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Well-Posedness and Stability Results for a Nonlinear Damped Porous–Elastic System with Infinite Memory and Distributed Delay Terms

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Abstract: In the present paper, we consider an important problem from the application perspective in science and engineering, namely, one-dimensional porous–elastic systems with nonlinear damping, infinite memory and distributed delay terms. A new minimal conditions, placed on the nonlinear term and the relationship between the weights of the different damping mechanisms, are used to show the well-posedness of the solution using the semigroup theory. The solution energy has an explicit and optimal decay for the cases of equal and nonequal speeds of wave propagation.

Keywords: well-posedness; general decay; infinite memory; nonlinear damping; porous–elastic system; distributed delay term

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1. Introduction

As introduced in [1], the one-dimensional porous–elastic model constitutes a system of two partial differential equations with unknown (u, φ) given by

$$\begin{aligned} &\rho_0 u_{tt} = \mu u_{xx} + \beta \varphi_x, \text{ in } (0,l) \times (0,L), \\ &\rho_0 k \varphi_{tt} = \alpha \varphi_{xx} - \beta u_x - \tau \varphi_t - \xi \varphi, \text{ in } (0,l) \times (0,L), \end{aligned} \tag{1}$$

where $l, L > 0$ the constant ρ is the mass density, κ is the equilibrated inertia and the constants $\mu, \alpha, \beta, \tau, \xi$ are assumed to satisfy the appropriate conditions. This type of problem has been studied by many authors and a lot of results have been shown (please see [1–9]). The pioneering contribution was made by [10] for the problem (1). The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\rho u_{tt} = T_x, J\phi_{tt} = H_x + G, \tag{2}$$

where T is the stress tensor, H is the equilibrated stress vector and G is the equilibrated body force. The variables u and ϕ are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are

$$T = \mu u_x + b\phi, H = \delta \phi_x - \int_0^t g(t-s)\phi_x(s)ds, G = -bu_x - \xi \phi. \tag{3}$$

A porous–elastic system was considered by [11] in the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,1) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds = 0, \text{ in } (0,1) \times (0,\infty). \end{cases} \tag{4}$$

System (4) subjected Neumann–Dirichlet boundary conditions, where g is the relaxation function; the authors obtained a general decay result for the case of equal speeds of wave propagation (See [12,13]). In [14], the authors improved the case of non-equal speed of wave propagation. In [15] the authors considered the following system with memory and distributed delay terms

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(s)\phi_{xx}(t-s)ds \\ \qquad + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x,t-\varrho)d\varrho = 0. \end{cases} \tag{5}$$

The exponential stability results of systems with memory and distributed delay terms, for the case of equal speeds of wave propagation under a suitable assumptions, are proved. In [16], the following system was considered

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + a\phi + \int_0^\infty g(s)\phi_{xx}(t-s)ds + \alpha(t)f(\phi_t) = 0. \end{cases} \tag{6}$$

The authors proved the global well-posedness and stability results of (6), which has been extended in [17] for the case of nonequal speeds of wave propagation. Very recently, one-dimensional equations of an homogeneous and isotropic porous–elastic solid with an interior time-dependent delay term feedbacks was treated by Borges Filho and M. Santos in [1].

The result in [10] for system (1) was improved by Apalara to exponential stability in [18]. For more papers related to our paper, please see [19–22].

Motivated by all the above papers, we investigate the well-posedness and stability results with distributed delay for the cases of equal and nonequal speeds of wave propagation, under additional conditions of the following system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^\infty g(p)\phi_{xx}(t-p)dp \\ \qquad + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x,t-\varrho)d\varrho + \alpha(t)f(\phi_t) = 0, \end{cases} \tag{7}$$

where

$$(x,\varrho,t) \in (0,1) \times (\tau_1,\tau_2) \times (0,\infty),$$

with the Neumann–Dirichlet boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad t \geq 0, \tag{8}$$

and the initial data

$$\begin{aligned} &u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,1) \\ &\phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \quad x \in (0,1) \\ &\phi_t(x,-t) = f_0(x,t), \quad (x,t) \in (0,1) \times (0,\tau_2). \end{aligned} \tag{9}$$

Here, $\rho, \mu, J, b, \delta, \xi$ and μ_1 are positive constants satisfying $\mu\xi > b^2$, the term $\alpha(t)f(\phi_t)$, where the functions α and f are specified later, represent the nonlinear damping term. The term $\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x, t-\varrho)d\varrho$ is a distributed delay that acts only on the porous equation and τ_1, τ_2 are two real numbers with $0 \le \tau_1 \le \tau_2$, where μ_2 is an L^∞ function, and the function g is called the relaxation function. We first state the following assumptions:

Hypothesis 1 (H1). *$g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying*

$$g(0) > 0, \quad \delta - \int_0^\infty g(p)dp = l > 0, \quad \int_0^\infty g(p)dp = g_0. \tag{10}$$

Hypothesis 2 (H2). *There exists a non-increasing differentiable function $\alpha, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$g'(t) \le -\eta(t)g(t), \quad t \ge 0, \tag{11}$$

and

$$\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0. \tag{12}$$

Hypothesis 3 (H3). *$f \in C^0(\mathbb{R}, \mathbb{R})$ is non-decreasing such that there exist $v_1, v_2, \varepsilon > 0$ and a strictly increasing function $G \in C^1([0, \infty))$, with $G(0) = 0$ and G is a linear or strictly convex C^2-function on $(0, \varepsilon]$, such that*

$$\begin{aligned} s^2 + f^2(s) \le G^{-1}(sf(s)), \quad &\forall |s| < \varepsilon \\ v_1|s| \le |f(s)| \le v_2|s|, \quad &\forall |s| \ge \varepsilon. \end{aligned} \tag{13}$$

which implies that $sf(s) > 0$ for all $s \ne 0$. The function f satisfies

$$|f(\psi_2) - f(\psi_1)| \le k_0(|\psi_2|^\beta + |\psi_1|^\beta)|\psi_2 - \psi_1|, \quad \psi_1, \psi_2 \in \mathbb{R}, \tag{14}$$

where $k_0, \beta > 0$.

Hypothesis 4 (H4). *The bounded function $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$, satisfying*

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho < \mu_1. \tag{15}$$

Now, as in [23], taking the following new variable

$$y(x, \rho, \varrho, t) = \phi_t(x, t - \varrho\rho),$$

then we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ y(x, 0, \varrho, t) = \phi_t(x, t). \end{cases}$$

As in [24], we introduce the following new variable

$$\eta^t(x, s) = \phi(x, t) - \phi(x, t - s), \quad (x, t, s) \in (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+,$$

where η^t is the relative history of ϕ satisfies

$$\eta_t^t + \eta_s^t = \phi_t(x, t), \quad (x, t, s) \in (0, 1) \times (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Consequently, the problem (7) is equivalent to

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ J\phi_{tt} - l\phi_{xx} + bu_x + \xi\phi + \int_0^\infty g(p)\eta_{xx}^t(p)dp \\ \qquad + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)d\varrho + \alpha(t)f(\phi_t) = 0 \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0 \\ \eta_t^t + \eta_s^t = \phi_t(x,t), \end{cases} \tag{16}$$

where

$$(x,\rho,\varrho,t) \in (0,1) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty),$$

with the following boundary and initial conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0, \tag{17}$$

$$\begin{aligned} & u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,1) \\ & \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \quad x \in (0,1) \\ & y(x,\rho,\varrho,0) = f_0(x,\rho\varrho), \quad x \in (0,1), \rho \in (0,1), \varrho \in (0,\tau_2) \\ & \eta^t(x,0) = 0, \eta^0(x,s) = \eta_0(x,s), \quad (x,s) \in (0,1) \times \mathbb{R}_+. \end{aligned}$$

Meanwhile, from $(7)_1$ *and (9), it follows that*

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx = 0. \tag{18}$$

Therefire, by solving (18) and using the initial data of u*, we get*

$$\int_0^1 u(x,t)dx = t\int_0^1 u_1(x)dx + \int_0^1 u_0(x)dx.$$

Consequently, if we let

$$\bar{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)dx - \int_0^1 u_0(x)dx, \tag{19}$$

we get

$$\int_0^1 \bar{u}(x,t)dx = 0, \quad \forall t \geq 0.$$

Therefore, the use of Poincaré's inequality for $\bar{u}$ *is justified. In addition, a simple substitution shows that* $(\bar{u},\phi,y,\eta^t)$ *satisfies system (7). Hence, we work with* $\bar{u}$ *instead of* u*, but write* u *for simplicity of notation.*

By imposing new appropriate conditions (H3), with the help of some special results, we obtain an unusual, weaker decay result using Lyaponov functiona, extending some earlier results known in the existing literature. The main results in this manuscript are as follows: Theorem 1 for the existence and uniqueness of solution and Theorem 2 for the general stability estimates.

2. Well-Posedness

In this section, we prove the existence and uniqueness result of the system (16)–(18) using the semigroup theory. To achieve our goal, we first introduce the vector function

$$U = (u, u_t, \phi, \phi_t, y, \eta^t)^T,$$

and the new dependent variables $v = u_t, \psi = \phi_t, \varphi = \eta^t$; then, the system (16) can be written as follows

$$\begin{cases} U_t = \mathcal{A}U + \Gamma(U) \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, f_0, \eta_0)^T, \end{cases} \tag{20}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} :\to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi_x \\ \psi \\ \frac{l}{J}\psi_{xx} + \frac{b}{J}u_x - \frac{\xi}{J}\phi_x + \frac{1}{J}\int_0^\infty g(p)\varphi_{xx}(p)dp \\ \qquad - \frac{\mu_1}{J}\psi - \frac{1}{J}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho \\ -\frac{1}{\varrho}y_\rho \\ -\varphi_s + \psi \end{pmatrix}, \tag{21}$$

and

$$\Gamma(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\alpha(t)}{J}f(\psi) \\ 0 \\ 0 \end{pmatrix}, \tag{22}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \times L_g(0,1),$$

where

$$\begin{aligned} L^2_*(0,1) &= \{\Phi \in L^2(0,1) \ / \ \int_0^1 \Phi(x)dx = 0\}, \\ H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \\ L_g(0,1) &= \{\Phi : \mathbb{R}_+ \to H^1_0(0,1), \ \int_0^1 \int_0^\infty g(s)\Phi_x^2(p)dp < \infty\}, \end{aligned}$$

where the space $L_g(0,1)$ is endowed with the following inner product

$$\langle \Phi_1, \Phi_2 \rangle_{L_g(0,1)} = \int_0^1 \int_0^\infty g(p)\Phi_{1x}(p)\Phi_{2x}(p)dp.$$

For any

$$U = (u, v, \phi, \psi, y, \varphi)^T \in \mathcal{H}, \quad \widehat{U} = (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\psi}, \widehat{y}, \widehat{\varphi})^T \in \mathcal{H}.$$

The space $\mathcal{H}$ equipped with the inner product is defined by

$$\begin{aligned}\langle U,\widehat{U}\rangle_{\mathcal{H}} &= \rho\int_0^1 v\widehat{v}dx + \mu\int_0^1 u_x\widehat{u}_x dx + J\int_0^1 \psi\widehat{\psi}dx \\ &+ \xi\int_0^1 \phi\widehat{\phi}dx + l\int_0^1 \phi_x\widehat{\phi}_x dx + b\int_0^1 (u_x\widehat{\phi} + \widehat{u}_x\phi)dx \\ &+ \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y\widehat{y}d\varrho d\rho dx + \langle\varphi,\widehat{\varphi}\rangle_{L_g(0,1)}. \end{aligned} \tag{23}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} U\in\mathcal{H} \ \ / \ \ u\in H_*^2\cap H_*^1, \ \ \phi\in H^2\cap H_0^1, \\ v\in H_*^1, \ \ \psi\in H_0^1(0,1), \ \ \varphi\in L_g(0,1), \\ y, y_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \ \ y(x,0,\varrho,t)=\psi, \end{array}\right\}$$

where

$$H_*^2(0,1) = \Big\{\Phi\in H^2(0,1) \ \ / \ \ \Phi_x(1) = \Phi_x(0) = 0\Big\}.$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can state and prove the existence result.

Theorem 1. *Let $U_0\in\mathcal{H}$ and assume that (10)–(15) hold. Then, there exists a unique solution $U\in C(\mathbb{R}_+,\mathcal{H})$ of problem (20). Moreover, if $U_0\in\mathcal{D}(\mathcal{A})$, then*

$$U\in C(\mathbb{R}_+,\mathcal{D}(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0\in\mathcal{D}(\mathcal{A})$ and by using (23), we have

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} &= -\mu_1\int_0^1\psi^2dx - \int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\psi y(x,1,\varrho,t)d\varrho dx \\ &- \int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y_\rho y d\varrho d\rho dx - \int_0^1\int_0^\infty g(p)\varphi_{xp}(p)\varphi_x(p)dpdx. \end{aligned} \tag{24}$$

For the third term of the RHS of (24), we have

$$\begin{aligned} -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y_\rho y d\varrho d\rho dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1|\mu_2(\varrho)|\frac{d}{d\rho}y^2 d\rho d\varrho dx \\ &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,0,\varrho,t)d\varrho dx. \end{aligned} \tag{25}$$

Using Young's inequality, we obtain

$$\begin{aligned} -\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|\psi y(x,1,\varrho,t)d\varrho dx &\le \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\psi^2dx \\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{26}$$

By integrating the last term of the right-hand side of (24), we have

$$-\int_0^1\int_0^\infty g(p)\varphi_{xp}(p)\varphi_x(p)dpdx = \frac{1}{2}\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx. \tag{27}$$

Substituting (25), (26) and (27) into (24), using the fact that $y(x,0,\varrho,t) = \psi(x,t)$ and (15), we obtained

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\right) \int_0^1 \psi^2 dx + \frac{1}{2}\int_0^1 \int_0^\infty g'(p)\varphi_x^2(p) dp dx \\ &\leq 0. \end{aligned} \tag{28}$$

Hence, the operator $\mathcal{A}$ is dissipative.
Next, we prove that the operator $\mathcal{A}$ is maximal. This is enough to show that the operator $(\lambda I - \mathcal{A})$ is surjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, we prove that there is a unique $V = (u, v, \phi, \psi, y, \varphi) \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})V = F. \tag{29}$$

That is

$$\begin{cases} \lambda u - v = f_1 \in H_*^1(0,1) \\ \rho\lambda v - \mu u_{xx} - b\phi_x = \rho f_2 \in L_*^2(0,1) \\ \lambda\phi - \psi = f_3 \in H_0^1(0,1) \\ J\lambda\psi - l\phi_{xx} + bu_x + \xi\phi - \int_0^\infty g(p)\varphi_{xx}(p) dp \\ \qquad + \mu_1\psi + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho = J f_4 \in L^2(0,1) \\ \lambda\varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = \varrho f_5 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ \lambda\varphi + \varphi_s - \psi = f_6 \in L_g(0,1). \end{cases} \tag{30}$$

We note that the equation $(30)_5$ with $y(x,0,\varrho,t) = \psi(x,t)$ has a unique solution, given by

$$y(x,\rho,\varrho,t) = e^{-\lambda\rho\varrho}\psi + \varrho e^{\lambda\varrho\rho} \int_0^\rho e^{\lambda\varrho\sigma} f_5(x,\sigma,\varrho,t) d\sigma, \tag{31}$$

then

$$y(x,1,\varrho,t) = e^{-\lambda\varrho}\psi + \varrho e^{\lambda\varrho} \int_0^1 e^{\lambda\varrho\sigma} f_5(x,\sigma,\varrho,t) d\sigma, \tag{32}$$

and we infer from $(30)_6$ that

$$\varphi = e^{\lambda s} \int_0^s e^{\tau}(\psi + f_6(\tau)) d\tau, \tag{33}$$

and we have

$$v = \lambda u - f_1, \quad \psi = \lambda\phi - f_3. \tag{34}$$

Inserting (32), (33) and (34) in $(30)_2$ and $(30)_4$, we get

$$\begin{cases} \rho\lambda^2 u - \mu u_{xx} - b\phi_x = h_1 \in L_*^2(0,1) \\ \mu_3\phi - \mu_4\phi_{xx} + bu_x = h_2 \in L^2(0,1), \end{cases} \tag{35}$$

where

$$\begin{cases} \mu_3 = J\lambda^2 + \xi + \lambda\mu_1 + \lambda \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\lambda\varrho} d\varrho \\ \mu_4 = l + \int_0^\infty g(p)(1 - e^{\lambda p}) dp, \\ h_1 = \rho(\lambda f_1 + f_2) \\ h_2 = (J\lambda + \mu_1 + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\lambda\varrho} d\varrho) f_3 + J f_4 \\ \qquad - \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| e^{\lambda\varrho} \int_0^1 e^{\lambda\varrho\sigma} f_5(x,\sigma,\varrho,t) d\sigma d\varrho \\ \qquad + \int_0^\infty g(p) e^{\lambda p} \int_0^p e^{\tau} (\psi + f_6(\tau))_{xx} d\tau dp. \end{cases} \tag{36}$$

We multiply (35) by $\widehat{u}, \widehat{\phi}$, respectively and integrate their sum over $(0,1)$ to obtain the following variational formulation

$$B((u,\phi),(\widehat{u},\widehat{\phi})) = \Upsilon(\widehat{u},\widehat{\phi}), \tag{37}$$

where

$$B : (H^1_*(0,1) \times H^1_0(0,1))^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned} B((u,\phi),(\widehat{u},\widehat{\phi})) &= \lambda^2\rho \int_0^1 u\widehat{u}dx + \mu_3 \int_0^1 \phi\widehat{\phi}dx + \mu \int_0^1 u_x\widehat{u}_x dx \\ &\quad + \mu_4 \int_0^1 \phi_x\widehat{\phi}_x dx + b \int_0^1 (u_x\widehat{\phi} + \phi\widehat{u}_x) dx, \end{aligned} \tag{38}$$

and

$$\Upsilon : (H^1_*(0,1) \times H^1_0(0,1)) \to \mathbb{R},$$

is the linear functional given by

$$\Upsilon(\widehat{u},\widehat{\phi}) = \int_0^1 h_1\widehat{u}dx + \int_0^1 h_2\widehat{\phi}dx \tag{39}$$

Now, for $V = H^1_*(0,1) \times H^1_0(0,1)$, equipped with the norm

$$\|(u,\phi)\|_V^2 = \|u\|_2^2 + \|\phi\|_2^2 + \|u_x\|_2^2 + \|\phi_x\|_2^2,$$

we have

$$\begin{aligned} B((u,\phi),(u,\phi)) &= \lambda^2\rho \int_0^1 u^2 dx + \mu_3 \int_0^1 \phi^2 dx + \mu \int_0^1 u_x^2 dx \\ &\quad + 2b \int_0^1 u_x\phi dx + \mu_4 \int_0^1 \phi_x^2 dx. \end{aligned} \tag{40}$$

On the other hand, we can write

$$\begin{aligned} \mu u_x^2 + 2bu_x\phi + \mu_3\phi^2 &= \frac{1}{2}\Big[\mu\Big(u_x + \frac{b}{\mu}\phi\Big)^2 + \mu_3\Big(\phi + \frac{b}{\mu_3}u_x\Big)^2 \\ &\quad + \Big(\mu - \frac{b^2}{\mu_3}\Big)u_x^2 + \Big(\mu_3 - \frac{b^2}{\mu}\Big)\phi^2\Big]. \end{aligned}$$

Since $\mu\xi > b^2$, we deduce that

$$\mu u_x^2 + 2bu_x\phi + \mu_3\phi^2 > \frac{1}{2}\Big[\Big(\mu - \frac{b^2}{\mu_3}\Big)u_x^2 + \Big(\mu_3 - \frac{b^2}{\mu}\Big)\phi^2\Big],$$

then, for some $M_0 > 0$

$$B((u,\phi),(u,\phi)) \geq M_0\|(u,\phi)\|_V^2. \tag{41}$$

Thus, B is coercive, similarly,

$$\Upsilon(\widehat{u},\widehat{\phi}) \geq M_1\|(\widehat{u},\widehat{\phi})\|_V^2. \tag{42}$$

Consequently, using Lax–Milgram theorem, we conclude that (16) has a unique solution

$$(u,\phi) \in H_*^1(0,1 \times H_0^1(0,1).$$

Substituting u,ϕ into (32), (33) and (34), respectively, we have

$$\begin{aligned} &v \in H_*^1(0,1), \quad \psi \in H_0^1(0,1), \quad \varphi \in L_g(0,1) \\ &y, y_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)). \end{aligned} \tag{43}$$

Moreover, if we take $\widehat{u} = 0 \in H_*^1(0,1)$ in (37) to obtain

$$\mu_3\int_0^1 \phi\widehat{\phi}dx + b\int_0^1 u_x\widehat{\phi}dx + \mu_4\int_0^1 \phi_x\widehat{\phi}_xdx = \int_0^1 h_2\widehat{\phi}d, \tag{44}$$

we get

$$\mu_4\int_0^1 \phi_x\widehat{\phi}_xdx = \int_0^1 (h_2 - \mu_3\phi - bu_x)\widehat{\phi}dx, \quad \forall\widehat{\phi} \in H_0^1(0,1), \tag{45}$$

which yields

$$\mu_4\phi_{xx} = (h_2 - \mu_3\phi - bu_x) \in L^2(0,1). \tag{46}$$

Thus,

$$\phi \in H^2(0,1) \cap H_0^1(0,1). \tag{47}$$

Consequently, (45) takes the following form

$$\int_0^1 (-\mu_4\phi_{xx} - h_2 + \mu_3\phi + bu_x)\widehat{\phi}dx = 0, \quad \forall\widehat{\phi} \in H_0^1(0,1).$$

Hence, we get

$$-\mu_4\phi_{xx} + \mu_3\phi + bu_x = h_2.$$

This give $(35)_2$. Similarly, if we take $\widehat{\phi} = 0 \in H_0^1(0,1)$ in (37) to obtain

$$\mu\int_0^1 u_x\widehat{u}_xdx + b\int_0^1 \phi\widehat{u}_xdx + \lambda^2\rho\int_0^1 u\widehat{u}dx = \int_0^1 h_1\widehat{u}dx,$$

we obtain

$$\mu\int_0^1 u_x\widehat{u}_xdx = \int_0^1 (h_1 + b\phi_x - \lambda^2\rho u)\widehat{u}dx, \quad \forall\widehat{u} \in H_*^1(0,1), \tag{48}$$

which yields

$$-\mu u_{xx} = (h_1 + b\phi_x - \lambda^2\rho u) \in L_*^2(0,1). \tag{49}$$

Consequently, (48) takes the following form

$$\int_0^1 (-\mu u_{xx} - h_1 - b\phi_x + \lambda^2\rho u)\widehat{u}dx = 0, \quad \forall\widehat{u} \in H_*^1(0,1).$$

Hence, we obtain

$$-\mu u_{xx} - b\phi_x + \lambda^2\rho u = h_1.$$

This give $(35)_1$.
Moreover, (48) also holds for any $\Phi \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\mu \int_0^1 u_x \Phi_x dx + \int_0^1 (-h_1 - b\phi_x + \lambda^2 \rho u)\Phi dx = 0, \quad \forall \Phi \in C^1([0,1]). \tag{50}$$

Then, we obtain for any $\Phi \in C^1([0,1])$

$$u_x(1)\Phi(1) - u_x(0)\Phi(0) = 0. \tag{51}$$

Since Φ is arbitrary, we obtain that $u_x(0) = u_x(1) = 0$. Hence, $u \in H^2_*(0,1) \cap H^1_*(0,1)$. Therefore, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (29) is satisfied. Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Now, we prove that the operator Γ defined in (22) is locally Lipschitz in $\mathcal{H}$. Let

$$U = (u, v, \phi, \psi, y, \varphi)^T \in \mathcal{H}, \widehat{U} = (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\psi}, \widehat{y}, \widehat{\varphi})^T \in \mathcal{H}.$$

Then, we have

$$\|\Gamma(U) - \Gamma(\widehat{U})\|_{\mathcal{H}} \leq M_3 \|f(\psi) - f(\widehat{\psi})\|_{L^2(01)}.$$

By using (14) and Holder and Poincaré's inequalities, we can obtain

$$\begin{aligned} \|f(\psi) - f(\widehat{\psi})\|_{L^2(01)} &\leq k_0 (\|\psi\|_{2\beta}^{\beta} + \|\widehat{\psi}\|_{2\beta}^{\beta}) \|\psi - \widehat{\psi}\| \\ &\leq k_1 \|\psi_x - \widehat{\psi}_x\|_{L^2(01)}, \end{aligned}$$

which gives us

$$\|\Gamma(U) - \Gamma(\widehat{U})\|_{\mathcal{H}} \leq M_4 \|U - \widehat{U}\|_{\mathcal{H}}.$$

Then, the operator Γ is locally Lipschitz in $\mathcal{H}$. Consequently, the well-posedness result follows from the Hille–Yosida theorem. The proof is completed. □

3. Stability Result

In this section, we state and prove our decay result for the energy of the system (16)–(18) using the multiplier technique. We need the following Lemmas.

Lemma 1. *The energy functional $\mathcal{E}$, defined by*

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + l\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right] dx \\ &+ \frac{1}{2} \int_0^1 \int_0^\infty g(p)\varphi_x^2(p) dp dx \\ &+ \frac{1}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x, \rho, \varrho, t) d\varrho d\rho dx, \end{aligned} \tag{52}$$

satisfies

$$\begin{aligned} \mathcal{E}'(t) &\leq -\eta_0 \int_0^1 \phi_t^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(p)\varphi_x^2(p) dp dx + \alpha(t) \int_0^1 \phi_t f(\phi_t) dx \\ &\leq 0, \end{aligned} \tag{53}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho > 0$ and $\varphi(s) = \eta^t = \phi(x,t) - \phi(x, t-p)$.

Proof. Multiplying $(16)_1$ by u_t and $(16)_2$ by ϕ_t, then integration by parts over $(0,1)$ and using (17), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho u_t^2+\mu u_x^2+J\phi_t^2+\delta\phi_x^2+\xi\phi^2+2bu_x\phi\right]dx\\
&-\int_0^1\phi_{xt}\int_0^\infty g(p)\varphi_x(p)dpdx+\mu_1\int_0^1\phi_t^2dx\\
&+\int_0^1\phi_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx+\alpha(t)\int_0^1\phi_t f(\phi_t)dx=0.
\end{aligned}\tag{54}$$

The last term in the LHS of (54) is estimated as follows

$$\begin{aligned}
\int_0^1\phi_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx \quad\leq\quad & \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\phi_t^2dx \\
&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx,
\end{aligned}\tag{55}$$

and

$$\begin{aligned}
-\int_0^1\phi_{xt}\int_0^\infty g(p)\varphi_x(p)dpdx \quad\leq\quad & \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx\\
&-\frac{1}{2}\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx.
\end{aligned}\tag{56}$$

Now, multiplying the equation $(16)_3$ by $y|\mu_2(\varrho)|$ and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\quad\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx\\
&=\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|yy_\rho(x,\rho,\varrho,t)d\varrho d\rho dx\\
&=\quad -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\frac{d}{d\rho}y^2(x,\rho,\varrho,t)d\varrho d\rho dx\\
&=\quad \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|(y^2(x,0,\varrho,t)-y^2(x,1,\varrho,t))d\varrho dx\\
&=\quad \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\int_0^1\phi_t^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx.
\end{aligned}\tag{57}$$

Now, using (54), (55), (56) and (57), we have

$$\begin{aligned}
\mathcal{E}'(t)\quad\leq\quad & -\Big(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\phi_t^2dx+\frac{1}{2}\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx\\
&-\alpha(t)\int_0^1\phi_t f(\phi_t)dx,
\end{aligned}\tag{58}$$

then, by (10), there exists a positive constant η_0, such that

$$\mathcal{E}'(t)\leq-\eta_0\int_0^1\phi_t^2dx+\frac{1}{2}\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx-\alpha(t)\int_0^1\phi_t f(\phi_t)dx,\tag{59}$$

hence, by (11)–(15) we obtain $\mathcal{E}$ is a non-increasing function. □

Remark 1. *Using ($\mu\xi > b^2$), we conclude that the energy $\mathcal{E}(t)$ definie by (52) satisfies*

$$\begin{aligned}\mathcal{E}(t) &> \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \widehat{\mu} u_x^2 + J\phi_t^2 + l\phi_x^2 + \widehat{\xi}\phi^2\right]dx \\ &+\frac{1}{2}\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx, \end{aligned} \tag{60}$$

where

$$\widehat{\mu} = \frac{1}{2}(\mu - \frac{b^2}{\xi}) > 0, \quad \widehat{\xi} = \frac{1}{2}(\xi - \frac{b^2}{\mu}) > 0,$$

then $\mathcal{E}(t)$ is positive function.

Lemma 2. *The functional*

$$D_1(t) := J\int_0^1 \phi_t\phi dx + \frac{b\rho}{\mu}\int_0^1 \phi \int_0^x u_t(y)dydx + \frac{\mu_1}{2}\int_0^1 \phi^2 dx, \tag{61}$$

satisfies

$$\begin{aligned}D_1'(t) &\le -\frac{l}{2}\int_0^1 \phi_x^2 dx - \widehat{\mu}\int_0^1 \phi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx \\ &+c\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx + c\int_0^1 f^2(\phi_t)dx \\ &+c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx, \end{aligned} \tag{62}$$

where $\widehat{\mu} = \xi - \frac{b^2}{\mu} > 0$.

Proof. Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$\begin{aligned}D_1'(t) &= -l\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + \frac{b\rho}{\mu}\int_0^1 \phi_t \int_0^x u_t(y)dydx \\ &+\int_0^1 \phi_x \int_0^\infty g(p)\varphi_x(p)dpdx + \alpha(t)\int_0^1 \phi f(\phi_t)dx \\ &+J\int_0^1 \phi_t^2 dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx \\ &\le -l\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\ &+\varepsilon_1\int_0^1\left(\int_0^x u_t(y)dy\right)^2 dx + \int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)dpdx \\ &-\int_0^1 \phi\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx + \alpha(t)\int_0^1 \phi f(\phi_t)dx. \end{aligned} \tag{63}$$

According to Cauchy–Schwartz inequality, it is clear that

$$\int_0^1\left(\int_0^x u_t(y)dy\right)^2 dx \le \int_0^1\left(\int_0^1 u_t dx\right)^2 dx \le \int_0^1 {u_t}^2 dx.$$

Therefore, estimate (63) becomes

$$D_1'(t) \leq -\delta\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx + \varepsilon_1\int_0^1 u_t{}^2 dx - \int_0^1 \phi\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx + \int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)dpdx + \alpha(t)\int_0^1 \phi f(\phi_t)dx. \quad (64)$$

The last term in the RHS of (64) is estimated as follows

$$\int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)dpdx \leq c\delta_1\int_0^1 \phi_x^2 dx + \frac{c}{4\delta_1}\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx, \quad (65)$$

where we used Cauchy–Schwartz, Young and Poincaré's inequalities, for $\delta_1, \delta_2, \delta_3 > 0$. By substituting (65) into (63), we obtain

$$D_1'(t) \leq -(l - c\delta_1 - \mu_1 c\delta_2 - c\delta_3)\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx + \frac{c}{4\delta_1}\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx + \frac{1}{4\delta_2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx + \frac{1}{4\delta_3}\int_0^1 f^2(\phi_t)dx. \quad (66)$$

Bearing in mind that $\mu\xi > b^2$ and letting $\delta_1 = \frac{l}{6c}$, $\delta_2 = \frac{l}{6c\mu_1}$ and $\delta_3 = \frac{l}{6c}$, we obtain estimate (62). □

Lemma 3. *Then, for any $\varepsilon_2 > 0$ the functional*

$$D_2(t) := \int_0^1 \phi_x u_t dx + \int_0^1 \phi_t u_x dx - \frac{\rho}{\mu J}\int_0^1 u_t\int_0^\infty g(p)\phi_x(t-p)dpdx,$$

satisfies

$$D_2'(t) \leq -\frac{b}{2J}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\varepsilon_2\int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx + c\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx - \frac{c}{\varepsilon_2}\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx + c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx + c\int_0^1 f^2(\phi_t)dx + \left(\frac{\delta}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x\phi_{xx}dx. \quad (67)$$

Proof. By differentiating D_2, then using (16), integration by parts and (17) we obtain

$$D_2'(t) = -\frac{b}{J}\int_0^1 u_x^2 dx + \left(\frac{l+g_0}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x\phi_{xx}dx + (\frac{b}{\rho} - \frac{bg_0}{\mu J})\int_0^1 \phi_x^2 dx - \frac{\xi}{J}\int_0^1 u_x\phi dx - \frac{b}{\mu J}\int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)dpdx - \frac{\rho}{\mu J}\int_0^1 u_t\int_0^\infty g'(p)\varphi_x(p)dpdx - \frac{\alpha(t)}{\mu J}\int_0^1 u_x f(\phi_t)dx - \frac{\mu_1}{J}\int_0^1 \phi_t u_x dx - \frac{1}{J}\int_0^1 u_x\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx. \quad (68)$$

In what follows, we estimate the last six terms in the RHS of (68), using Young, Cauchy–Schwartz and Poincaré's inequalities. For $\delta_4, \delta_5, \varepsilon_2 > 0$, we have

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{\xi}{J}\delta_4\int_0^1 u_x^2 dx + \frac{\xi}{4J\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \frac{b}{6\xi}$, using Poincaré's inequality, we get

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{b}{6J}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx, \tag{69}$$

and by Young and Chauchy–Schawrz's inequalities, we get

$$-\frac{b}{\mu J}\int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)dpdx \leq c\delta_5\int_0^1 \phi_x^2 dx + \frac{c}{4\delta_5}\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx.$$

By letting $\delta_5 = \frac{b}{6cJ}$, we obtain

$$-\frac{b}{\mu J}\int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)dpdx \leq \frac{b}{6J}\int_0^1 \phi_x^2 dx + c\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx. \tag{70}$$

Similarly, $\forall\varepsilon_2 > 0$ we have

$$\frac{\rho}{\mu J}\int_0^1 u_t\int_0^\infty g'(p)\varphi_x(p)dpdx \leq c\varepsilon_2\int_0^1 u_t^2 dx + \frac{c}{\varepsilon_2}\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx, \tag{71}$$

and

$$-\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx \leq \frac{\mu_1\delta_6}{2J}\int_0^1 u_x^2 dx + \frac{\mu_1}{2J\delta_6}\int_0^1 \phi_t^2 dx, \tag{72}$$

and

$$\begin{aligned}\frac{1}{J}\int_0^1 u_x\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx \quad &\leq \quad \frac{\delta_7\mu_1}{2J}\int_0^1 u_x^2 dx \\ &+\frac{1}{2J\delta_7}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho,\end{aligned} \tag{73}$$

and

$$-\frac{\alpha(t)}{J}\int_0^1 u_x f(\phi_t)dx \leq \frac{\alpha(0)\delta_8}{2J}\int_0^1 u_x^2 dx + \frac{\alpha(0)}{2J\delta_8}\int_0^1 f^2(\phi_t)dx. \tag{74}$$

Replacing (69)–(74) into (68) and letting $\delta_6 = \delta_7 = \frac{b}{6\mu_1}$ and $\delta_8 = \frac{b}{6\alpha(0)}$, yields (67). □

Lemma 4. *The functional*

$$D_3(t) := -\rho\int_0^1 u_t u dx,$$

satisfies

$$D_3'(t) \leq -\rho\int_0^1 u_t^2 dx + \frac{3\mu}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \tag{75}$$

Proof. Direct computations give

$$D_3'(t) = -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\int_0^1 u_x\phi dx.$$

The estimate (75) easily follows using Young and Poincaré inequalities.

$$\begin{aligned} D_3'(t) &\leq -\rho\int_0^1 u_t^2dx + \mu\int_0^1 u_x^2dx + b\varepsilon\int_0^1 u_x^2dx + \frac{b}{4\varepsilon}\int_0^1 \phi^2dx \\ &\leq -\rho\int_0^1 u_t^2dx + \mu\int_0^1 u_x^2dx + b\varepsilon\int_0^1 u_x^2dx + \frac{bc}{4\varepsilon}\int_0^1 \phi_x^2dx, \end{aligned}$$

by taking $\varepsilon = \frac{\mu}{2b}$, we obtain (75). □

Lemma 5. *The functional*

$$D_4(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho}|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx,$$

satisfies

$$\begin{aligned} D_4'(t) \leq\; & -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx + \mu_1\int_0^1 \phi_t^2dx \\ & -\eta_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx, \end{aligned} \tag{76}$$

where η_1 is a positive constant.

Proof. By differentiating D_4, with respect to t, using the Equation $(16)_3$, we have

$$\begin{aligned} D_4'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_2(\varrho)|yy_\rho(x,\rho,\varrho,t)d\varrho d\rho dx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho}|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|[e^{-\varrho}y^2(x,1,\varrho,t) - y^2(x,0,\varrho,t)]d\varrho dx. \end{aligned}$$

Using the fact that $y(x,0,\varrho,t) = \phi_t(x,t)$ and $e^{-\varrho} \leq e^{-\varrho\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} D_4'(t) =\; & -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\ & -\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx + \int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\int_0^1 \phi_t^2dx. \end{aligned}$$

□

Since $-e^{-\varrho}$ is an increasing function, we have $-e^{-\varrho} \leq -e^{-\tau_2}$, for all $\varrho \in [\tau_1,\tau_2]$. Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (15), we obtain (76). We are now ready to prove the main result.

Theorem 2. *Assume (10)–(15) hold. Let $h(t) = \alpha(t).\ \eta(t)$ be a positive non-increasing function. Then, for any $U_0 \in \mathcal{D}(\mathcal{A})$, satisfying, for some $c_0 > 0$*

$$max\left\{\int_0^1 \phi_{0x}^2(x,s)dx, \int_0^1 \phi_{0sx}^2(x,s)dx\right\} \leq c_0, \quad \forall s > 0, \tag{77}$$

there are positive constants β_1, β_2 and β_3 such that the energy functional given by (52) satisfies

$$\mathcal{E}(t) \leq \beta_1 G_0^{-1}\left(\frac{\beta_2 + \beta_3\int_0^t h(p)\varpi(p)dp}{\int_0^t h(p)dp}\right), \tag{78}$$

where

$$G_0(t) = tG'(\varepsilon_0 t), \forall \varepsilon_0 \geq 0, \ \ and \ \ \varpi(s) = \int_s^\infty g(\sigma)d\sigma. \tag{79}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := N\mathcal{E}(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \tag{80}$$

where N, N_1, N_2, and N_4 are positive constants, to be chosen later. By differentiating (80) and using (53), (62), (67), (75), (76), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2 - c\right]\int_0^1 \phi_x^2 dx - [\rho - N_1\varepsilon_1 - N_2 c\varepsilon_2]\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx + c[N_1 + N_2]\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\
& -\left[\eta_0 N - cN_1(1 + \frac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4\right]\int_0^1 \phi_t^2 dx \\
& -[N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho dx \\
& -N_1\widehat{\mu}\int_0^1 \phi^2 dx + \left[\frac{N}{2} - \frac{cN_2}{\varepsilon_2}\right]\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
& +c[N_1 + N_2]\int_0^1 f^2(\phi_t)dx + N_2\chi\int_0^1 u_{xx}\phi_x dx,
\end{aligned}$$

where $\chi = (\frac{\mu}{\rho} - \frac{\delta}{J})$ and by setting

$$\varepsilon_1 = \frac{\rho}{4N_1}, \varepsilon_2 = \frac{\rho}{4cN_2},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2(1 + N_2) - c\right]\int_0^1 \phi_x^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx + c[N_1 + N_2]\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\
& -[\eta_0 N - cN_1(1 + N_1) - cN_2 - \mu_1 N_4]\int_0^1 \phi_t^2 dx \\
& -[N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho dx \\
& -N_1\widehat{\mu}\int_0^1 \phi^2 dx + \left[\frac{N}{2} - cN_2^2\right]\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(s)| y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
& +c[N_1 + N_2]\int_0^1 f^2(\phi_t)dx + N_2\chi\int_0^1 u_{xx}\phi_x dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose a N_2 that is large enough that

$$\alpha_1 = \frac{bN_2}{2J} - \frac{3\mu}{2} > 0,$$

then, we choose a large enough N_1 that

$$\alpha_2 = \frac{lN_1}{4} - cN_2(1+N_2) - c > 0,$$

then we choose a large enough N_4 that

$$\alpha_3 = N_4\eta_1 - cN_1 - cN_2 > 0,$$

thus, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq\ & -\alpha_2 \int_0^1 \phi_x^2 dx - \alpha_0 \int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx - \alpha_1 \int_0^1 u_x^2 dx \\ & -[\eta_0 N - c]\int_0^1 \phi_t^2 dx + \left[\frac{N}{2} - c\right]\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\ & +c\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx - \alpha_3\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ & -\alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\ & +c\int_0^1 f^2(\phi_t)dx + \alpha_5\int_0^1 u_{xx}\phi_x dx. \end{aligned} \tag{81}$$

where $\alpha_0 = \widehat{\mu}N_1 = \left(\xi - \frac{b^2}{\mu}\right)N_1,\ \ and\ \ \alpha_5 = N_2\chi = N_2(\frac{\mu}{\rho} - \frac{\delta}{J})$. On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t),$$

then

$$\begin{aligned} |\mathfrak{L}(t)| \leq\ & JN_1\int_0^1|\phi\phi_t|dx + \frac{b\rho N_1}{\mu}\int_0^1\left|\phi\int_0^x u_t(y)dy\right|dx \\ & +N_2\int_0^1\left|\phi_x u_t + u_x\phi_t - \frac{\rho}{\mu J}u_t\int_0^\infty g(p)\phi_x(t-p)dp\right|dx \\ & +\rho\int_0^1|u_t u|dx + N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho e^{-\varrho\rho}|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx. \end{aligned}$$

Exploiting Young, Cauchy–Schwartz and Poincaré inequalities, we obtain

$$\begin{aligned} |\mathfrak{L}(t)| \leq\ & c\int_0^1\left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2\right)dx + c\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ & +c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(s)|y^2(x,\rho,\varrho,t)d\varrho d\rho \\ \leq\ & c\mathcal{E}(t). \end{aligned}$$

Consequently, we obtain

$$|\mathfrak{L}(t)| = |\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t),$$

that is

$$(N-c)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+c)\mathcal{E}(t). \tag{82}$$

Now, by choosing a large enough N that

$$\frac{N}{2} - c > 0, N - c > 0, N\eta_0 - c > 0,$$

and exploiting (52), estimates (81) and (82), respectively, we obtain

$$c_2\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_3\mathcal{E}(t), \forall\, t \geq 0, \tag{83}$$

and

$$\begin{aligned} \mathcal{L}'(t) \;&\leq\; -k_1\mathcal{E}(t) + k_2\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &\quad +k_3\int_0^1(\phi_t^2 + f^2(\phi_t))dx + \alpha_5\int_0^1 u_{xx}\phi_x dx, \end{aligned} \tag{84}$$

for some $k_1, k_2, k_3, c_2, c_3 > 0$.

Case 1. If $\chi = (\frac{\mu}{\rho} - \frac{\delta}{J}) = 0$, in this case, (84) takes the form

$$\begin{aligned} \mathcal{L}'(t) \;&\leq\; -k_1\mathcal{E}(t) + k_2\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &\quad +k_3\int_0^1(\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{85}$$

By multiplying (85) by $h(t) = \alpha(t).\eta(t)$, we obtain

$$\begin{aligned} h(t)\mathcal{L}'(t) \;&\leq\; -k_1h(t)\mathcal{E}(t) + k_2h(t)\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &\quad +k_3h(t)\int_0^1(\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{86}$$

We distinguish two cases

- G is linear on $[0,\varepsilon]$. In this case, using the assumption $(13)_1$ and (53), we can write

$$k_3h(t)\int_0^1(\phi_t^2 + f^2(\phi_t))dx \leq k_3h(t)\int_0^1 \phi_t f(\phi_t))dx \leq -k_3\eta(t)\mathcal{E}'(t), \tag{87}$$

and by (11) we have

$$\begin{aligned} h(t)\int_0^1\int_0^t g(p)\varphi_x^2(p)dpdx \;&=\; \alpha(t)\int_0^1\int_0^t \eta(s)g(p)\varphi_x^2(p)dpdx \\ &\leq\; -\alpha(t)\int_0^1\int_0^t g'(p)\varphi_x^2(p)dpdx \\ &\leq\; -\alpha(t)\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\ &\leq\; -2\alpha(t)\mathcal{E}'(t), \end{aligned} \tag{88}$$

and by (77) we obtain

$$\begin{aligned} \int_0^1 \varphi_x^2(s)dx \;&=\; 2\int_0^1 \phi_x^2(x,t)dx + 2\int_0^1 \phi_x^2(x,t-s)dx \\ &\leq\; 4\sup_{s>0}\int_0^1 \phi_x^2(x,s)dx + 2\sup_{\tau>0}\int_0^1 \phi_{0x}^2(x,\tau)dx \\ &\leq\; \frac{8\mathcal{E}(0)}{l} + 2c_0, \end{aligned} \tag{89}$$

then, we obtain

$$h(t)\int_0^1\int_t^\infty g(p)\varphi_x^2(p)dpdx \leq (\frac{8\mathcal{E}(0)}{l} + 2c_0)h(t)\int_t^\infty g(p)dp. \tag{90}$$

Hence,

$$h(t)\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \leq -2\alpha(t)\mathcal{E}'(t) + (\frac{8\mathcal{E}(0)}{l}+2c_0)h(t)\varpi(t). \tag{91}$$

Inserting (87) and (91) in (86). Since $h'(t)\leq 0, \alpha'(t)\leq 0, \eta'(t)\leq 0$. Then, we have

$$\mathcal{L}_1'(t) \leq -k_1h(t)\mathcal{E}(t)+\gamma h(t)\varpi(t), \tag{92}$$

and

$$m_1\mathcal{E}(t)\leq \mathcal{L}_1(t)\leq m_2\mathcal{E}(t), \tag{93}$$

with

$$m_1=\tau_1, \quad m_2=c_2h(0)+k_3\eta(0)+2k_2\alpha(0)+\tau_1,$$

where

$$\begin{aligned}\mathcal{L}_1(t) &= h(t)\mathcal{L}(t)+(k_3\eta(t)+2k_2\alpha(t)+\tau_1)\mathcal{E}(t)\sim\mathcal{E}(t), \\ \gamma &= (\frac{8\mathcal{E}(0)}{l}+2c_0), \quad \tau_1>0 \ \ and\ \ \varpi(t)=\int_t^\infty g(p)dp.\end{aligned} \tag{94}$$

Since $\mathcal{E}'(t)\leq 0, \forall\, t\geq 0$. By using (92), we have

$$\mathcal{E}(T)\int_0^T h(t)dt \leq \left(\frac{\mathcal{L}_1(0)}{k_1}+\frac{\gamma}{k_1}\int_0^T h(t)\varpi(t)dt\right). \tag{95}$$

Using the fact that G_0^{-1} is linear. Then,

$$\mathcal{E}(T)\leq \zeta G_0^{-1}\left(\frac{\frac{\mathcal{L}_1(0)}{k_1}+\frac{\gamma}{k_1}\int_0^T h(t)\varpi(t)dt}{\int_0^T h(t)dt}\right). \tag{96}$$

with $\beta_1=\zeta, \quad \beta_2=\frac{\mathcal{L}_1(0)}{k_1}, \quad \beta_3=\frac{\gamma}{k_1}$. This completes the proof.

- G is nonlinear on $[0,\varepsilon]$, we choose $0\leq\varepsilon_1\leq\varepsilon$ and we consider

$$I_1(t)=\{x\in(0,1), \ \ |\phi_t|\leq\varepsilon_1\}, \ \ I_2=\{x\in(0,1), \ \ |\phi_t|>\varepsilon_1\},$$

we define

$$I=\int_{I_1}\phi_t f(\phi_t)dt.$$

Using Jensen's inequality and the assumption $(13)_1$, we have

$$\begin{aligned}k_3h(t)\int_0^1(\phi_t^2+f^2(\phi_t))dx &\leq k_3h(t)\int_0^1\phi_t f(\phi_t))dx \\ &\leq k_3'h(t)G^{-1}(I(t))-k_3'\eta(t)\mathcal{E}'(t).\end{aligned} \tag{97}$$

Inserting (97) in (86), since $\alpha'(t)\leq 0, \eta'(t)\leq 0$ and $\mathcal{E}'(t)\leq 0$, we obtain

$$\mathcal{L}_2'(t)\leq -k_1h(t)\mathcal{E}(t)+\gamma h(t)\varpi(t)+k_3'h(t)G^{-1}(I(t)). \tag{98}$$

and

$$m_3\mathcal{E}(t)\leq\mathcal{L}_2(t)\leq m_4\mathcal{E}(t), \tag{99}$$

with

$$m_3=\tau_1, \quad m_4=c_2h(0)+k_3'\eta(0)+2k_2\alpha(0)+\tau_1,$$

where

$$\mathcal{L}_2(t) = h(t)\mathcal{L}(t) + (k_3'\eta(t) + 2k_2\alpha(t) + \tau_1)\mathcal{E}(t) \sim \mathcal{E}(t).$$

Now, for $\varepsilon_0 < \varepsilon_1$ and by using $\mathcal{E}'(t) \leq 0, G' > 0$ and $G'' > 0$ on $(0, \varepsilon]$, we define the functional $\mathcal{L}_3(t)$ by,

$$\mathcal{L}_3(t) = G'(\varepsilon_0\mathcal{E}(t))\mathcal{L}_2(t) + \tau_2\mathcal{E}(t) \sim \mathcal{E}(t), \quad \tau_2 > 0,$$

satisfies

$$\begin{aligned} \mathcal{L}_3'(t) &= \mathcal{E}'(t)(\varepsilon_0 G'(\varepsilon_0\mathcal{E}(t))\mathcal{L}_2(t) + \tau_2) + \mathcal{L}_2'(t)G'(\varepsilon_0\mathcal{E}(t)) \\ &\leq -k_1h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t) \\ &\quad + k_3'h(t)G'(\varepsilon_0\mathcal{E}(t))G^{-1}(I(t)). \end{aligned} \tag{100}$$

To estimate the last term of (92), using the general Young's inequality

$$AB \leq G^*(A) + G(B), \quad if \ \ A \in (0, G'(\varepsilon)), \ \ B \in (0, \varepsilon), \tag{101}$$

where

$$G^*(A) = s(G')^{-1}(s) - G((G')^{-1}(s)), \quad if \ \ s \in (0, G'(\varepsilon)),$$

satisfies

$$k_3'h(t)G'(\varepsilon_0\mathcal{E}(t))G^{-1}(I(t)) \leq k_3'\varepsilon_0 h(t)G_0(\mathcal{E}(t)) - k_3'\eta(t)\mathcal{E}'(t). \tag{102}$$

Inserting (102) in (92) and letting $\varepsilon_0 = \frac{k_1}{2k_3'}$ we get

$$\mathcal{L}_3'(t) + k_3'\eta(t)\mathcal{E}'(t) \leq -k_1h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t). \tag{103}$$

Since $\eta'(t) \leq 0$, then

$$\mathcal{L}_4'(t) \leq -k_1h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t),$$

where

$$\mathcal{L}_4(t) = \mathcal{L}_3(t) + k_3'\eta(t)\mathcal{E}(t) \sim \mathcal{E}(t).$$

Since $\alpha(t), G_0(\mathcal{E}(t)), G'(\varepsilon_0\mathcal{E}(t))$ are non-increasing functions,
then, for any $T > 0$

$$\begin{aligned} k_1G_0(\mathcal{E}(T))\int_0^T h(t)dt &\leq k_1\int_0^T h(t)G_0(\mathcal{E}(t))dt \\ &\leq \mathcal{L}_4(0) + \gamma G'(\varepsilon_0\mathcal{E}(0))\int_0^T h(t)\varpi(t)dt, \end{aligned}$$

which gives (78) with $\beta_1 = 1, \ \ \beta_2 = \dfrac{\mathcal{L}_4(0)}{k_1}$ and $\beta_3 = \dfrac{\gamma G'(\varepsilon_0\mathcal{E}(0))}{k_1}$.
The proof is now completed.

Case 2. If $\chi = (\frac{\mu}{\rho} - \frac{\delta}{J}) \neq 0$ and

$$\begin{cases} |\chi| < \dfrac{k_1\mu^2 l}{2N_2(l\rho + b\mu)} & if \quad \chi < 0 \\ |\chi| < \dfrac{k_1\mu^2}{2N_2\rho} & if \quad \chi > 0. \end{cases}$$

This case is more important from the physical perspective, where waves are not necessarily of equal speeds. Let

$$\mathcal{E}(t) = \mathcal{E}(u, \phi, y, \varphi) = \mathcal{E}_1(t).$$

Denotes the first-order energy defined in (52) and

$$\mathcal{E}_2(t) = \mathcal{E}(u_t, \phi_t, y_t, \varphi_t).$$

Denotes the second-order energy; then, we have

$$\begin{aligned} \mathcal{E}_2'(t) &\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(p)\varphi_{tx}^2(p)dp \\ &\quad -\alpha'(t) \int_0^1 \phi_{tt} f(\phi_t)dx - \alpha(t) \int_0^1 \phi_{tt}^2 f'(\phi_t)dx \\ &= -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(p)\varphi_{tx}^2(p)dp \\ &\quad +\alpha(t)\left(\frac{-\alpha'(t)}{\alpha(t)} \int_0^1 \phi_{tt} f(\phi_t)dx - \int_0^1 \phi_{tt}^2 f'(\phi_t)dx\right). \end{aligned} \tag{104}$$

Since f, g are non-decreasing functions, $\alpha(t)$ is a positive function and $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$, we deduce that

$$\begin{aligned} \mathcal{E}_2'(t) &\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(s)\varphi_{tx}^2 \\ &\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx, \end{aligned} \tag{105}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho > 0$.
The last term in (84), by using $(16)_1$, Young's inequality and by setting $K = \frac{\chi N_2 \rho}{\mu} = \frac{\alpha_5 \rho}{\mu}$ and $\alpha_5 = \chi N_2$ as follows

$$\begin{aligned} \alpha_5 \int_0^1 u_{xx}\phi_x dx &= \frac{\alpha_5 \rho}{\mu} \int_0^1 \phi_x u_{tt} dx - \frac{b\alpha_5}{\mu} \int_0^1 \phi_x^2 dx \\ &= K\left(\frac{d}{dt}\left[\int_0^1 \phi_t u_x dx + \int_0^1 \phi_x u_t dx\right]\right) \\ &\quad -K \int_0^1 u_x \phi_{tt}^2 dx - \frac{b\alpha_5}{\mu} \int_0^1 \phi_x^2 dx \\ &\leq K\left(\frac{d}{dt}\left[\int_0^1 \phi_t u_x dx + \int_0^1 \phi_x u_t dx\right]\right) \\ &\quad + \frac{|K|}{4} \int_0^1 \phi_{tt}^2 dx + |K| \int_0^1 u_x^2 dx. \end{aligned} \tag{106}$$

Let

$$\mathcal{N}(t) = \left(\int_0^1 \phi_t u_x dx + \int_0^1 \phi_x u_t dx\right),$$

then (84)

$$\begin{aligned}\mathcal{L}'(t)+K\mathcal{N}'(t) &\le -k_1\mathcal{E}_1(t)+k_2\int_0^1\int_0^\infty g(p)\varphi_p^2dpdx+\frac{|K|}{4}\int_0^1\phi_{tt}^2dx\\ &\quad +|K|\int_0^1 u_x^2dx+k_3\int_0^1(\phi_t^2+f^2(\phi_t))dx\\ &\le -k_4\mathcal{E}_1(t)+k_2\int_0^1\int_0^\infty g(p)\varphi_p^2dpdx\\ &\quad +\frac{|K|}{4}\int_0^1\phi_{tt}^2dx+k_3\int_0^1(\phi_t^2+f^2(\phi_t))dx, \end{aligned} \tag{107}$$

where

$$k_4=k_1-2\frac{|K|}{\mu}>0.$$

Let

$$\mathcal{R}(t)=\mathcal{L}(t)+K\mathcal{N}(t)+N_5(\mathcal{E}_1(t)+\mathcal{E}_2(t)). \tag{108}$$

Indeed, by using Young's inequality, we obtain

$$\begin{aligned}|\mathcal{N}(t)| &= |\int_0^1\phi u_{xt}dx|+|\int_0^1\phi_t u_x dx|\\ &\le \frac{1}{2}\int_0^1 u_t^2dx+\frac{1}{2}\int_0^1\phi_t^2dx+\frac{1}{2}\int_0^1\phi_x^2dx+\frac{1}{2}\int_0^1 u_x^2dx\\ &\le C_0\mathcal{E}_1(t), \end{aligned} \tag{109}$$

where $C_0=max\{\frac{1}{J},\frac{1}{\xi},\frac{1}{\rho},\frac{1}{\mu}\}$.
By (83) and (109), we obtain

$$|\mathcal{R}(t)-N_5(\mathcal{E}_1(t)+\mathcal{E}_2(t))|\le(c_3+C_0)\mathcal{E}_1(t)\le c(\mathcal{E}_1(t)+\mathcal{E}_2(t)), \tag{110}$$

and

$$(N_5-c)(\mathcal{E}_1(t)+\mathcal{E}_2(t))\le\mathcal{R}(t)\le(N_5+c)(\mathcal{E}_1(t)+\mathcal{E}_2(t)), \tag{111}$$

and by using (105), (107) and (3), we obtain

$$\begin{aligned}\mathcal{R}'(t) &= \mathcal{L}'(t)+K\mathcal{N}'(t)+N_5(\mathcal{E}_1'(t)+\mathcal{E}_2'(t))\\ &\le -k_4\mathcal{E}_1(t)+k_2\int_0^1\int_0^\infty g(p)\varphi_p^2dpdx\\ &\quad +k_3\int_0^1(\phi_t^2+f^2(\phi_t))dx-(\eta_0N_5-\frac{|K|}{4})\int_0^1\phi_{tt}^2dx. \end{aligned} \tag{112}$$

We choose a large enough N_5 that

$$\eta_0N_5-\frac{|K|}{4}>0,\quad N_5-c>0,$$

we obtain

$$\mathcal{R}(t)\sim(\mathcal{E}_1(t)+\mathcal{E}_2(t)), \tag{113}$$

and

$$\begin{aligned}\mathcal{R}'(t) &\le -k_4\mathcal{E}_1(t)+k_2\int_0^1\int_0^\infty g(p)\varphi_p^2dpdx\\ &\quad +k_3\int_0^1(\phi_t^2+f^2(\phi_t))dx. \end{aligned} \tag{114}$$

By multiplying (114) by $h(t) = \alpha(t).\eta(t)$, we obtain

$$\begin{aligned} h(t)\mathcal{R}'(t) &\leq -k_4 h(t)\mathcal{E}(t) + k_2 h(t)\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &+k_3 h(t)\int_0^1 (\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{115}$$

We distinguish two cases

- G is linear on $[0,\varepsilon]$. In the same way as in the previous case, we obtain

$$\mathcal{R}_1'(t) \leq -k_4 h(t)\mathcal{E}(t) + \gamma h(t)\varpi(t), \tag{116}$$

and

$$m_1(\mathcal{E}_1(t) + \mathcal{E}_2(t)) \leq \mathcal{R}_1(t) \leq m_2(\mathcal{E}_1(t) + \mathcal{E}_2(t)), \tag{117}$$

with

$$m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\begin{aligned} \mathcal{R}_1(t) &= h(t)\mathcal{R}(t) + (k_3\eta(t) + 2k_2\alpha(t) + \tau_1)\mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)) \\ \gamma &= (\frac{8\mathcal{E}(0)}{l} + 2c_0), \ \ \tau_1 > 0 \ \ and \ \ \varpi(t) = \int_t^\infty g(p)dp. \end{aligned}$$

Since $\mathcal{E}'(t) \leq 0, \forall t \geq 0$. By using (116), we have

$$\mathcal{E}(T)\int_0^T h(t)dt \leq \left(\frac{\mathcal{R}_1(0)}{k_4} + \frac{\gamma}{k_4}\int_0^T h(t)\varpi(t)dt\right). \tag{118}$$

Using the fact that G_0^{-1} is linear. Then,

$$\mathcal{E}(T) \leq \zeta G_0^{-1}\left(\frac{\frac{\mathcal{R}_1(0)}{k_4} + \frac{\gamma}{k_4}\int_0^T h(t)\varpi(t)dt}{\int_0^T h(t)dt}\right), \tag{119}$$

with $\beta_1 = \zeta, \quad \beta_2 = \frac{\mathcal{R}_1(0)}{k_4}, \quad \beta_3 = \frac{\gamma}{k_4}$. This completes the proof.

- G is nonlinear on $[0,\varepsilon]$, we choose $0 \leq \varepsilon_1 \leq \varepsilon$. In a similar way to that in the previous case, we have

$$\mathcal{R}_2'(t) \leq -k_1 h(t)\mathcal{E}(t) + \gamma h(t)\varpi(t) + k_3' h(t)G^{-1}(I(t)), \tag{120}$$

and

$$m_3(\mathcal{E}_1(t) + \mathcal{E}_2(t)) \leq \mathcal{R}_2(t) \leq m_4(\mathcal{E}_1(t) + \mathcal{E}_2(t)), \tag{121}$$

with

$$m_3 = \tau_1, \quad m_4 = c_2 h(0) + k_3'\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\mathcal{R}_2(t) = h(t)\mathcal{R}(t) + (k_3'\eta(t) + 2k_2\alpha(t) + \tau_1)\mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)).$$

Now, for $\varepsilon_0 < \varepsilon_1$, and by using $\mathcal{E}'(t) \leq 0, G' > 0$ and $G'' > 0$ on $(0,\varepsilon]$, we define the functional $\mathcal{L}_3(t)$ by,

$$\mathcal{R}_3(t) = G'(\varepsilon_0\mathcal{E}(t))\mathcal{R}_2(t) + \tau_2\mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)), \ \ \tau_2 > 0,$$

satisfies

$$\begin{aligned} \mathcal{R}_3'(t) &= \mathcal{E}'(t)(\varepsilon_0 G'(\varepsilon_0 \mathcal{E}(t))\mathcal{R}_2(t) + \tau_2) + \mathcal{R}_2'(t)G'(\varepsilon_0\mathcal{E}(t)) \\ &\leq -k_4 h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t) \\ &\quad + k_3' h(t)G'(\varepsilon_0\mathcal{E}(t))G^{-1}(I(t)). \end{aligned} \tag{122}$$

To estimate the last term of (122), again using the general Young's inequality (101). Inserting (122) in (121) and letting $\varepsilon_0 = \frac{k_1}{2k_3'}$, we get

$$\mathcal{R}_3'(t) + k_3'\eta(t)\mathcal{E}'(t) \leq -k_4 h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t). \tag{123}$$

Since $\eta'(t) \leq 0$, then

$$\mathcal{R}_4'(t) \leq -k_4 h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t),$$

where

$$\mathcal{R}_4(t) = \mathcal{R}_3(t) + k_3'\eta(t)\mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)).$$

Since $\alpha(t), G_0(\mathcal{E}(t)), G'(\varepsilon_0\mathcal{E}(t))$ are non-increasing functions, then, for any $T > 0$

$$\begin{aligned} k_4 G_0(E(T))\int_0^T h(t)dt &\leq k_4\int_0^T h(t)G_0(\mathcal{E}(t))dt \\ &\leq \mathcal{R}_4(0) + \gamma G'(\varepsilon_0\mathcal{E}(0))\int_0^T h(t)\varpi(t)dt, \end{aligned}$$

which gives (78) with $\beta_1 = 1,\ \ \beta_2 = \frac{\mathcal{R}_4(0)}{k_4}$ and $\beta_3 = \frac{\gamma G'(\varepsilon_0\mathcal{E}(0))}{k_4}$.
The proof is completed.

□

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References

1. Santos, M.L.; Junior, D.A. On porous-elastic system with localized damping. *Z. Angew. Math. Phys.* **2016**, *67*, 63. [CrossRef]
2. Bahri, N.; Beniani, A.; Zennir, K.; Zhang, H.W. Existence and exponential stability of solutions for laminated viscoelastic Timoshenko beams. *Appl. Sci.* **2020**, *22*, 1–16.
3. Bouzettouta, L.; Zennir, K.; Zitouni, S. Uniform Decay for a Viscoelastic Wave Equation with Density and Time-Varying Delay in $\mathbf{R}^n$. *Filomat* **2019**, *33*, 961–970.
4. Djebabla, A.; Choucha, A.; Ouchenane, D.; Zennir, K. Explicit Stability for a Porous Thermoelastic System with Second Sound and Distributed Delay. *Int. J. Appl. Comput. Math.* **2021**, *7*, 50. [CrossRef]
5. Dridi, H.; Zennir, K. Well-posedness and energy decay for some thermoelastic systems of Timoshenko type with Kelvin–Voigt damping. *SeMA* **2021**, *78*, 385–400. [CrossRef]
6. Ouchenane, D.; Zennir, K.; Guidad, D. Well-posedness and a general decay for a nonlinear damped porous thermoelastic system with second sound and distributed delay terms. *J. Appl. Nonlinear Dyn.* in press.
7. Ouchenane, D.; Zennir, K. General decay of solutions in one-dimensional porous-elastic system with memory and distributed delay term with second sound. *Commun. Optim. Theory* **2020**, *2020*, 18.
8. Rivera, J.E.M.; Pamplona, P.X.; Quintanilla, R. On the decay of solutions for porous-elastic system with history. *J. Math. Anal. Appl.* **2011**, *379*, 682–705.

9. Zennir, K. Stabilization for Solutions of Plate Equation with Time-Varying Delay and Weak-Viscoelasticity in $\mathbf{R^n}$. *Russ. Math.* **2020**, *64*, 21–33. [CrossRef]
10. Quintanilla, R. Slow decay in one-dimensional porous dissipation elasticity. *Appl. Math. Lett.* **2003**, *16*, 487–491. [CrossRef]
11. Apalara, T.A. General decay of solution in one-dimensional porous-elastic system with memory. *J. Math. Anal. Appl.* **2017**, *469*, 457–471. [CrossRef]
12. Zennir, K.; Ouchenane, D.; Choucha, A.; Biomy, M. Well-posedness and stability for Bresse–Timoshenko type systems with thermodiffusion effects and nonlinear damping. *AIMS Math.* **2021**, *6*, 2704–2721. [CrossRef]
13. Hebhoub, F.; Zennir, K.; Miyasita, T.; Biomy, M. Blow up at well defined time for a coupled system of one spatial variable Emden–Fowler type in viscoelasticities with strong nonlinear sources. *AIMS Math.* **2021**, *6*, 442–455. [CrossRef]
14. Feng, B. On the decay rates for a one-dimensional porous elasticity system with past history. *Commun. Pure Appl. Anal.* **2019**, *19*, 2905–2921. [CrossRef]
15. Choucha, A.; Ouchenane, D.; Zennir, K. General Decay of Solutions in One-Dimensional Porous-Elastic with Memory and Distributed Delay Term. *Tamkang J. Math.* **2021**, *52*, 1–18. [CrossRef]
16. Khochemane, H.; Bouzettouta, L.; Zitouni, S. General decay of a nonlinear damping porous-elastic system with past history. *Ann. Univ. Ferrara* **2019**, *65*, 249–275. [CrossRef]
17. Khochemane, H.; Djebabla, A.; Zitouni, S.; Bouzettouta, L. Well-posedness and general decay of a nonlinear damping porous-elastic system with infinite memory. *J. Math. Phys.* **2020**, *61*, 021505. [CrossRef]
18. Apalara, T.A. Exponential decay in one-dimensional porous dissipation elasticity. *Q. J. Mech. Appl. Math.* **2017**, *70*, 363–372, Correction in **2017**, *70*, 553–555. [CrossRef]
19. Apalara, T.A.; Soufyane, A. Energy decay for a weakly nonlinear damped porous system with a nonlinear delay. *Appl. Anal.* **2021**. [CrossRef]
20. Apalara, T.A. A general decay for a weakly nonlinearly damped porous system. *J. Dyn. Control Syst.* **2019**, *25*, 311–322. [CrossRef]
21. Dos Santos, M.J.; Feng, B.; Junior, D.S.A.; Santos, M.L. Global and exponential attractors for a nonlinear porous elastic system with delay term. *Discrete Contin. Dyn. Syst. B* **2021**, *26*, 2805.
22. Freitas, M.M.; Santos, M.L.; Langa, J.A. Porous elastic system with nonlinear damping and sources terms. *J. Diff. Equ.* **2018**, *264*, 2970–3051. [CrossRef]
23. Nicaise, A.S.; Pignotti, C. Stabilization of the wave equation with boundary or internal distributed delay. *Differ. Integral Equ.* **2008**, *21*, 935–958.
24. Fernandez Sare, H.D.; Munnoz Rivera, J.E. Stability of Timoshenko systems with past history. *J. Math. Anal. Appl.* **2008**, *339*, 482–502.