Journal of Advances in Mathematics and Computer Science

Volume 38, Issue 6, Page 6-15, 2023; Article no.JAMCS.97543
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Fixed Points of ξ-(α, β)-Contractive Mappings in b-Metric Spaces

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Author's contribution

This work was carried out in collaboration among all authors. All the authors have equally contributed in the planning, execution and analysis of study.

Article Information

DOI: 10.9734/JAMCS/2023/v38i61764

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/97543

Received: 10/01/2023
Accepted: 16/03/2023
Published: 22/03/2023

Original Research Article

Abstract

In the paper [*Some new observations on Geraghty and Ćirić type results in b-metric spaces*, Mathematics, **7**, (2019), doi: 10.3390/math7070643] Mlaiki *et al.* introduced (α, β)-type contraction in order to generalize the contraction mapping defined by Pant and Panicker. Also, in the paper [*Some fixed point results in b-metric spaces and b-metric-like spaces with new contractive mappings,* Axioms, **10(2)**, (2021), 15 pages, doi: 10.3390/axioms10020055] Jain and Kaur presented the concepts of ξ-contractive mappings. Now, the aim of the present article is to introduce ξ-(α, β)-contractive mappings in b-metric spaces by combining the concepts (α, β)-type contraction and ξ-contractive mappings. Also, we establish some fixed point results for newly defined mappings. Our results generalize various theorems in literature. In support, we provide an example.

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Keywords: b-metric space; (α,β)-admissible mappings; (α,β)-type contraction; ξ-contractive mappings; Cauchy Sequence; fixed point.

2010 Mathematics Subject Classification: 47H10, 54H25.

1 Introduction and Preliminaries

In 1922, Banach [1] initiated the study of constructive theory in metric space. Banach Contraction Principle (BCP) is one of the most useful and important theorems in classical functional analysis. This theorem has been extended and generalized in many directions. In 1969, Nadlar [2] proved BCP for a multivalued mapping. Various other extensions of this principle can be found in [3, 4, 5, 6, 7, 8]. Also, in the last decade Samet *et al.* [9] defined α-ψ-contractive mapping to generalize BCP. Later, in 2015, Chandok [10] introduced the concept of (α,β)-admissible mappings. In 2016, Pant and Panicker [11] obtained some results for (α,β)-Geraghty type contractive mapping. Mlaiki *et al.* [12] generalize the results of Pant and Panicker in 2019. Recently, Jain and Kaur [13] presented the concept of ξ-contractive mappings. As an extension of metric space, Bakhtin [14] has studied the concept of b-metric space (by weakening the triangle inequality in metric space) as:

Definition 1.1. [14] Let X be a non-empty set. Then a mapping $d : X \times X \to [0,+\infty)$ is called a *b-metric*, if for all $x, y, z \in X$, the following conditions hold:
$(bm1)$ $d(x,y) = 0$ if and only if $x = y$;
$(bm2)$ $d(x,y) = d(y,x)$;
$(bm3)$ $d(x,y) \leq 2(d(x,z) + d(z,y))$.
The pair (X,d) is called a *b-metric space*.

In 1998, Czerwik [15] presented the concept of b-metric space in the more general form as shown below:

Definition 1.2. [15] Let X be a non-empty set and $s \geq 1$ be a given real. Then a mapping $d : X \times X \to [0,+\infty)$ is called a *b-metric*, if for all $x, y, z \in X$, the following conditions hold:
$(b1)$ $d(x,y) = 0$ if and only if $x = y$;
$(b2)$ $d(x,y) = d(y,x)$;
$(b3)$ $d(x,y) \leq s(d(x,z) + d(z,y))$.
The pair (X,d) is called a *b-metric space*.
In 2010, Khamsi and Hussain [16] used the notion of b-metric space under the name *metric type space*. More on b-metric spaces can be studied in [17, 18, 19, 20, 21, 22, 23, 24, 25].

To prove our results, we need the following concepts and results of literature.

Definition 1.3. [26] Let (X,d) be a b-metric space. Then, a sequence $\{x_n\}$ in X is called :
(i) *Cauchy sequence*, if for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, for all $n, m \geq n_0$.
(ii) *Convergent*, if there exists $l \in X$ such that for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, l) < \epsilon$, for all $n \geq n_0$. In this case, sequence $\{x_n\}$ is said to *converge* to l.

Definition 1.4. [26] A b-metric space (X,d) is said to be *complete* if every Cauchy sequence is convergent in it.

Lemma 1.1. *[27] Let (X,d) be a b-metric space with $s \geq 1$ and suppose that sequences $\{x_n\}$ and $\{y_n\}$ converge to x and $y \in X$, respectively. Then*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to+\infty} d(x_n, y_n) \leq \limsup_{n\to+\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to+\infty} d(x_n, y_n) = 0$.
Moreover, for any $z \in X$,

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to+\infty} d(x_n, z) \leq \limsup_{n\to+\infty} d(x_n, z) \leq s d(x,z).$$

Lemma 1.2. *[28, 29] Every sequence $\{x_n\}$ of elements from a b-metric space (X,d), having the property that there exists $\lambda \in [0,1)$ such that $d(x_n,x_{n+1}) \leq \lambda d(x_{n-1},x_n)$ for every $n \in \mathbb{N}$, is Cauchy.*

Definition 1.5. [10] Let (X,d) be a b-metric space, $T : X \to X$ and $\alpha,\beta : X \times X \to [0,+\infty)$. Then mapping T is said to be (α,β)-*admissible*, if $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ and $\beta(Tx,Ty) \geq 1$, for all $x,y \in X$.

Definition 1.6. [10] Let (X,d) be a b-metric space, $T : X \to X$ and $\alpha,\beta : X \times X \to [0,+\infty)$. Then X is said to be (α,β)-*regular*, if $\{x_n\}$ is a sequence in X such that $\{x_n\}$ converges to x, $\alpha(x_n,x_{n+1}) \geq 1$, $\beta(x_n,x_{n+1}) \geq 1$ for all n, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_{n_k+1}) \geq 1$, $\beta(x_{n_k},x_{n_k+1}) \geq 1$ for all k and $\alpha(x,Tx) \geq 1$, $\beta(x,Tx) \geq 1$.

Definition 1.7. [10] Let Ψ be the set of all functions $\psi : [0,+\infty) \to [0,+\infty)$ such that ψ is continuous, strictly increasing and $\psi(0) = 0$.

Definition 1.8. [10] Let Θ be the set of all functions $\theta : [0,+\infty) \to [0,1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ implies $t_n \to 0$.

In 2016, Pant and Panicker [11] introduced the following contractive mapping.

Definition 1.9. [11] Let (X,d) be a b-metric space with $s \geq 1$. Then a mapping $T : X \to X$ is said to be (α,β)-*Geraghty type contractive mapping*, if there exists $\alpha,\beta : X \times X \to [0,+\infty)$, $\psi \in \Psi, \theta \in \Theta$ such that

$$\alpha(x,Tx)\beta(y,Ty)\psi(s^3 d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y)), \tag{1.1}$$

for all $x,y \in X$, where $N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2s}\right\}$.

In 2019, Mlaiki *et al.* [12] established the following contraction mapping which generalizes the contractive mapping defined by Pant and Panicker.

Definition 1.10. [12] Let (X,d) be a b-metric space with $s > 1$. Then mapping $T : X \to X$ is said to be an (α,β)-*type contraction,* if there exists $\alpha,\beta : X \times X \to [0,+\infty)$, $\psi \in \Psi$, $\epsilon > 1$ such that

$$\alpha(x,Tx)\beta(y,Ty)\psi(s^{\epsilon} d(Tx,Ty)) \leq \psi(N(x,y)), \tag{1.2}$$

for all $x,y \in X$, where $N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2s}\right\}$.

Theorem 1.3. *[12] Let (X,d) be a complete b-metric space with $s > 1$ and $T : X \to X$ be a mapping such that the following hold:*
(i) T is (α,β)-type contraction;
(ii) T is (α,β)-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$;
(iv) either T is continuous or X is (α,β)-regular;
Then T has a unique fixed point.

Following concepts and results were introduced by Jain and Kaur [13].

Definition 1.11. [13] For any $m \in \mathbb{N}$, define Ξ_m to be the set of all functions $\xi : [0,+\infty)^m \to [0,+\infty)$ such that
(ξ_1) $\xi(t_1,t_2,\cdots,t_m) < \max\{t_1,t_2,\cdots,t_m\}$ if $(t_1,t_2,\cdots,t_m) \neq (0,0,\cdots,0)$;
(ξ_2) if $\left\{t_i^{(n)}\right\}_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are m sequences in $[0,+\infty)$ such that
$\limsup\limits_{n\to+\infty} t_i^{(n)} = t_i < +\infty$ for all $i = 1$ to m, then
$\liminf\limits_{n\to+\infty} \xi\left(t_1^{(n)}, t_2^{(n)}, \cdots, t_m^{(n)}\right) \leq \xi(t_1,t_2,\cdots,t_m)$.

Definition 1.12. [13] Let (X, d) be a b-metric space with $s \geq 1$. The mapping $T : X \to X$ is said to be an *ξ-contractive mapping of type-I*, if there exists $\xi \in \Xi_4$ and

$$d(Tx, Ty) \leq \frac{1}{s}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s}\right), \tag{1.3}$$

for all $x, y \in X$.

Theorem 1.4. *[13] Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be an ξ-contractive mapping of type-I. Then T has a unique fixed point.*

Definition 1.13. [13] Let (X, d) be a b-metric space with $s \geq 1$. The mapping $T : X \to X$ is said to be an *ξ-contractive mapping of type-II*, if there exists $\xi \in \Xi_5$ and

$$d(Tx, Ty) \leq \frac{1}{s}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, d(Tx, y)\right), \tag{1.4}$$

for all $x, y \in X$.

Theorem 1.5. *[13] Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be an ξ-contractive mapping of type-II. Then T has a unique fixed point.*

2 Main Results

In the present work, first we introduce ξ-(α, β)-contractive mapping of type-I and ξ-(α, β)-contractive mapping of type-II in b-metric spaces. After that, we prove some unique fixed point theorems which generalize various results in literature.

Definition 2.1. Let (X, d) be a b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping. We say that T is an *ξ-(α, β)-contractive mapping of type-I*, if there exists $\alpha, \beta : X \times X \to [0, +\infty)$, $\psi \in \Psi$, $\xi \in \Xi_4$, $\epsilon > 1$ such that

$$\begin{aligned}&\alpha(x, Tx)\beta(y, Ty)\psi(s^{\epsilon} d(Tx, Ty))\\ \leq\ &\psi\left(s^{\epsilon-1}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s}\right)\right),\end{aligned} \tag{2.1}$$

for all $x, y \in X$.

Definition 2.2. Let (X, d) be a b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping. We say that T is an *ξ-(α, β)-contractive mapping of type-II*, if there exists $\alpha, \beta : X \times X \to [0, +\infty)$, $\psi \in \Psi$, $\xi \in \Xi_5$, $\epsilon > 1$ such that

$$\begin{aligned}&\alpha(x, Tx)\beta(y, Ty)\psi(s^{\epsilon} d(Tx, Ty))\\ \leq\ &\psi\left(s^{\epsilon-1}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, d(Tx, y)\right)\right),\end{aligned} \tag{2.2}$$

for all $x, y \in X$.

Theorem 2.1. *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping such that the following hold:*
(i) T is ξ-(α, β)-contractive mapping of type-I;
(ii) T is (α, β)-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(iv) either T is continuous or X is (α, β)-regular with respect to T;
Then T has a fixed point. If $Tx = x$ implies $\beta(x, x) \geq 1$, then fixed point of T is unique.

Proof. Define a sequence $\{x_n\}$ in X as $x_n = Tx_{n-1}$ for all $n \geq 1$. Assume that any two consecutive terms of sequence $\{x_n\}$ are distinct, otherwise T has a fixed point. Now by condition (ii), (iii) and using induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \beta(x_n, x_{n+1}) \geq 1 \text{ for all } n \geq 1.$$

First, we prove that $\{x_n\}$ is a Cauchy sequence. For this, let $n \in \mathbb{N}$.
Consider

$$\begin{aligned} &\psi(s^\epsilon d(x_n, x_{n+1})) \\ \leq\ & \alpha(x_{n-1}, x_n)\beta(x_n, x_{n+1})\psi(s^\epsilon d(x_n, x_{n+1})) \\ =\ & \alpha(x_{n-1}, Tx_{n-1})\beta(x_n, Tx_n)\psi(s^\epsilon d(Tx_{n-1}, Tx_n)) \\ \leq\ & \psi\left(s^{\epsilon-1}\xi\left(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\right)\right) \end{aligned}$$

which implies that

$$s^\epsilon d(x_n, x_{n+1}) \leq s^{\epsilon-1}\xi\left(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right),$$

i.e.,

$$d(x_n, x_{n+1}) \leq \frac{1}{s}\xi\left(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right) \tag{2.3}$$

$$\begin{aligned} &< \frac{1}{s}\max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right\} \\ &= \frac{1}{s}\max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1})}{2s}\right\} \\ &\leq \frac{1}{s}\max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\}, \end{aligned} \tag{2.4}$$

which implies that

$$d(x_n, x_{n+1}) < \frac{1}{s}d(x_{n-1}, x_n), \qquad \text{for all } n \geq 1. \tag{2.5}$$

Case 1: If $s > 1$, then by Lemma 1.2 in view of (2.5), $\{x_n\}$ is a Cauchy sequence.
Case 2: If $s = 1$, then by (2.5), sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing and also it is bounded below, therefore, $d(x_n, x_{n+1}) \to k$ for some $k \geq 0$. Suppose that $k > 0$. Now taking $\liminf\ n \to +\infty$ in (2.3), we have $k \leq \xi(k, k, k, k')$,
where

$$k' = \limsup_{n\to+\infty} \frac{d(x_{n-1}, x_{n+1})}{2} \leq \limsup_{n\to+\infty} \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} = k.$$

Now, $k \leq \xi(k, k, k, k') < \max\{k, k, k, k'\} = k$, a contradiction, therefore,

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Suppose that $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ such that for any $r \in \mathbb{N}$, there exists $m_r > n_r \geq r$ such that

$$d(x_{m_r}, x_{n_r}) \geq \varepsilon. \tag{2.7}$$

Also, assume that m_r is smallest natural greater than n_r such that (2.7) holds. Now

$$\begin{aligned} \varepsilon &\leq d(x_{m_r}, x_{n_r}) \\ &\leq d(x_{m_r}, x_{m_r-1}) + d(x_{m_r-1}, x_{n_r}) \\ &< d(x_{m_r}, x_{m_r-1}) + \varepsilon \\ &< d(x_r, x_{r-1}) + \varepsilon, \end{aligned}$$

so by using (2.6) and taking lim $r \to +\infty$, we get

$$\lim_{r\to+\infty} d(x_{m_r}, x_{n_r}) = \varepsilon. \tag{2.8}$$

Consider

$$\begin{aligned}
&\psi(d(x_{m_r+1}, x_{n_r+1}))\\
&\le \alpha(x_{m_r}, x_{m_r+1})\beta(x_{n_r}, x_{n_r+1})\psi(d(x_{m_r+1}, x_{n_r+1}))\\
&= \alpha(x_{m_r}, Tx_{m_r})\beta(x_{n_r}, Tx_{n_r})\psi(d(Tx_{m_r}, Tx_{n_r}))\\
&\le \psi\left(\xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2}\right)\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
&d(x_{m_r+1}, x_{n_r+1})\\
&\le \xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2}\right).
\end{aligned}$$

Now

$$\begin{aligned}
&d(x_{m_r}, x_{n_r})\\
&\le d(x_{m_r}, x_{m_r+1}) + d(x_{m_r+1}, x_{n_r+1}) + d(x_{n_r+1}, x_{n_r})\\
&\le d(x_{m_r}, x_{m_r+1}) + d(x_{n_r+1}, x_{n_r}) +\\
&\quad \xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2}\right),
\end{aligned}$$

thus, by taking lim inf $r \to +\infty$ on both sides and also using (2.6) and (2.8), we get $\varepsilon \le 0 + 0 + \xi(\varepsilon, 0, 0, \varepsilon')$, where

$$\begin{aligned}
\varepsilon' &= \limsup_{r\to+\infty} \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2}\\
&\le \limsup_{r\to+\infty} \frac{d(x_{m_r}, x_{n_r}) + d(x_{n_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{m_r}) + d(x_{m_r}, x_{n_r})}{2}\\
&= \frac{\varepsilon + 0 + 0 + \varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $\varepsilon \le \xi(\varepsilon, 0, 0, \varepsilon') < \max\{\varepsilon, 0, 0, \varepsilon'\} = \varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in (X, d). But (X, d) be a complete b-metric space, therefore, there exists $x \in X$ such that $x_n \to x$.
If T is continuous, then $x_n \to x$ implies $x_{n+1} = Tx_n \to Tx$. But, as limit is unique, therefore, $Tx = x$.
If X is (α, β)-regular with respect to T, then $\alpha(x, Tx) \ge 1$ and $\beta(x, Tx) \ge 1$. Suppose that $Tx \ne x$. Now

$$\begin{aligned}
\psi(s^{\epsilon} d(Tx_n, Tx)) &\le \alpha(x_n, Tx_n)\beta(x, Tx)\psi(s^{\epsilon} d(Tx_n, Tx))\\
&\le \psi\left(s^{\epsilon-1}\xi\left(d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2s}\right)\right),
\end{aligned}$$

which implies that

$$s^{\epsilon} d(Tx_n, Tx) \le s^{\epsilon-1}\xi\left(d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s}\right),$$

i.e.,

$$d(x_{n+1}, Tx) \leq \frac{1}{s}\xi\left(d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s}\right).$$

Taking $\liminf\ n \to +\infty$ on both sides and using Lemma 1.1, we get

$$\frac{1}{s}d(x, Tx) \leq \frac{1}{s}\xi\left(0, 0, d(x, Tx), l\right),$$

i.e.,

$$d(x, Tx) \leq \xi\left(0, 0, d(x, Tx), l\right),$$

where,

$$l = \limsup_{n\to+\infty} \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s} \leq \limsup_{n\to+\infty} \frac{sd(x, Tx) + 0}{2s} = \frac{d(x, Tx)}{2}.$$

Hence,

$$d(x, Tx) \leq \xi\left(0, 0, d(x, Tx), l\right) < \max\{0, 0, d(x, Tx), l\} = d(x, Tx),$$

which is a contradiction, therefore, $Tx = x$.
If $Ty = y$ for some $y \in X$, then by the given condition, $\beta(y, y) \geq 1$. Now suppose that $x \neq y$, and consider

$$\begin{aligned} \psi(s^{\epsilon}d(x, y)) &= \psi(s^{\epsilon}d(Tx, Ty)) \\ &\leq \alpha(x, Tx)\beta(y, Ty)\psi(s^{\epsilon}d(Tx, Ty)) \\ &\leq \psi\left(s^{\epsilon-1}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right)\right), \end{aligned}$$

which implies that

$$s^{\epsilon}d(x, y) \leq s^{\epsilon-1}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right),$$

i.e.,

$$\begin{aligned} d(x, y) &\leq \frac{1}{s}\xi\left(d(x, y), 0, 0, \frac{d(x, y)}{s}\right) \\ &< \frac{1}{s}\max\left\{d(x, y), 0, 0, \frac{d(x, y)}{s}\right\} \\ &= \frac{d(x, y)}{s}, \end{aligned}$$

a contradiction, therefore, $x = y$. ■

Example 2.2. *Let* $A = \left\{\frac{1}{\sqrt{n}} \mid n \in \mathbb{N}\right\} \cup \{0\}$, $B = \{n+1 \mid n \in \mathbb{N}\}$ *and* $X = A \cup B$. *Define* $d : X \times X \to [0, +\infty)$ *by* $d(x, y) = |x - y|^2$ *for all* $x, y \in X$. *Then* d *is a b-metric on* X *with* $s = 2$.
Define $T : X \to X$ *by* $T\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{2(n+1)}}$, $T(n + 1) = n + 2$ *for all* $n \in \mathbb{N}$ *and* $T(0)=0$. *Define* $\alpha, \beta : X \times X \to [0, +\infty)$, $\psi \in \Psi$, $\xi \in \Xi_4$ *as:*

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1, & \text{if } x, y \in A, \\ 0, & \text{otherwise}, \end{cases} \tag{2.9}$$

$\psi(t) = t$, *and*

$$\xi(t_1, t_2, t_3, t_4) = \begin{cases} \frac{\max\{t_1, t_2, t_3, t_4\}}{1+t_1}, & \text{if } t_1 > 0, \\ \frac{1}{2}\max\{t_2, t_3, t_4\}, & \text{otherwise}. \end{cases}$$

Now for all $x, y \in X$, *(2.1) is satisfied. Also* T *is* (α, β)*-admissible,* X *is* (α, β)*-regular with respect to* T *and for* $x_0 = 1 \in X$, $\alpha(x_0, Tx_0) \geq 1$ *and* $\beta(x_0, Tx_0) \geq 1$. *Thus, all the conditions of Theorem 2.1 are satisfied and* 0 *is only fixed point of* T.

Now following remark improves our main result Theorem 2.1.

Remark 2.1. Theorem 2.1 is also valid, if term $\frac{d(x,Ty)+d(Tx,y)}{2s}$ in (2.1) is replaced by $\frac{d(x,Ty)+d(Tx,y)}{\delta s}$, where δ is a real number defined by

$$\delta = \begin{cases} 2, & \text{if } s = 1, \\ \delta', & \text{if } 1 < s \leq 2, \\ 1, & \text{if } s > 2, \end{cases}$$

where δ' is any number in $\left(\frac{2}{s}, 1 + \frac{1}{s}\right)$.

Corollary 2.3. *(Theorem 1.3) Let (X, d) be a complete b-metric space with $s > 1$ and $T : X \to X$ be a mapping such that the following hold:*
(i) T is (α, β)-type contraction;
(ii) T is (α, β)-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(iv) either T is continuous or X is (α, β)-regular;
Then T has a unique fixed point.

Proof. Take $\xi \in \Xi_4$ defined by $\xi(t_1, t_2, t_3, t_4) = \frac{1}{s^{\epsilon-1}} \max\{t_1, t_2, t_3, t_4\}$, then by Theorem 2.1, we get the result. ■

Corollary 2.4. *(Theorem 1.4) Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be an ξ-contractive mapping of type-I. Then T has a unique fixed point.*

Proof. Take $\alpha, \beta : X \times X \to [0, +\infty)$ such that $\alpha(x, y) = \beta(x, y) = 1$, for all $x, y \in X$, and $\psi \in \Psi$ by $\psi(t) = t$. Then by Theorem 2.1, T has a unique fixed point. ■

Remark 2.2. Example 2.2 is applicable for Theorem 2.1, but not for Corollary 2.4.

Remark 2.3. In view of Remark 2.1, Corollaries 2.3 and 2.4 are also valid, if term $\frac{d(x,Ty)+d(Tx,y)}{2s}$ is replaced by $\frac{d(x,Ty)+d(Tx,y)}{\delta s}$, where δ is same as defined in Remark 2.1.

Proof of our next result is on a similar manner as the proof of Theorem 2.1.

Theorem 2.5. *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping such that the following conditions hold:*
(i) T is ξ-(α, β)-contractive mapping of type-II;
(ii) T is (α, β)-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(iv) either T is continuous or X is (α, β)-regular with respect to T;
Then T has a fixed point. If $Tx = x$ implies $\beta(x, x) \geq 1$, then fixed point of T is unique.

Corollary 2.6. *(Theorem 1.5) Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be an ξ-contractive mapping of type-II. Then T has a unique fixed point.*

Proof. Take $\alpha, \beta : X \times X \to [0, +\infty)$ such that $\alpha(x, y) = \beta(x, y) = 1$, for all $x, y \in X$, and $\psi \in \Psi$ by $\psi(t) = t$. Then by Theorem 2.5, T has a unique fixed point. ■

Acknowledgement

Authors are thankful to the reviewers for their suggestions to improve the presentation of this paper.

Competing Interests

The authors declare no conflict of interest.

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