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A New Preconditioner with Two Variable Relaxation Parameters for Saddle Point Linear Systems with Highly Singular (1,1) Blocks

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Abstract

In this paper, we provide new preconditioner for saddle point linear systems with (1,1) blocks that have a high nullity. The preconditioner is block triangular diagonal with two variable relaxation paremeters and it is extension of results in [1] and [2]. Theoretical analysis shows that all eigenvalues of preconditioned matrix is strongly clustered. Finally, numerical tests confirm our analysis.

Keywords: Saddle Point Linear Systems, Block Triangular Preconditioner, Krylov Subspace Methods

1. Introduction

Consider the following saddle point linear system

$$Ax \equiv \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \qquad (1)$$

where $F \in R^{n\times n}$ is symmetric and positive semidefinite with nullity r, $B \in R^{m\times n}\left(m \le n\right)$ has full row rank, $f \in R^n$ and $g \in R^n$, u and p are unknown. Note that the assumption that A is nonsingular, *i.e.*, the system (1) has a unique solution implies that null(F)/null(B) = 0, which we use in our analysis below. Saddle point linear systems of form (1) can arise, for example, from constraint optimization [3], mixed nite element formulation for the stokes problem [4], and discrete time harmonic Maxwell equations in mixed form [5].

There has many techniques for solving Saddle point linear systems of form (1), see [6] for a comprehensive survey. However, when F is singular, it cannot be inverted and the Schur complement does not exist. In this case, one possible way of dealing with system is by augmentation [7]. Another way we can refer to [8] where Grief and SchÄotzau exploited a preconditioning technique for solving time-harmonic Maxwell equations in mixed form.

Recently, Rees and Grief [2] extend the work by Grief and SchÄotzau [8] to interior point methods for optimization problems. The preconditioner has attractive property of improved eigenvalue clustering with ill-conditioned the (1,1) block of saddle point systems. Based on the basic of above work, Huang etc. costructed two block triangular preconditioners for solving saddle point systems (1) [1].

In this paper we are devoted to give new block triangular preconditioner for solving saddle point systems of (1) with an ill-conditioned (1,1) blocks. The preconditioner is involving two parameters, and they are extension of recent work in Grief and SchÄotzau [8], Rees and Grief [2], and Cheng etc. [1].

2. New Preconditioner and Spectral Analysis

Rees and Grief [9] provided the following preconditioner for the symmetric saddle point systems (1)

$$M = \begin{pmatrix} F + B^T W^{-1} B & tB^T \\ 0 & W \end{pmatrix},$$

where t is a scalar and $W \in R^{m\times m}$ is symmetric positive weight matrix.

Recently, Huang etc. [6] established the following preconditioners for the saddle point systems:

$$M_t = \begin{pmatrix} F + B^T W^{-1} B & \left(1-t\right)B^T \\ 0 & tW \end{pmatrix},$$

where $t \ne 0$ is a parameter and

$$\hat{M}_t = \begin{pmatrix} F + tB^T W^{-1} B & tB^T \\ 0 & \dfrac{\left(1-t\right)}{t} W \end{pmatrix},$$

where $1 \neq t > 0$.

In this section, we introduce the following preconditioner involving two parameters:

$$M_{\eta,\varepsilon} = \begin{pmatrix} F + \eta B^T W^{-1} B & (1-\eta\varepsilon) B^T \\ 0 & \varepsilon W \end{pmatrix}, \quad (2)$$

for the saddle point systems (1), where $\eta > 0$ and $\varepsilon \neq 0$. We note that when the parameter $\eta = 1$, and $\varepsilon = t$, $M_{\eta\varepsilon}$ reduce to M_t; when $\eta = t$ and $\varepsilon = \frac{1-t}{t}$, $M_{\eta\varepsilon}$ reduce to M_t.

Theorem 2.1. The matrix $M_{\eta\varepsilon}^{-1} A$ has two distinct eigenvalues, given by

$$\lambda_1 = 1 \text{ and } \lambda_2 = -1/\eta\varepsilon,$$

with the algebraic multiplicities *n* and *r*, respectively. The remaining $m-r$ eigenvalues satisfy the relation

$$\lambda = -\frac{\mu}{\varepsilon(\eta\mu + 1)}, \quad (3)$$

where μ are positive generalized eigenvalues of

$$B^T W^{-1} B u = \mu F u, \quad (4)$$

Let $\{x_i\}_{i=1}^{r}$ be a basis of the null space of *F*, $\{z_i\}_{i=1}^{n-m}$ be a basis of the null space of *B*, and $\{y_i\}_{i=1}^{n-m}$ a set of linearly independent vectors that complete null(*F*) ∪ null(*B*) to a basis of R^n, Then the r vectors $\left(x_i; \frac{1}{\varepsilon} W^{-1} B x_i\right)$, the $n-m$ vectors $(z_i; 0)$, and the $m-r$ vectors $\left(y_i, \frac{1}{\varepsilon} W^{-1} B y_i\right)$ are linearly independent eigenvectors associated with $\lambda = 1$, and the r vectors $\left(x_i; -\eta W^{-1} B x_i\right)$ are eigenvectors associated with $\lambda = -\frac{1}{\eta\varepsilon}$.

Proof: Suppose that λ is an eigenvalue of $\hat{M}_{\eta\varepsilon}^{-1} A$, whose eigenvector is $\left(u^T, p^T\right)^T$. Then the corresponding eigenvalue problem is

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \lambda \begin{pmatrix} F + \eta B^T W^{-1} Bu & (1-\eta\varepsilon) B^T \\ 0 & \varepsilon W \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}.$$

From the second row we can obtain $p = \frac{1}{\varepsilon\lambda} W^{-1} B u$. By substituting it into the first row we have

$$(1-\lambda)\left[\lambda F u + \left(\frac{1}{\varepsilon\lambda} + \lambda\eta\right) B^T W^{-1} B u\right] = 0. \quad (5)$$

If $\lambda = 1$, then (5) is satisfied for any $u \in R^n$, and hence $\left(u; W^{-1} B u\right)$ is an eigenvector of $M_{\eta\varepsilon}^{-1} A$.

If $u \in$ null(*F*), then from (5) we obtain

$$(1-\lambda)\left(\frac{1}{\varepsilon\lambda} + \lambda\eta\right) B^T W^{-1} B u = 0.$$

From which it follows that $\left(u; \frac{1}{\varepsilon} W^{-1} B u\right)$ and $\left(u; -\eta W^{-1} B u\right)$ are eigenvectors associated $\lambda = 1$ and $\lambda - \frac{1}{\eta\varepsilon}$.

Next, suppose $\lambda = 1$ and $\lambda = -1/\eta\varepsilon$. We divide (5) by $(1-\lambda)(1/\varepsilon + \lambda\eta)$, which yields (3), with *u* defined in (4).

Now we can find a specific set of linearly independent eigenvectors for $\lambda = 1$ and $\lambda = -1/\eta\varepsilon$. Since null(*F*) ∩ null(*B*) = 0, the vectors $\{x_i\}_{i=1}^{r}$ and $\{z_i\}_{i=1}^{n-m}$ defined above are linearly independent and form a subspace of R^n of dimension $n-m+r$. Let $\{y_i\}_{i=1}^{n-m}$ complete this set to a basis of *Rn*. It follows that $\left(x_i, \varepsilon^{-1} W^{-1} B x_i\right)$, $(z_i, 0)$, and $\left(y_i, \varepsilon^{-1} W^{-1} B y_i\right)$ are eigenvectors associated with $\lambda = 1$. The r vectors $\left(x_i, -\eta W^{-1} B x_i\right)$ are igenvectors associated with $\lambda = -1/\eta\varepsilon$.

When the parameters satisfy $-1/\eta\varepsilon = 1$, we can obtain the following corollary from Theorem 2.1.

Corollary 2.2. Let $-1/\eta\varepsilon = 1$. Then the matrix $M_{\eta\varepsilon}^{-1} A$ has one eigenvalue which given by $\lambda = 1$ with algebraic multiplicity $n + r$. The remaining $m-r$ eigenvalues satisfy the relation

$$\lambda = -\eta\mu/(\eta\mu + 1),$$

where μ are positive generalized eigenvalues of

$$B^T W^{-1} B u = \mu F u.$$

Theorem 2.3. The matrix $M_{\eta\varepsilon}^{-1} A$ has two distinct eigenvalues, given by

$$\lambda_1 = 1 \text{ and } \lambda_2 = -1/\eta\varepsilon,$$

with the algebraic multiplicities *n* and *r*, respectively. The remaining $m-r$ eigenvalues lie in the interval

$$(0, -1/\eta\varepsilon)(\varepsilon < 0), \text{or } (-1/\eta\varepsilon, 0)(\varepsilon > 0).$$

Proof: From Theorem 2.1 we obtain that the matrix $M_{\eta\varepsilon}^{-1} A$ has two distinct eigenvalues, given by

$$\lambda_1 = 1 \text{ and } \lambda_2 = -1/\eta\varepsilon,$$

with the algebraic multiplicities *n* and *r*, respectively.

Taking inner product of (5) with u, we can obtain that the remaining $m-r$ eigenvalues satisfy

$$\lambda = \frac{-\left\langle B^T W^{-1} B u, u\right\rangle / (\eta\varepsilon)}{\langle F u, u\rangle / \eta + \left\langle B^T W^{-1} B u, u\right\rangle}, \quad (6)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product, $u \notin$ null(*F*) and $u \notin$ null(*B*). By (6), we have that the remaining $m-r$ eigenvalues lie in interval

$$\left(0,-1/\eta\varepsilon\right)\left(\varepsilon<0\right), \text{or } \left(-1/\eta\varepsilon,0\right)\left(\varepsilon>0\right).$$

When the parameters satisfy $-1/\eta\varepsilon=1$, we can obtain the following corollary from Theorem 2.3.

Corollary 2.4. Let $-1/\eta\varepsilon=1$. Then the matrix $M_{\eta\varepsilon}^{-1}A$ has one eigenvalue which given by $\lambda = 1$ with algebraic multiplicity $n + r$. The remaining $m - r$ eigenvalues lie in the interval (0, 1).

3. Numerical Experiments

We consider the following finite element discretization of the time-harmonic Maxwell equations ($k^2 = 0$) [5,8]. The following two-dimensional Model problem is considered: find u and p that satisfy

$$\begin{aligned} \nabla\times\nabla\times u+\nabla p=f \ \text{ in } \ \Omega \\ \nabla\cdot u=0 \ \text{ in } \ \Omega \\ u\times \mathbf{n}=0 \ \text{ on } \ \partial\Omega \\ p=0 \ \text{ on } \ \partial\Omega \end{aligned} \qquad (7)$$

Here $\Omega\in R^2$ is a simply connected polyhedron domain with a connected boundary $\partial\Omega$, and $\sim n$ denotes the outward unit normal on $\partial\Omega$. The datum f is a given source (not necessarily divergence free). Using the lowest order N'ed'elec elements of first kind [9,10] for the approximation of the vector field and standard nodal elements for multiplier yields the following saddle-point linear system

$$Ax\equiv\begin{pmatrix}F & B^T\\ B & 0\end{pmatrix}\begin{pmatrix}u\\ p\end{pmatrix}=\begin{pmatrix}g\\ 0\end{pmatrix}\equiv b.$$

Experiments were done in a square domain ($0 \le x \le 1$; $0 \le y \le 1$). And we set the right-hand side function so that the exact solution is given by

$$u(x,y)=\left(y(1-y),x(1-x)\right)^T.$$

In our numerical experiments the matrix W in the augmentation block preconditioner is taken as $W = \mathrm{I}$.

We consider three meshes with different values of n and m in **Table 1**. **Table 2** shows iteration counts for different η and meshes, applying BiCGSTAB of block-triangular preconditioner, and $-1/\eta\varepsilon=1$. We observe that for a fixed mesh, When $\eta = 6$, We spent the least iteration counts. In particular, when $\eta > 2$, the iteration we spent are less than $\eta = 2$ (corresponding to $M2$ [1]). It shows that preconditioner $\lambda=M_{\eta\varepsilon}$ is more efficient than M_t and $\hat{M}_t$.

Table 1. Values of *n* and *m* and size of the linear systems for three meshes.

h	n	m	$n+m$
$\frac{1}{8}$	176	49	225
$\frac{1}{16}$	225	736	961
$\frac{1}{32}$	961	3008	3969

Table 2. Iteration counts for different and meshes, using BiCGSTAB for solving the saddle point system with preconditioner $\mathcal{M}_{\eta\varepsilon}$, the iteration was stopped once $\|r^{(k)}\|/\|r^{(0)}\|\le 10^{-13}$.

h	0.1	0.5	2	4	6
$h=\frac{1}{8}$	3	3	1	1	0.5
$h=\frac{1}{16}$	3	2	2	1	1
$h=\frac{1}{32}$	1	0.5	0.5	0.5	0.5

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