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On the Proof Complexities of Strongly Equal Non-classical Tautologies

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Abstract

The strong equality of classical tautologies and their proof complexities comparative analysis in certain proof systems were given by first author in previous studies. Here we introduce the analogous notions of strong equality for non-classical (intuitionistic and minimal) tautologies and investigate the relations between the proof complexity measures of strongly equal non-classical tautologies in some proof systems. We prove that 1) the strongly equal tautologies have the same proof complexities in some proof systems and 2) there are such proof systems, in which some measures of proof complexities for strongly equal tautologies are the same, while the other measures differ from each other only as a function of the sizes of tautologies.

Keywords: Determinative conjunct, strongly equal tautologies, proof complexity

1 Introduction

The research regarding the lengths of proofs in systems of propositional calculus is important because of its relations with some of the main problems of computational complexity theory: $NP \stackrel{?}{=} co\text{-}NP$, $PSPACE \stackrel{?}{=} NP$. The investigations of proof complexity started for the systems of Classical Propositional Logic (CPL), however natural real conclusions have a constructive character in most cases. Therefore, the investigations of proofs complexities are important also for the systems of Intuitionistic Propositional Logic (IPL) and in some cases also for Minimal (Johanssons) Propositional Logic (MPL). Information about proof complexity in IPL and MPL can be important, in particular, for logical programming.

The traditional assumption that all tautologies are equal to each other is not fine-grained enough to support a sharp distinction among tautologies. The authors of [1] have provided a different picture of the equality of classical tautologies. They suggested revising the notion of equivalence between

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tautologies in such way that it takes into account an appropriate measure of their complexity. They introduced in [1] the notion of *strong equality* of classical tautologies on the basis of the notion of determinative conjunct, defined in [2].

By analogy with the notions of determinative conjuncts in CPL, we introduce the same notions for non-classical tautologies and give the algorithms for construction of I-determinative and M-determinative conjuncts for intuitionistic and minimal tautologies accordingly. On the base of introduced non-classical determinative conjuncts we introduce the notion of strong equality for non-classical tautologies and compare different measures of proof complexity for them in some proof systems of IPL and MPL. We prove that 1) the strongly equal tautologies have the same proof complexities in some proof systems and 2) there are such proof systems, in which some measures of proof complexities for strongly equal tautologies are the same, the other measures differ from each other only by the sizes of tautologies.

The study of mentioned relations for some other more interesting systems of CPL, IPL, MPL and other logics (monotone, positive, fuzzy, modal etc) is in progress.

2 Preliminaries

We begin with reviewing of some notions and results for CPL.

We use the current concepts of the unit Boolean cube (E^n), a propositional formula, a classical tautology and proof complexity. The particular choice of a language for presenting propositional formulas is irrelevant for the purpose of our paper. However, because for technical reasons we assume that the language contains the propositional variables $p_i (i \geq 1)$ and (or) $p_{ij} (i \geq 1; j \geq 1)$, logical connectives $\neg, \&, \vee, \supset$ and parentheses $(,)$. Note that some parentheses can be omitted in generally accepted cases.

By $|\varphi|$ we denote the size of a formula φ, defined as the number of all symbols of φ.

Following the usual terminology we call the variables and negated variables *literals*. The conjunct K can be represented simply as a set of literals (no conjunct contains a variable and its negation simultaneously).

In [1] the following notions were introduced.

Each of the following trivial identities for a propositional formula ψ is called *replacement-rule*:

$$\begin{array}{llll} 0\&\psi = 0, & \psi\&0 = 0, & 1\&\psi = \psi, & \psi\&1 = \psi, \\ 0 \vee \psi = \psi, & \psi \vee 0 = \psi, & 1 \vee \psi = 1, & \psi \vee 1 = 1, \\ 0 \supset \psi = 1, & \psi \supset 0 = \overline{\psi}, & 1 \supset \psi = \psi, & \psi \supset 1 = 1, \\ \overline{0} = 1, & \overline{1} = 0, & \overline{\overline{\psi}} = \psi. & \end{array}$$

Application of a replacement-rule to some word consists of replacing some its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

Let φ be a propositional formula, $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of φ, and $P' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\}$ $(1 \leq m \leq n)$ be some subset of P.

Definition *Given* $\sigma = \{\sigma_1, \ldots, \sigma_m\} \subset E^m$*, the conjunct* $K^\sigma = \{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\}$ *is called* $\varphi - 1$*-determinative (*$\varphi - 0$*-determinative) if assigning* σ_j $(1 \leq j \leq m)$ *to each* p_{i_j} *and successively using replacement-rule we obtain the value of* φ *(1 or 0) independently of the values of the remaining variables.*

Definition *DNF* $D = \{K_1, K_2, \ldots, K_j\}$ *is called determinative DNF (dDNF) for* φ *if* $\varphi = D$ *and every conjunct* K_i $(1 \leq i \leq j)$ *is 1-determinative for* φ*.*

Some arguments for the following definition were given in [1].

Main Definition 1. The classical tautologies φ and ψ are strongly equal if every φ-determinative conjunct is also ψ-determinative and vice versa.

The comparative analysis of proof complexities for strong equality tautologies in some proof systems of CPL is given in [3]. Three of the systems under consideration are based on the dDNF: cut-free Frege system, which repeats Calmars proof of classical Frege systems completeness, the systems E and E(lin), which are dual systems for resolution system R and resolution over linear equations R(lin) respectively. The fourth system is the well-known cut-free sequent system.

As the non-classical analogies of the **system** E play important role in our consideration, we recall the definition of E.

The axioms of E aren't fixed, but for every formula φ each conjunct from some dDNF of φ can be considered as an axiom.

The *elimination rule* (e-rule) infers $K^{'} \cup K^{''}$ from conjuncts $K^{'} \cup \{p\}$ and $K^{''} \cup \{\overline{p}\}$, where $K^{'}$ and $K^{''}$ are conjuncts and p is a variable.

The proof in E is a finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of E or is inferred from earlier conjuncts in the sequence by e-rule. It is obvious that DNF $D = \{K_1, K_2, \ldots, K_l\}$ is classical tautology if using e-rule the empty conjunction ($\emptyset$) can be proved from the axioms $\{K_1, K_2, \ldots, K_l\}$.

As the intuitionistic (minimal) validity is determined by derivability in some intuitionistic (minimal) propositional proof system, the above definition of dDNF for non-classical tautologies is not applicable. Author of [2] gives some algorithm for construction of dDNF for classical tautologies on the base of their resolution refutations. The analogous algorithms for non-classical tautologies are given in [4].

In the theory of proof complexity four main characteristics of the proof are: t-complexity, defined as the number of proof steps (time), l-complexity, defined as total number of proof symbols (size), s-complexity (space), informally defined as maximum of the minimal number of formulas on blackboard, needed to verify all steps in the proof (formal definitions are for example in [6]), and w-complexity (width), defined as the maximum of widths of proof formulas.

Let Φ be a proof system and φ be a tautology. We denote by t_{φ}^{Φ} (l_{φ}^{Φ}, s_{φ}^{Φ}, w_{φ}^{Φ}) the minimal possible value of t-complexity (l-complexity, s-complexity, w-complexity) for all proofs of tautology φ in Φ.

3 Main Notions for IPL and MPL

Let us recall some of proof systems of IPL and MPL.

1) The resolution systems RI (RM)

The system RI (resolution for IPL) is described by Mints in [5].

The axioms are the sequents

$$p \to\ p \text{ and } \perp \to\ p.$$

The rules of inference (resolution rules) are:

$$\left\{ \begin{array}{l} \dfrac{(p\ \supset\ q) \to r;\ \Sigma, p \to \perp}{\Sigma\ \to r} \quad (1) \\[2ex] \dfrac{(p\ \supset\ q) \to r;\ \Sigma, p \to q}{\Sigma\ \to r} \\[2ex] \dfrac{(p\ \supset\ q) \to r;\ \Sigma \to \perp}{\Sigma\ \to r} \\[2ex] \dfrac{(p\ \supset\ \perp) \to r;\ \Sigma, p \to \perp}{\Sigma\ \to r} \end{array} \right\} \qquad (\supset^{-})$$

$$\frac{p \rightarrow q \vee r;\ \Gamma \rightarrow p;\ \Sigma q \rightarrow s^*;\ \Pi,\ r \rightarrow s^{**}}{\Gamma,\ \Sigma,\ \Pi \rightarrow s} \qquad (\vee^-)$$

$$\frac{p,\ q \rightarrow r^*;\ \ \Gamma \rightarrow p;\ \Sigma \rightarrow q}{\Gamma,\ \Sigma \rightarrow r^*}(\text{cut})\ \frac{p \rightarrow q;\ \Gamma \rightarrow p}{\Gamma \rightarrow q}$$

(2) $\frac{\rightarrow \bot}{\rightarrow p}$ $(\bot)$, where p^* for some propositional variable p can be p or $\bot$.

The corresponding system for MPL are defined as follow: RM is obtained from RI by dropping the rules (1) and (2) [4].

It is necessary to make some comments about the system RI (RM). Let φ be some formula and $\{p_1, p_2, ..., p_n\}$ is the set of its distinct variables (later we call these variables the *main variables*). Associating a new variable with every non-elementary subformula of φ, we can construct the system of disjuncts by employing the well-known above mentioned method [5]. The disjuncts of this system can be represented as the following sequents

$$p \rightarrow q \vee r; (p \supset q^*) \rightarrow r; q_1, q_2, \ldots, q_k \rightarrow r^*. \qquad (\diamondsuit)$$

Let s be the variable, associated with φ itself. Mints has shown that the sequent $\rightarrow s$ is proved in RI from the axioms and from above set ($\diamondsuit$) of disjuncts, constructed for φ, iff the sequent $\rightarrow \varphi$ is proved in the system NI (natural system for IPL [5]). The same result can be proved for the systems RM and NM (corresponding natural system for MPL, which is obtained from NI by dropping the rule $\frac{\Gamma \rightarrow \bot}{\Gamma \rightarrow A}$). Later for every formula φ each of disjuncts of the set ($\diamondsuit$) is called the *additional axiom*. The axiom (additional or not) is called the *main axiom* if it contains at least one main variable.

In order to describe for IPL and MPL the systems, which are the analogies of mentioned system E for CPL, we must define the notion of φ-determinative conjunct for IPL and MPL.

Recall that there is a well-known notion of positive and negative occurrences of subformulas (or variables) in the formula or in the sequent [2]. If a variable p has negative occurrence in some subformula, which in its turn has negative occurrence in the formula, we say that the variable p has double negative occurrence in this formula.

It is not difficult to see that any occurrence of a variable in the axioms (additional or not) or inference rules of system RI, is either positive, negative or double negative, and since $\bar{\bar{p}} \equiv p$ is not derivable in IPL (MPL), then not only variable or variable with negation, but also variables with double negation can serve as literal for φ-determinative conjunction in IPL.

The analogies of the φ-determinative DNF for IPL and MPL ($\varphi - I$-determinative DNF and $\varphi - M$-determinative DNF respectively) can be constructed using the following algorithm.

Let W be the proof of $\rightarrow s$ in RI (RM) with the minimal steps. The steps for the construction of the $\varphi - I$-determinative ($\varphi - M$-determinative) DNF are the following:

a) We transform the proof W into tree-like proof W^{tree} of $\rightarrow s$. Let k be the number of the paths of this tree.

b) For every path $i (1 \le i \le k)$ between two vertices, one associated with the main axiom and another with $\rightarrow s$, we construct the conjunct K_i as the set of negations of all main variables (or their negations, or double negations), which have positive (negative or double negative) occurrence in the sequents of this path.

(Note that $\bar{\bar{\bar{p}}}$ and $\bar{p}$ are equal in intuitionistic logic, therefore only variables with one or double negations are the literals of above constructed conjuncts.)

The DNF $D = \{K_{i_1}, K_{i_2}, ..., K_{i_t}\}(t \le k)$, consisting of all the distinct non contradictory conjuncts constructed as specified above, is called $\varphi - I$-determinative for RI -proof ($\varphi - M$-determinative for RM-proof).

Main Definition 2. The intuitionistic (minimal) tautologies φ and ψ are strongly equal if every φ - I-determinative (- M-determinative) DNF is also ψ - I-determinative (- M-determinative) DNF and vice versa.

By analogy the corresponding proof system EI (EM) can be constructed for IPL (MPL).

The axioms consist of every I-determinative (M-determinative) conjunct from some I-determinative (M-determinative) DNF.

For EI (EM) we take the following inference rule

$$\frac{K^{'} \cup \bar{\bar{p}} \quad K^{''} \cup \bar{p}}{K^{'} \cup K^{''}} \quad I - elimination - rule$$

$$\left(\frac{K^{'} \cup (p \supset \bot) \supset \bot \quad K^{''} \cup p \supset \bot}{K^{'} \cup K^{''}} \quad M - elimination - rule\right)$$

where $K^{'}$ and $K^{''}$ are conjuncts and p is a variable.

2) Multi-succedent cut-free sequent systems LI$^{-}_{mc}$ and LM$^{-}_{mc}$

The system LI$^{-}_{mc}$ is the following [7]

$$\frac{}{\Gamma,\ A\ \vdash A,\ \Delta}\ ax. \qquad \frac{}{\Gamma,\ \bot \vdash A,\ \Delta}\ \bot\ ax.$$

$$\frac{\Gamma \vdash A,\ B,\ \Delta}{\Gamma \vdash A\ \vee\ B,\ \Delta}\ \vee r \qquad \frac{\Gamma,\ A \vdash \Delta \quad \Gamma,\ B \vdash \Delta}{\Gamma,\ A \vee B \vdash \Delta}\ \vee l$$

$$\frac{\Gamma \vdash A,\ \Delta \quad \Gamma \vdash B,\ \Delta}{\Gamma \vdash A\ \wedge\ B,\ \Delta}\ \wedge r \qquad \frac{\Gamma,\ A,\ B \vdash \Delta}{\Gamma,\ A \wedge B \vdash \Delta}\ \wedge l$$

$$\frac{\Gamma,\ A \vdash B}{\Gamma \vdash A \supset B, \Delta}\ \supset r \qquad \frac{\Gamma, A \supset B \vdash A, \Delta \quad \Gamma,\ B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}\ \supset l$$

$$\frac{\Gamma,\ A \vdash}{\Gamma \vdash \neg A, \Delta}\ \neg r \qquad \frac{\Gamma, \neg A \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg l$$

The system LM$^{-}_{mc}$ is obtained from LI$^{-}_{mc}$ by imposing that the succedent of the conclusion in the left negation rule must be empty.

4 Main Results

Here we give the main theorems.

Theorem 1 *Strongly equal intuitionistic (minimal) tautologies have the same t, l, s, w complexities in the systems EI (EM).*

The proof is based on the fact that refutations in the systems EI (EM) deal exclusively with the conjuncts of I-determinative (M-determinative) DNF.

Theorem 2 *If φ and ψ are strongly equal intuitionistic (minimal) tautologies, $k = \frac{|\varphi|}{|\psi|}$ and Φ is the system LI^{-}_{mc} (LM^{-}_{mc}), then*

$$\begin{aligned} &1. \quad t^{\Phi}_{\varphi} = kt^{\Phi}_{\psi} \\ &2. \quad l^{\Phi}_{\varphi} = k^2 l^{\Phi}_{\psi} \\ &3. \quad w^{\Phi}_{\varphi} = kw^{\Phi}_{\psi} \\ &4. \quad s^{\Phi}_{\varphi} = s^{\Phi}_{\psi} \end{aligned}$$

Proof of Theorem 2

The constructions of the proof of sequent $\rightarrow\ \varphi$ in LI$^{-}_{mc}$ (LM$^{-}_{mc}$) are the same as for CPL [2], but they are based on the above-described $\varphi - I$-determinative DNF ($\varphi - M$-determinative DNF). Indeed from this construction we have $t^{\Phi}_{\varphi} = c_1 t^{E}_{\varphi}|\varphi|$ for some constant c_1 and $l^{\Phi}_{\varphi} = c_2 t^{E}_{\varphi}|\varphi|^2$ for some constant c_2. Statement of point 3. is obvious. To verify the statement of point 4. note that for the proof of formula φ from every determinative conjunct we must use step by step one or two subformulas independently of the formula size.

The study of mentioned relations for some other systems of CPL, IPL, MPL and other logics is in progress.

5 Conclusion

The main results, obtained in this paper for the proof complexities of strongly equal non-classical tautologies are the same, as the results on strongly equal classical tautologies given for corresponding systems by first author in previous studies. It will be interesting to investigate the relation between the proof complexities of strongly equal tautologies in the other systems, for example in Frege systems of mentioned logics.

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Competing Interests

The authors declare that no competing interests exist.

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