \frac{1}{k^{3n}}\, a(3k^3+k^4-2k) \\
&= \frac{ak^3(3k^3+k^4-2k)}{(k^3-1)} \times \frac{1}{k^{3\ell}} \\
&= \frac{ak^9(3k^3+k^4-2k)}{(k^3-1)}\left(|x|^3+|y|^3\right).
\end{aligned}$$
Thus f satisfies (4.13) for all $x, y \in \mathbb{K}$ with $0 < |x|^3 + |y|^3 < \frac{1}{k^3}$.

We claim that the cubic functional equation (1.8) is not stable for $s = 3$ in condition (ii) of Corollary 4.2. Suppose on the contrary that there exist a cubic mapping $\mathcal{C} : \mathbb{K} \to \mathbb{K}$ and a constant $b > 0$ satisfying (4.14). Since f is bounded and continuous for all $x \in \mathbb{K}$, C is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 4.1, $\mathcal{C}$ must have the form $\mathcal{C}(x) = cx^3$ for any x in $\mathbb{K}$. Thus we obtain that

$$|f(x)| \leq (b + |c|)\,|x|^3. \tag{4.16}$$

But we can choose a positive integer m with $ma > b + |c|$.

If $x \in \left(0, \frac{1}{k^{m-1}}\right)$, then $k^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\zeta(k^n x)}{k^{3n}} \geq \sum_{n=0}^{m-1} \frac{a(k^n x)^3}{k^{3n}} = max^3 > (b + |c|)\,x^3$$

which contradicts (4.16). Therefore the cubic functional equation (1.8) is not stable in sense of Ulam, Hyers and Rassias if $s = 3$, assumed in the condition (ii) of (4.12). □

The following example is to illustrate that the functional equation (1.8) is not stable for $s = \frac{3}{2}$ in condition (iii) of Corollary 4.2.

Example 4.2. Let $\zeta : \mathbb{K} \to \mathbb{K}$ be a function defined by

$$\zeta(x) = \begin{cases} ax^3, & \text{if } |x| < \dfrac{3}{2} \\ \dfrac{3a}{2}, & \text{otherwise} \end{cases}$$

where $a > 0$ is a constant, and define a function $f : \mathbb{K} \to \mathbb{K}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\zeta(k^n x)}{k^{3n}} \qquad for\ all \quad x \in \mathbb{K}.$$

Then F satisfies the functional inequality

$$|D_k f(x,y)| \leq \frac{3a\ k^9(3k^3 + k^4 - 2k)}{2(k^3 - 1)} \left(|x|^{\frac{3}{2}}\ |y|^{\frac{3}{2}} + \left\{|x|^3 + |y|^3\right\}\right) \tag{4.17}$$

for all $x, y \in \mathbb{K}$. Then there do not exist a cubic mapping $C : \mathbb{K} \to \mathbb{K}$ and a constant $b > 0$ such that

$$|f(x) - \mathcal{C}(x)| \leq b|x| \qquad for\ all \quad x \in \mathbb{K}. \tag{4.18}$$

Proof. The proof of the example is similar to that of Example 4.1. □

5 Stability Results: Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the $k-$ type cubic functional equation (1.8).

Now, we present the following theorem due to [39] for fixed point theory.

Theorem 5.1. *[39] Suppose that for a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L*. Then, for each given* $x \in \Omega$ *, either*

$$d(T^n x, T^{n+1} x) = \infty, \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 *such that the properties hold:*

(FP1) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*

(FP2) The sequence $(T^n x)$ *is convergent to a fixed to a fixed point* y^* *of* T*;*

(FP3) y^* *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;

(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Delta$.

Using the above theorem, we obtain the Hyers - Ulam stability of (1.8).

Throughout this section let $\mathcal{U}$ be a normed space and V a (β, p) Banach space with $p-$norm $\|.\|_{\mathcal{V}}$. Define a mapping $D_k f : \mathcal{U} \to \mathcal{V}$ by

$$D_k f(x, y) = kf(x + ky) - f(kx + y) - \frac{k(k^2-1)}{2}\left[f(x+y) + f(x-y)\right] - (k^4 - 1)f(y) + 2k(k^2-1)f(x)$$

for all $x, y \in \mathcal{U}$.

Theorem 5.2. *Let* $D_k f : \mathcal{U} \to \mathcal{V}$ *be a mapping for which there exists a function* $\zeta : \mathcal{U}^2 \to [0, \infty)$ *with the condition*

$$\lim_{n\to\infty} \frac{1}{\rho_i^{3n}} \zeta_{\rho_i^n y}^{\rho_i^n x} = 0 \tag{5.1}$$

where

$$\rho_i = \begin{cases} k & if \quad i = 0, \\ \frac{1}{k} & if \quad i = 1 \end{cases} \tag{5.2}$$

such that the functional inequality

$$\|D_k f(x, y)\|_{\mathcal{V}} \leq \zeta_y^x \tag{5.3}$$

holds for all $x, y \in \mathcal{U}$*. Assume that there exists* $L = L(i)$ *such that the function*

$$x \to Z_0^x = \zeta_0^{\frac{x}{k}},$$

has the property

$$\frac{1}{\rho_i^3} Z_0^x = L\, Z_0^{\rho_i x}. \tag{5.4}$$

Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ *satisfying the functional equation (1.8) and*

$$\| f(x) - \mathcal{C}(x) \|_{\mathcal{V}}^p \leq \left(\frac{L^{1-i}}{1-L}\right)^p (Z_0^x)^p \tag{5.5}$$

for all $x \in \mathcal{U}$*.*

Proof. Consider the set

$$\Omega = \{h/h : \mathcal{U} \to \mathcal{V},\ h(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(h, g) = \inf\{M \in (0, \infty) : \| h(x) - g(x) \|_{\mathcal{V}} \leq M\ Z_0^x, x \in \mathcal{U}\}.$$

It is easy to see that (Ω, d) is complete. Define $J : \Omega \to \Omega$ by

$$Jh(x) = \frac{1}{\rho_i^3} h(\rho_i x),$$

for all $x \in \mathcal{U}$. Now $h, g \in \Omega$,

$$\begin{aligned} d(h,g) \leq M \Rightarrow & \parallel h(x) - g(x) \parallel_{\mathcal{V}} \leq M Z_0^x, x \in \mathcal{U}, \quad or \\ & \left\| \frac{1}{\rho_i^3} h(\rho_i x) - \frac{1}{\rho_i^3} g(\rho_i x) \right\|_{\mathcal{V}} \leq \frac{1}{\rho_i^3} M Z_0^{\rho_i x}, x \in \mathcal{U}, \quad or \\ & \left\| \frac{1}{\rho_i^3} h(\rho_i x) - \frac{1}{\rho_i^3} g(\rho_i x) \right\|_{\mathcal{V}} \leq L M Z_0^x, x \in \mathcal{U} \quad or, \\ & \parallel Jh(x) - Jg(x) \parallel_{\mathcal{V}} \leq L M Z_0^x, x \in \mathcal{U}, \quad or \\ & d(h,g) \leq LM. \end{aligned}$$

This implies $d(Jh, Jg) \leq L d(h,g)$. i.e., J is a strictly contractive mapping on Ω with Lipschitz constant L. It follows from (4.5) that

$$\left\| f(kx) - k^3 f(x) \right\|_{\mathcal{V}} \leq \zeta_0^x \tag{5.6}$$

for all $x \in \mathcal{U}$. Using property $(QB2)$ in (5.6), we obtain

$$\left\| \frac{f(kx)}{k^3} - f(x) \right\|_{\mathcal{V}} \leq \frac{\zeta_0^x}{k^{3\beta}} \tag{5.7}$$

for all $x \in \mathcal{U}$. Using (5.7) for the case $i = 0$ it reduces to

$$\|Jf(x) - f(x)\|_{\mathcal{V}} \leq L\ Z_0^x$$

for all $x \in \mathcal{U}$,

$$\text{i.e.,} \quad d(Jf, f) \leq L = \frac{1}{k^{3\beta}} \Rightarrow d(Jf, f) \leq L = L^1 < \infty. \tag{5.8}$$

Again replacing $x = \frac{x}{k}$ in (5.6), we get

$$\left\| f(x) - k^3 f\left(\frac{x}{k}\right) \right\|_{\mathcal{V}} \leq \zeta_0^{\frac{x}{k}} \tag{5.9}$$

for all $x \in \mathcal{U}$. Using (5.9) for the case $i = 1$, we get

$$\|f(x) - Jf(x)\|_{\mathcal{V}} \leq Z_0^x$$

for all $x \in \mathcal{U}$,

$$\text{i.e.,} \quad d(f, Jf) \leq 1 \Rightarrow d(f, Jf) \leq 1 = L^0 < \infty. \tag{5.10}$$

Thus, from (5.8) and (5.10), we reach

$$d(f, Jf) \leq L^{1-i} < \infty. \tag{5.11}$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{C}$ of J in Ω such that

$$\mathcal{C}(x) = \lim_{n\to\infty} \frac{1}{\rho_i^{3n}} f(\rho_i^n x) \tag{5.12}$$

for all $x \in \mathcal{U}$. In order to show that $\mathcal{C}$ satisfies (1.8), replacing (x, y) by $(\rho_i^n x, \rho_i^n y)$ and dividing by ρ_i^{3n} in (5.3), we have

$$\|\mathcal{C}(x,y)\|_{\mathcal{V}}^p = \lim_{n\to\infty} \frac{1}{\rho_i^{3np}} \|D_k f(\rho_i^n x, \rho_i^n y)\|_{\mathcal{V}}^p \leq \lim_{n\to\infty} \frac{1}{\rho_i^{3np}} \left(\zeta_{\rho_i^n y}^{\rho_i^n x}\right)^p = 0$$

for all $x, y \in \mathcal{U}$, i.e., $\mathcal{C}$ satisfies the functional equation (1.8).

By property (FP3), $\mathcal{C}$ is the unique fixed point of J in the set $\Delta = \{C \in \Omega : d(f, C) < \infty\}, \mathcal{C}$ is the unique function such that

$$\|f(x) - \mathcal{C}(x)\|_{\mathcal{V}} \leq M Z_0^x$$

for all $x \in \mathcal{U}$. Finally by property (FP4), we obtain

$$d(f,\mathcal{C}) \leq \frac{1}{1-L} d(f,Jf)$$

this implies

$$d(f,\mathcal{C}) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(x) - \mathcal{C}(x) \|_{\mathcal{V}}^p \leq \left(\frac{L^{1-i}}{1-L}\right)^p (Z_0^x)^p$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5.2 concerning the stability of (1.8).

Corollary 5.3. *Let $D_k f : \mathcal{U} \to \mathcal{V}$ be a mapping. If there exist real numbers λ and s such that*

$$\|D_k f(x,y)\|_{\mathcal{V}} \leq \begin{cases} \lambda, & \\ \lambda \left\{||x||^s + ||y||^s\right\}, & s \neq 3; \\ \lambda \left\{||x||^s ||y||^s + \left\{||x||^{2s} + ||y||^{2s}\right\}\right\}, & s \neq \frac{3}{2}; \end{cases} \tag{5.13}$$

for all $x, y \in \mathcal{U}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that

$$\|f(x) - \mathcal{C}(x)\|_{\mathcal{V}}^p \leq \begin{cases} \left(\dfrac{k^3 \lambda}{k^{3\beta}|k^3 - 1|}\right)^p, \\ \left(\dfrac{k^3 \lambda ||x||^s}{k^{3\beta}|k^3 - k^{\beta s}|}\right)^p, \\ \left(\dfrac{k^3 \lambda ||x||^{2s}}{k^{3\beta}|k^3 - k^{2\beta s}|}\right)^p \end{cases} \tag{5.14}$$

for all $x \in \mathcal{U}$.

Proof. Let

$$\zeta_y^x = \begin{cases} \lambda, \\ \lambda \{||x||^s + ||y||^s\}, \\ \lambda \left\{||x||^s ||y||^s + \left\{||x||^{2s} + ||y||^{2s}\right\}\right\} \end{cases}$$

for all $x, y \in \mathcal{U}$. Now

$$\frac{1}{\rho_i^n} \zeta_{\rho_i^n y}^{\rho_i^n x} = \begin{cases} \dfrac{\lambda}{\rho_i^n}, \\ \dfrac{\lambda}{\rho_i^n} \{||\rho_i^n x||^s + ||\rho_i^n y||^s\}, \\ \dfrac{\lambda}{\rho_i^n} \left\{||\rho_i^n x||^s \, ||\rho_i^n y||^s + \left\{||\rho_i^n x||^{2s} + ||\rho_i^n y||^{2s}\right\}\right\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (5.1) holds. But, we have

$$Z_0^x = \zeta_0^{\frac{x}{k}}$$

has the property

$$\frac{1}{\rho_i^3} Z_0^x = L \, Z_0^{\rho_i x}$$

for all $x \in \mathcal{U}$. Hence

$$Z_0^x = \zeta_0^{\frac{x}{k}} = \begin{cases} \lambda, \\ \lambda \left\|\dfrac{x}{k}\right\|^s \\ \lambda \left\|\dfrac{x}{k}\right\|^{2s} \end{cases} = \begin{cases} \lambda, \\ \dfrac{\lambda}{k^{\beta s}} \|x\|^s, \\ \dfrac{\lambda}{k^{2\beta s}} \|x\|^{2s}. \end{cases}$$

Now,

$$\frac{1}{\rho_i^3}Z_0^x = \begin{cases} \dfrac{\lambda}{\rho_i^3}, \\ \dfrac{\lambda}{\rho_i^3}||\rho_i x||^s, \\ \dfrac{\lambda}{\rho_i^3}||\rho_i x||^{2s}, \end{cases} = \begin{cases} \dfrac{\lambda}{\rho_i^3}, \\ \dfrac{\rho_i^{\beta s}\lambda}{\rho_i^3}||x||^s, \\ \dfrac{\rho_i^{2\beta s}\lambda}{\rho_i^3}||x||^{2s}, \end{cases} = \begin{cases} \rho_i^{-3}Z_0^x, \\ \rho_i^{\beta s-3}Z_0^x, \\ \rho_i^{2\beta s-3}Z_0^x. \end{cases}$$

Hence, the inequality (5.5) holds for

(i). either $L = k^{-3}$ if $i = 0$ and $L = \frac{1}{k^{-3}}$ if $i = 1$

(ii). either $L = k^{\beta s-3}$ for $s < 3$ if $i = 0$ and $L = \frac{1}{k^{\beta s-3}}$ for $s > 3$ if $i = 1$

(iii). either $L = k^{2\beta s-3}$ for $s > \frac{3}{2}$ if $i = 0$ and $L = \frac{1}{k^{2\beta s-3}}$ for $s > \frac{3}{2}$ if $i = 1$

Now, from (5.5), we prove the following cases for condition (ii).

Case:1 $L = k^{\beta s-3}$ for $s < 3$ if $i = 0$

$$\begin{aligned} \|f(x) - \mathcal{C}(x)\| &\leq \frac{\left(k^{\beta s-3}\right)^{1-0}}{1 - k^{\beta s-3}} \frac{\lambda\, ||x||^s}{k^{\beta s}} \\ &= \frac{\left(k^{\beta s}\right)}{k^3 - k^{\beta s}} \frac{\lambda\, ||x||^s}{k^{\beta s}} \\ &= \frac{\lambda\, ||x||^s}{k^3 - k^{\beta s}}. \end{aligned}$$

Case:2 $L = \frac{1}{k^{\beta s-3}}$ for $s > 3$ if $i = 1$

$$\begin{aligned} \|f(x) - \mathcal{C}(x)\| &\leq \frac{\left(\frac{1}{k^{\beta s-3}}\right)^{1-1}}{1 - \frac{1}{k^{\beta s-3}}} \frac{\lambda\, ||x||^s}{k^{\beta s}} \\ &= \frac{\left(k^{\beta s}\right)}{k^{\beta s} - k^3} \frac{\lambda\, ||x||^s}{k^{\beta s}} \\ &= \frac{\lambda\, ||x||^s}{k^{\beta s} - k^3}. \end{aligned}$$

Similarly, one can prove the other two conditions. Hence the proof is complete. □

6 Concluding Remarks

In this section, we present some important cases related to our $k-$ type cubic functional equation (1.8).

(I) Setting (K, β, p, k, j) by $(1, 1, 1, 3, 1)$ in Theorem 4.1, we arrive the results of Theorem 3.1. of [34].

(II) Replacing (K, β, p, k, j) by $(1, 1, 1, 3, -1)$ in Theorem 4.1, we arrive the results of Theorem 3.2. of [34].

(III) Again replacing (K, β, p, k) by $(1, 1, 1, 3)$ in Corollary 4.2 of Condition (i), we arrive the results of Corollary 3.1 of [34].

(IV) Finally, setting (K, β, p, k) by $(1, 1, 1, 3)$ in Corollary 4.2 of Condition (ii), we arrive the results of Corollary 3.2 of [34].

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Competing Interests

The authors declare that no competing interests exist.

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