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A New Jarque-Bera Type Omnibus Goodness-of-fit Test for Multivariate Normality

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper, a new Jarque-Bera type statistic for assessing the multivariate normality of a multivariate datasets is obtained. The affine-invariant and consistent statistic is shown to follow, asymptotically, a chi-square distribution with 2 degrees of freedom. The critical values of the test were evaluated empirically through extensive simulation studies for different sample sizes and different random vector dimensions. Also, the empirical type-I-error rates and empirical powers of the proposed test were compared with some other well-known competing statistics in the literature. The results obtained showed that the proposed omnibus statistic is a powerful tool for assessing multivariate normality (MVN) of multivariate datasets.

Keywords: Asymptotic null distribution; empirical critical values; omnibus test; skewness and kurtosis measures; test for multinormality.

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1 Introduction

Skewness and kurtosis are two important characteristics of every statistical distribution as they determine the shape of a distribution. They measure the symmetry and peakedness of a distributionrespectively. Given a random variable X which is said to have a distribution $F(x)$, with a density function, $f(x)$, the skewness and kurtosis of X are defined respectively by:

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} \tag{1.1}$$

where μ_2, μ_3, and μ_4 are the second, third, and fourth moments of X with the ith moment as $\mu_i = E(X-\mu)^i$; $i = 1, 2, \ldots$. For a random sample of size n which is obtained from X, the measures of skewness and kurtosis in (1.1) are estimated by:

$$\sqrt{b_1} = \frac{\hat{\mu}_3}{\hat{\mu}_2^{3/2}} \text{ and } b_2 = \frac{\hat{\mu}_4}{\hat{\mu}_2^2} \tag{1.2}$$

where $\hat{\mu}_2$, $\hat{\mu}_3$, and $\hat{\mu}_4$ are the second, third and fourth sample moments of the sample, with the ith sample moment obtained as $\hat{\mu}_i = n^{-1}\sum_{j=1}^{n}\left(X_j - \overline{X}\right)^i$; $i = 1, 2, \ldots$; $\overline{X} = n^{-1}\sum_{j=1}^{n} X_j$.

Now, it is well known that a random variable X, which is normally distributed with parameters μ and σ^2, has skewness and kurtosis measures of zero and 3 respectively, irrespective of the values of μ and σ^2. As a result, any random sample with these measures in (1.2) deviating significantly from 0 and 3, respectively, can be said to have come from a distribution other than normal. Based on this statement of fact, Jaque and Bera [1, 2] used the methodology of Bowman and Shenton [3] to obtain a goodness-of-fit statistic for assessing normality of datasets. The statistic is given by:

$$JB_n = n\left[\frac{\left(\sqrt{b_1}\right)^2}{6} + \frac{(b_2-3)^2}{24}\right] \tag{1.3}$$

The test rejects normality of datasets for large values of the statistic in (1.3) which asymptotically follows a χ_2^2, with an assumption of independence of the $\sqrt{b_1}$ and b_2.

In multivariate case, Mardia [4,5] and Srivastava [6] have differently obtained measures of skewness and kurtosis. Suppose a $d-$component random vector $\mathbf{x}$ is said to have a distribution $F(\mathbf{x})$ with a density function $f(\mathbf{x})$, Mardia [4, 5] obtained the measures of skewness and kurtosis of $\mathbf{x}$ respectively as:

$$\beta_{1,d} = E\left\{(\boldsymbol{x}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\}^3 \text{ and } \beta_{2,d} = E\left\{(\boldsymbol{x}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}^2 \tag{1.4}$$

where $\boldsymbol{y}$ is independent and identically distributed (iid) with $\mathbf{x}$, $\boldsymbol{\mu}$ is the $d-$component mean vector of $\mathbf{x}$ and $\boldsymbol{\Sigma}$ is the dxd variance - covariance matrix of $\mathbf{x}$. For a random sample of size n from a $d-$dimensional distribution, Mardia [4, 5] obtained sample measures of skewness and kurtosis respectively as:

$$b_{1,d} = n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(\mathbf{x}_i-\overline{\mathbf{x}})^T\boldsymbol{S}^{-1}(\mathbf{x}_j-\overline{\mathbf{x}})\right]^3 \text{ and } b_{2,d} = n^{-1}\sum_{j=1}^{n}\left[(\mathbf{x}_j-\overline{\mathbf{x}})^T\boldsymbol{S}^{-1}(\mathbf{x}_j-\overline{\mathbf{x}})\right] \tag{1.5}$$

where $\mathbf{x}_j$; $j = 1, 2, \ldots, n$ is a single observation vector in the sample, $\overline{\mathbf{x}} = n^{-1}\sum_{j=1}^{n}\mathbf{x}_j$ and $S = n^{-1}\sum_{j=1}^{n}(\mathbf{x}_j-\overline{\mathbf{x}})(\mathbf{x}_j-\overline{\mathbf{x}})^T$.

Mardia [4] used the statistics in (1.5) to obtain two tests for multivariate normality with asymptotic null distributions. They are:

$$A_n = \frac{nb_{1,d}}{6} \sim \chi_v^2;\ v = \frac{d(d+1)(d+2)}{6}$$

and

$$B_n = \frac{[b_{2,d} - (d(d+2)(n-1))/(n+1)]}{[(8d(d+2))/n]^{1/2}} \sim N(0,\,1).$$

In a different approach, Srivastava [6] via principal components of a random sample of size n, obtained estimators of $\beta_{1,d}$ and $\beta_{2,d}$ in (1.4) as

$$b_{1,d} = \sqrt{d^{-1}\sum_{i=1}^{d}\left\{w_i^{-3/2}\frac{\sum_{j=1}^{n}(y_{ij}-\overline{y}_i)^3}{n}\right\}^2} \tag{1.6}$$

and

$$b_{2,d} = (nd)^{-1}\sum_{i=1}^{d} w_i^{-2}\sum_{j=1}^{n}(y_{ij}-\overline{y})^4 \tag{1.7}$$

where d is the dimension of the random vector, w_i is the ith highest eigenvalue of the sample covariance matrix, $\boldsymbol{S}$, y_{ij} is the jth observation of the ith principal component, $\overline{y}_i$ is the marginal mean of all the ith principal component observations and $\overline{y}$ is the overall sample mean of the d principal components data. Using the results in (1.6) and (1.7), Srivastava [6] obtained two statistics for assessing the MVN of datasets, which are given with their asymptotic null distributions as:

$$S_{1,n} = \left(\frac{nd}{6}\right) b_{1,d}^2 \sim \chi_d^2 \,\text{and}\, S_{2,d} = \left(\frac{nd}{24}\right)^{1/2}(b_{2,d}-3) \sim N(0,\,1);$$

where $b_{1,d}$ and $b_{2,d}$ are as obtained by Srivastava [6].

The tests for multivariate normality based on sample measures skewness and kurtosis are not the only tests for multivariate normality. In fact, there are no fewer than 100 tests for multinormality in the literature. These tests are developed using different characterizations of the multivariate normal distribution. Such tests include Henze and Zirkler [7], Henze and Wagner [8], Szekely and Rizzo [9], Madukaife and Okafor [10, 11], Henze and Visagie [12] as well as Dorr et al. [13, 14], to mention but a few. However, all the statistics for assessing the MVN of datasets developed by Mardia [4, 5] and Srivastava [6] are, no doubt, powerful and affine invaraint tests, except that they lack consistency. As a result, Koizumi et al. [15] proposed two Jaque-Bera type statistics for assessing MVN, by building in both sample skewness and kurtosis measures. The statistics are:

$$MJB_M = n\left\{\frac{b_{1,d}^M}{6} + \frac{(b_{2,d}^M - d(d+2))^2}{8d(d+2)}\right\} \tag{1.8}$$

and

$$MJB_S = nd\left\{\frac{b_{1,d}^S}{6} + \frac{(b_{2,d}^S - 3)^2}{24}\right\} \tag{1.9}$$

where $b_{1,d}^M$ and $b_{1,d}^S$ are the $b_{1,d}$ due to Mardia [4] and Srivastava [6] respectively, while $b_{2,d}^M$ and $b_{2,d}^S$ are the $b_{2,d}$ according to Mardia [4] and Srivastava [6] respectively.

The two statistics of Koizumi et al. [15] achieved consistency in addition to being affine invariant and having a good power performance. However, the principal components transform of Srivastava [6] is not the only univariate operator that can be employed to achieve marginal independence of a multivariate normal dataset. Villasenor Alva and Estrada [16] noted that if a set of normal random variables are not correlated, then they are independent. Based on that, they obtained a simple standardization transform of a $d-$component normal random vector, applied the Shapiro and Wilk [17] statistic to each coordinate of the random vector and obtained the average as a new Shapiro-Wilk type test for MVN. The test demonstrated a good control over type-I-error and an appreciable power performance. As a result, Kim [18] applied the standardization idea of Villasenor Alva and Estrada [16] to obtain a test for MVN based on the Jarque and Bera [1, 2] statistic. The statistic is given by:

$$KJB_n = n\sum_{i=1}^{d}\left\{\frac{(\sqrt{b_1})^2}{6} + \frac{(b_2-3)^2}{24}\right\} \tag{1.10}$$

The test procedure with the statistic in (1.10) which is asymptotically distributed as chi-square with $2d$ degrees of freedom rejects the MVN of datasets for large values of the statistic.

Now, Urzua [19] has critisized the statistic of Jarque and Bera [1, 2] as not having correct moments of the sample skewness and kurtosis. The work obtained correct moments of the measures as:

$$E\left(\sqrt{b_1}\right) = 0;\, var\left(\sqrt{b_1}\right) = \frac{6(n-2)}{(n+1)(n+3)};\, E\left(b_2\right) = \frac{3(n-1)}{(n+1)};\, var\left(b_2\right) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)} \quad (1.11)$$

Based on the correct expected values and variances in (1.11), the work obtained a Jarque-Bera type statistic for testing normality of univariate datasets, which is given by:

$$AJB_n = \frac{\left(\sqrt{b_1}\right)}{var\left(\sqrt{b_1}\right)} + \frac{\left(b_2 - E(b_2)\right)^2}{var\left(b_2\right)} \quad (1.12)$$

The statistic in (1.12) was shown to be more powerful than the Jarque and Bera [1, 2] statistic.

In this work therefore, the Villasenor Alva and Estrada [16] technique is employed to obtain another consistent and affine invariant test for MVN based on the Urzua [19] adjusted statistic. The power performance of the proposed statistic is expected to be better than that of Koizumi et al. [15] based on Srivastava [6] principal components sample measures of skewness and kurtosis as well as Kim [18]. The new statistic is developed in Section 2 with its properties. In Section 3, the empirical critical values of the statistic is computed. Also, the relative empirical power performance of the statistic is obtained in Section 4 while the work is concluded in Section 5.

2 The New Statistic for Assessing Multivariate Normality

Suppose a random vector, $\boldsymbol{x} \in \boldsymbol{R}^d$, is such that $\boldsymbol{x} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; $d \geq 1$. Then, it is well known that

$$\boldsymbol{z} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{x} - \boldsymbol{\mu}) \sim N_d(\mathbf{0}, \boldsymbol{I}_d), \quad (2.1)$$

where $\boldsymbol{\Sigma}^{-1/2}$ is the inverse of the positive definite square root matrix of the covariance matrix, $\boldsymbol{\Sigma}$. Now, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a random sample of size n from $\boldsymbol{x} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ such that $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated from the sample observations by $\overline{\mathbf{x}}_n = n^{-1}\sum_{j=1}^{n}\boldsymbol{x}_j$ and $\boldsymbol{S}_n = n^{-1}\sum_{j=1}^{n}(\boldsymbol{x}_j - \overline{\mathbf{x}}_n)(\boldsymbol{x}_j - \overline{\mathbf{x}}_n)^T$ respectively. It is also well known that the vector transformation

$$\boldsymbol{z}_j = \begin{pmatrix} Z_{1j} \\ Z_{2j} \\ \vdots \\ Z_{dj} \end{pmatrix} = \boldsymbol{S}_n^{-1/2}(\boldsymbol{x}_j - \overline{\mathbf{x}}_n) \quad (2.2)$$

approaches $N_d(\mathbf{0}, \boldsymbol{I}_d)$, especially at large sample sizes. As a result of (2.2), it is appropriate to state that each component of $\boldsymbol{z}_j$ is approximately from an independent standard normal. That is, $Z_{ij} \sim N(0, 1)$.

In order to test for the MVN of a random sample of observation vectors, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, it is appropriate to propose the moment and distribution corrected statistic:

$$MAJB_n = \frac{1}{d}\sum_{i=1}^{d}\left\{\frac{(\sqrt{b_1})^2}{var\left(\sqrt{b_1}\right)} + \frac{(b_2 - E(b_2))^2}{var\left(b_2\right)}\right\} \quad (2.3)$$

where $\sqrt{b_1}$ and b_2 are the sample measures of skewness and kurtosis respectively, as presented in (1.3), $E(b_2)$, $var\left(\sqrt{b_1}\right)$ and $var\left(b_2\right)$ are as presented in (1.11). Under the assumption of the MVN of datasets, the $MAJB_n$ is expected to be zero. Since the statistic is non-negative, the test rejects MVN of datasets for large values of the statistic.

Theorem 2.1: The asymptotic null distribution of the $MAJB_n$ is chi-square with 2 degrees of freedom.

Proof. Jaque and Bera [1,2] have established that

$$n\left\{\frac{(\sqrt{b_1})^2}{6}+\frac{(b_2-3)^2}{24}\right\}\sim\chi_2^2,$$

with $E(b_2)=3$, $var\left(\sqrt{b_1}\right)=\frac{6}{n}$, and $var\,(b_2)=\frac{24}{n}$. Based on the adjusted moments and additive (averaging) property of independent chi-square distributions, $d^{-1}\sum_{i=1}^{d}\chi_2^2\sim\chi_2^2$ since under the transformation of (2.2), $Z_{ij}'s$ are mutually independent. □

Table 1. Empirical critical values of the $MAJB_n$ statistic

n	$d=2$		$d=5$	
	α=0.05	α=0.01	α=0.05	α=0.01
10	6.5992	13.0314	5.1783	8.2230
15	6.4692	14.2114	5.2318	9.1183
20	6.2576	13.9267	5.2148	9.3438
25	6.2424	13.9002	5.1189	9.3796
30	6.0819	13.7001	5.0473	9.1768
35	6.0242	13.2260	4.9341	8.9495
40	5.9281	12.8348	4.9259	8.9476
45	5.9190	12.7971	4.8678	8.8010
50	5.9014	12.8441	4.7936	8.5957
60	5.8005	12.3223	4.7299	8.2822
70	5.7782	11.8789	4.7153	8.1856
80	5.6646	11.5320	4.5858	7.8161
90	5.6046	11.2110	4.5075	7.7016
100	5.5401	10.9181	4.5323	7.5629
200	5.2460	9.6323	4.1885	6.5702
300	5.1230	8.9623	4.0874	6.2193
400	5.0503	8.4879	4.0097	5.8334
500	4.9947	8.1878	3.9194	5.6017

3 Empirical Critical Values of the Test

Since the $MAJB_n$ statistic is asymptotically chi-square with 2 degrees of freedom, the critical values, c, of the statistic is such that $P(MAJB_n > c_{\alpha,n,d})=\alpha$ under the null distribution of MVN, where $c_{\alpha,n,d}$ is the upper $100(1-\alpha)$ percentile of the chi-square distribution with 2 degrees of freedom. As a result, the test can be applied by using the percentage points of the chi-square distribution, as tabulated in statistical tables such as the Biometrika Trust. In this sense, the MVN of a dataset is rejected if the realized value of the statistic is greater than the appropriate $c_{\alpha,n,d}$. In this work however, the critical values are computed empirically and presented in Table 1 for sample sizes, $n=10(5)50(10)100(100)500$ and number of variables, $d=2$ and 5 at $\alpha=0.05$ and 0.01. The $MAJB_n$ statistic is computed in each of 100,000 samples simulated from the standard multivariate normal distribution under the combinations of n and d. In each combination, the empirical $c_{\alpha,n,d}$ is obtained as the $100(1-\alpha)$ percentile of the 100,000 computed statistics.

4 Power Comparison

The power of a test is the ability of a test to reject the null hypothesis when it is wrong. Different tests for ascertaining the same thing in statistics often have varied degrees of this ability. In goodness of fit tests to statistical distributions for instance, power performance is one of the established ways of measuring how good a statistic is. The power performance of a statistic can be theoretical or empirical. In this work, the empirical power comparison is adopted. The empirical power performance of the proposed $MAJB_n$ statistic is compared with those of three other related statistics for assessing MVN. The related competing statistics include the HZ_n statistic of Henze and Zirkler [7]; the VE_n statistic of Villasenor Alva and Estrada [16] and the KJB_n statistic of Kim [18]. The HZ_n is given by:

$$HZ_n \;=\; n\left(4I\left\{\boldsymbol{S}_n \,\text{is singular}\right\} + D_{n,\beta} I\left\{\boldsymbol{S}_n \,\text{is nonsingular}\right\}\right)$$

where $D_{n,\beta} \;=\; \left(1+2\beta^2\right) + n^{-2}\sum_{j,k=1}^{n} exp\left\{-\frac{\beta^2\|y_j-y_k\|^2}{2}\right\} - 2\left(1+\beta^2\right)^{-d/2} n^{-1}\sum_{j=1}^{n} exp\left\{-\frac{\beta^2\|y_j\|^2}{2\left(1+\beta^2\right)}\right\};\ \beta>0$ and $I\{.\}$ is an indicator function. The test is universally consistent, affine invariant and rejects MVN of datasets for large values of the statistic, with appropriate $\beta \;=\; \frac{((2d+1)n/4)/(1/(d+4))}{\sqrt{2}}$. Also, the VE_n statistic is given by:

$$VE_n \;=\; \frac{1}{d}\sum_{i=1}^{d} W_{Z_i},$$

where W_{Z_i} is the Shapiro and Wilk [15] statistic evaluated on the ith coordinate of the transformed observations $Z_{i1},\, Z_{i2},\, \ldots,\, Z_{in};\ i \;=\; 1,\, 2,\, \ldots,\, d$ and d is the number of variables. Finally, the KJB_n statistic is given by (1.10).

The three statistics are chosen among the dozens of tests for MVN in the literature for two reasons. Firstly, they are all well known standard statistics for assessing MVN which have relatively high power performances , in addition to being consistent against fixed alternatives and invariant to changes in scale and location of the observation vectors. Secondly, all the statistics, except the HZ_n statistic, are obtained in somewhat similar manner or with similar intermediary statistics such as the $b_{1,d}$ and the $b_{2,d}$.

In order to execute the empirical power comparison, data are simulated from 17 different multivariate distributions. They are grouped into symmetric and skewed distributions. The symmetric distributions considered include the standard multivariate normal (SMVN); multivariate Cauchy (MVC); multivariate t with 2 degrees of freedom (MVt_2); the standard multivariate Laplace ($SMVL$); product of symmetric beta ($\text{Beta}_{1,1}^d$); product of Logistic ($\text{Logistic}_{0,1}^d$); and product of Laplace ($\text{Laplace}_{1,3}^d$) distributions. Also, the skewed distributions considered include product of the standard lognormal (SLN^d); product of the asymmetric beta ($\text{Beta}_{1,3}^d$); product of the standard exponential (Exponential_1^d); product of the gamma ($\text{Gamma}_{1,3}^d$); product of the Weibull ($\text{Weibull}_{1,3}^d$); product of half normal ($\text{Half}-\text{normal}_1^d$); product of inverse normal ($\text{Inverse}-\text{normal}_{1,5}^d$); product of the chi-square with 2 degrees of freedom (χ_2^2); and product of the inverse beta ($\text{Inverse}-\text{beta}_{1,3}^d$) distributions.

From each of the distributions, a total of 10,000 samples are simulated for each sample size, $n \;=\; 25$, 50 and 100 and variable dimension $d \;=\; 2$ and 5. The values of the four competing statistics are evaluated in each of the 10,000 samples under a combination of sample size and variable dimension and the power of each test is measured by the number of the 10,000 samples where MVN is rejected. In this work, the empirical powers are expressed in percentage. The results are presented in Tables 2 and 3 for symmetric and asymmetric distributions respectively.

The test for MVN under the first distribution in Table 2, the standard multivariate normal distribution, is actually a test of type-I-error rate. This is because the distribution is null and as a result, the power is expected to be the level of significance, which is 5%. A test for MVN is said to have a good control over type-I-error if its power does not exceed the level of significance under the null distribution. As a result, all the four test procedures compared in this work are observed to have good control over type-I-error in all the sample sizes and variable dimensions considered.

Table 2. Empirical power comparison of the $MAJB_n$ statistic with some other competing tests under symmetric distributions, $\alpha = 0.05$

Distributions	n	$d=2$				$d=5$			
		HZ_n	VE_n	KJB_n	$MAJB_n$	HZ_n	VE_n	KJB_n	$MAJB_n$
SMVN	25	4.0	5.2	5.1	4.6	2.9	4.8	5.1	5.1
	50	4.8	5.1	4.8	5.3	4.3	4.8	4.7	5.2
	100	5.2	4.9	5.1	4.9	5.1	5.3	4.7	4.6
MVC	25	98.8	96.8	96.3	97.1	99.8	97.5	95.0	98.0
	50	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0
	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
MVt_2	25	79.1	73.5	76.5	77.2	94.3	74.0	78.2	79.7
	50	97.3	95.4	96.0	96.5	99.9	98.9	99.2	99.3
	100	100.0	100.0	99.9	99.9	99.9	100.0	100.0	100.0
$Beta^d_{1,1}$	25	16.8	40.7	0.1	0.0	79	56.4	0.1	0.0
	50	60.8	93.2	0.7	0.0	42.0	99.7	1.4	0.0
	100	97.3	100.0	95.0	82.6	93.2	100.0	100.0	99.9
Student t^d_5	25	20.6	29.7	36.9	37.7	16.6	46.9	54.9	56.3
	50	33.9	51.6	59.6	60.7	35.0	75.8	82.4	84.5
	100	54.9	76.5	82.9	83.9	63.5	96.5	98.1	98.1
Logistic$^d_{0,1}$	25	10.4	16.2	22.3	22.3	8.4	22.3	31.3	30.4
	50	14.6	26.2	34.7	35.7	14.1	41.0	51.6	53.7
	100	24.9	43.4	53.2	54.9	24.5	67.9	78.1	79.8
MVLaplace	25	50.4	39.4	44.1	45.9	86.2	46.1	49.5	51.7
	50	83.9	69.8	71.0	72.6	99.7	86.2	86.5	88.2
	100	98.7	94.3	93.2	93.8	100.0	99.6	99.6	99.5
Laplace$^d_{1,3}$	25	35.2	42.6	46.8	49.2	29.1	63.5	66.4	68.9
	50	62.9	72.0	73.0	74.4	63.7	93.2	92.4	94.3
	100	91.3	95.0	94.2	94.7	94.4	99.9	99.8	99.9

The results under the remaining seven symmetric distributions considered show that the new statistic is generally more powerful than the other competing statistics. This however is with the exception of the symmetric beta distribution, beta$^d_{1,1}$, where the statistic recorded very poor power performance at $n = 25$ and 50.

Table 3 shows that the newly proposed statistic generally recorded power performances slightly inferior to the VE_n and KJB_n statistics, but slightly superior to the HZ_n statistic in all the variable dimensions considered especially at smaller sample sizes of 25 and 50 under almost all the asymmetric distributions considered. The power performance of the proposed test however improved to be almost at par with the slightly superior ones at sample size of 100. It is therefore expected that the proposed statistic can compete favourably with any known test for MVN at sample sizes beyond 100. As a result, it can be regarded as a powerful procedure for assessing multivariate normality of multivariate datasets.

4.1 Real-life application

The new statistic is applied to a chemical solvent dataset. The solvent dataset, retrieved from https://openmv.net/tag/multivariate, is a 9-component dataset which consists of physical properties of a sample of 103 chemical solvents. In this study, only three physical properties are extracted, which included the boiling point, dielectric and dipole moment, to form a 103x3 data matrix. The extracted dataset is tested for multivariate normality using the proposed $MAJB_n$ statistic at 5% level of significance and the statistic rejected the null distribution of multivariate normality of the dataset since the computed value of the statistic, 2906.04, is greater than the critical value of 5.0359. The result further shows the applicability of the proposed statistic.

Table 3. Empirical power comparison of the $MAJB_n$ statistic with some other competing tests under asymmetric distributions, $\alpha = 0.05$

Distributions	n	$d = 2$				$d = 5$			
		HZ_n	VE_n	KJB_n	$MAJB_n$	HZ_n	VE_n	KJB_n	$MAJB_n$
$\text{Lognormal}_{0,1}^d$	25	99.0	99.9	98.6	97.4	99.7	100.0	99.5	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$\text{Beta}_{1,3}^d$	25	53.4	81.6	34.6	25.7	37.2	95.5	49.7	36.1
	50	92.1	99.8	78.9	65.3	87.7	100.0	97.5	91.9
	100	100.0	100.0	100.0	99.9	99.9	100.0	100.0	100.0
Exponential_1^d	25	92.4	98.8	89.6	84.0	92.0	100.0	99.1	97.7
	50	99.9	100.0	99.9	99.8	100.0	100.0	100.0	100.0
	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$\text{Gamma}_{1,3}^d$	25	92.9	99.1	88.5	84.0	92.3	100.0	99.2	97.9
	50	99.9	100.0	99.9	99.7	100.0	100.0	100.0	100.0
	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$\text{Weibull}_{2,3}^d$	25	14.9	27.0	17.9	15.7	9.4	39.5	23.5	20.4
	50	31.5	60.9	35.2	30.4	24.2	88.0	53.6	47.7
	100	60.5	96.1	75.7	69.7	55.7	100.0	96.9	94.4
Halfnormal_1^d	25	47.9	77.2	43.3	36.8	35.0	94.1	62.8	53.7
	50	86.8	99.6	82.9	75.3	82.3	100.0	98.5	96.1
	100	99.8	100.0	99.9	99.9	99.7	100.0	100.0	100.0
$\text{Inverse normal}_{1,5}^d$	25	70.2	72.5	59.6	54.1	90.4	92.9	80.8	76.5
	50	97.8	98.0	90.9	88.2	100.0	100.0	99.6	99.3
	100	100.0	100.0	99.9	99.9	100.0	100.0	100.0	100.0
Chi square_2^d	25	92.8	99.1	89.9	84.1	92.0	100.0	99.1	97.8
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	100	100.0	100.0	99.9	99.7	100.0	100.0	100.0	100.0
$\text{Inverse beta}_{2,5}^d$	25	93.6	98.8	93.1	90.0	95.9	100.0	99.7	99.3
	50	100.0	100.0	99.9	99.9	100.0	100.0	100.0	100.0
	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

5 Conclusion

It is stated in earlier works on this subject that there is no shortage of procedures for assessing MVN of datasets. These procedures existing in the literature share different properties ranging from affine invariance, consistency, type-I-error rate, power performance, to asymptotic null distribution. No test however has been identified to be universally the best in terms of these properties. The test procedure developed in this work has shown to have good control over type-I-error, as well as having good power performance. It has also been shown, in this work, that the statistic has an asymptotic nuul distribution of chi square with 2 degrees of freedom. Its consistency and affine invariance are however not proved in this work, but it has been shown in the earlier works in the literature that tests based on skewness and kurtosis measures are affine invariant. Also, it well known that tests based on either skewness or kurtosis measures lack consistency but tests with combination of the two measures are consistent against any fixed alternative. As a result, the $MAJB_n$ statistic proposed in this work can be recommended as a powerful affine invariant and consistent test for multivariate normality.

Competing Interests

Author has declared that no competing interests exist.

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