

Inverse Differential Operators in Time and Space

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Abstract

When one function is defined as a differential operation on another function, it's often desirable to invert the definition, to effectively “undo” the differentiation. A Green's function approach is often used to accomplish this, but variations on this theme exist, and we examine a few such variations. The mathematical analysis of $\hat{A}u = f$ is sought in the form $u = \hat{A}^{-1}f$ if such an inverse operator exists, but physics is defined by both mathematical formula and ontological formalism, as I show for an example based on the Dirac equation. Finally, I contrast these “standard” approaches with a novel exact inverse operator for field equations.

Keywords

Matrix-Inverse Differential Operators, Green's Functions, Dirac's Equation, Wave Equations, Inverse Curl Operator

1. Introduction

Physics, for the most part, describes the behavior of physical entities, particles, or fields, in terms of interactions with other physical entities. The description is typically differential, that is, a difference in the entity of interest is related to a change in a relevant variable. When one such function is defined as a differential operation on another function, it's often desirable to invert the definition, that is, to “undo” the differentiation. This is frequently done via Green's function, but there are variations on this theme. The goal of this paper is to examine specific instances of the use of inverse differential operators, a specialized topic that often receives less emphasis and is generally more confusing than the original non-inverse relations. After reviewing inverse-based relations in *time* and in *3-space*, I compare these with a new inversion technique that I discovered. These

techniques can arise in any field, so we do not limit our interest to a specific physical field.

Most mathematical operations, after having been learned, seem quite natural and even intuitive, including differential and integral operations. Things become only slightly more complicated when these two are linked as in the antiderivative, for example, a function F is an antiderivative of f if

$$F'(x) = f(x) \quad (1)$$

$$F(x) = \int f(x) dx + C. \quad (2)$$

The definition of anti-derivative is beautifully simple, after one understands differentiation and integration. The actual solution, *i.e.*, finding the actual value of the anti-derivatives can be very complicated, and often, simply unsolvable. In fact, due to the infinite variety of mathematical functions, the exact solution is often impossible to obtain in closed form. Here we present several examples in the field of physics, with an example based on a scalar derivative, differentiation with respect to time, and another based on a vector derivative.

The paper is organized as follows. The introduction defines the terms involved and briefly defines the physical fields to which we apply inverse methods. Section 2 introduces Green's functions in a general way. Section 3 presents the Dirac equation and illustrates the use of a time-based inverse operator. The presentation up to this point is purely mathematical. When math is applied to physics, it is done in the context of ontology, *i.e.*, physical reality. Section 4 discusses Dirac's ontology and analyzes anomalous physical results based on his ontology. Section 5 reviews Jefimenko's analysis of the general wave equation for a vector field \mathbf{V} , deriving \mathbf{V} through an inverse procedure. Section 6 applies a new exact inverse operator for field equations and compares the \mathbf{V} derived in this way with the prior derivations. Finally, Section 7 presents the summary and conclusion.

Examples are presented in the context of the physics of fields. There are two real physical fields associated with electro-magnetism, \mathbf{E} and \mathbf{B} and two real fields associated with gravito-magnetism, \mathbf{G} and \mathbf{C} . We focus on Coulomb's law and Newton's laws. The electric field \mathbf{E} is the negative gradient of the electric potential ϕ_q , while the gravitic field \mathbf{G} is the negative gradient of the gravitic potential ϕ_m .

$$\mathbf{E} = -\nabla\phi_q(x) \quad \mathbf{G} = -\nabla\phi_m(x) \quad (3)$$

$$\nabla \cdot \mathbf{E} = \rho_q(x) \quad \nabla \cdot \mathbf{G} = -\rho_m(x) \quad (4)$$

both of which lead to Poisson's equation

$$-\nabla^2\phi_i(x) = \rho_i(x). \quad (5)$$

This equation is not limited to the above physical fields, but also applies to steady-state distributions of temperature in thermodynamic problems, including those with sources of heat.

2. Green's Function Solutions

Physically, the potential function $\phi(x)$ is a field induced by a source charge q or mass m in the above cases. Electric charge q has discrete value and is typically considered to be a point charge, so distribution of charge is viewed as point charges distributed over a region of space, with each point source described as $q\delta(x-x')$ where x' is the location of the source and x is varied over the region of interest, i.e., $\rho(x) = q\delta(x-x')$. From vector analysis we know that $-\nabla\left(\frac{1}{r}\right) = \frac{\mathbf{r}}{r^3}$ when \mathbf{r} is the vector from source point x' to point x : $\mathbf{r}(x-x')$.

It can be shown that $\nabla^2\left(\frac{1}{r}\right) = 0$ for $\mathbf{r} \neq 0$. Obviously, the point at $\mathbf{r} = 0$ is unique, and we can show that

$$\int_V \nabla^2\left(\frac{1}{r}\right) = -4\pi. \quad (6)$$

This is summarized by the delta function:

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}'). \quad (7)$$

We next define a class of functions, called Green's functions, as

$$G(\mathbf{x}, \mathbf{x}') = \frac{-1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \quad (8)$$

subject to $\nabla^2 F(\mathbf{x}, \mathbf{x}') = 0$ inside volume V . Hence $\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.

Define scalar Laplacian operator $\Delta = \nabla \cdot \nabla$, which occurs in the Poisson equation:

$$\Delta\phi = \rho. \quad (9)$$

The inverse operation yields

$$\phi(x) = [\Delta^{-1}]\rho(x) = -\int_{\mathbb{R}^3} \frac{\rho(x')}{4\pi|\mathbf{x}-\mathbf{x}'|} dx' \equiv 4\pi \int G(x, x')\rho(x') dx'. \quad (10)$$

Conceptually, we decompose the distributed problem into source "points", each of which contributes to the physical distribution. Physicists understand $\phi(x)$ as the potential at point \mathbf{x} due to a source at point \mathbf{y} and then they integrate over all space surrounding sources. While the above development of Green's function is based on the classical physics of fields, the technique is not limited to classical physics. Kauffmann points out [1]:

"...the integral transformation kernel $\frac{-1}{4\pi|\mathbf{x}-\mathbf{x}'|}$ is the coordinate representation inverse (i.e., Green's function) of the Hilbert space Laplacian operator $\hat{\nabla}^2 = -\frac{|\hat{\mathbf{p}}|^2}{\hbar^2}$, namely that

$$\langle \mathbf{r} | (\hat{\nabla}^2)^{-1} | \mathbf{r}' \rangle = \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (11)$$

for $\mathbf{p} = 0$, i.e. "a purely static state of affairs..."

3. Inverse Operator on Dirac Equation

The problems addressed above address field or potential distributions over spatial domains, including problems with initial conditions and boundary conditions. The physics involves changes over space, given the relevant conditions. Other physical problems focus on variation over time. For example, having developed the quantum equation for “spinless” particles, derived from the Klein-Gordon equation, [2] Adler, *et al.* next treat particles with spin based on Dirac’s relativistic equation [3]:

$$i\partial_t\psi = \beta m\psi + V + \boldsymbol{\alpha} \cdot \boldsymbol{\pi}\psi \quad (12)$$

with $\beta \equiv \gamma^0$, $\alpha \equiv \gamma^0\gamma^k$, and $\mathbf{p} = -i\nabla$. This equation is a 4-component wavefunction ψ that can be rewritten as two 2-component Pauli spin wavefunction

$$\psi = e^{-imt} \begin{pmatrix} \Psi \\ \varphi \end{pmatrix} \quad (13)$$

which leads to the coupled equations,

$$i\partial_t\Psi = V\Psi + (\boldsymbol{\sigma} \cdot \boldsymbol{\pi})\varphi \quad (14)$$

$$i\partial_t\varphi + 2m\varphi - V\varphi = (\boldsymbol{\sigma} \cdot \boldsymbol{\pi})\Psi. \quad (15)$$

They are interested in Ψ so they solve for φ , and obtain symbolically

$$i\partial_t\Psi = V\Psi + (\boldsymbol{\sigma} \cdot \boldsymbol{\pi})(2m - V + i\partial_t)^{-1}(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})\Psi \quad (16)$$

$$\varphi = (2m - V + i\partial_t)^{-1}(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})\Psi \quad (17)$$

the term V represents the potential the particle moves in. To simplify, consider the free particle, $V = 0$. We’re now dealing with the inverse operator

$$(2m - V + i\partial_t)^{-1} = (2m + i\partial_t)^{-1}. \quad (18)$$

The Green’s function approach is applied to the scalar Laplacian $\nabla \cdot \nabla$ which represents divergence over 3-space. For the time derivative, ∂_t , we apply a different inverse differential operator. Let

$$Af + \partial f = (A + \partial)f = F \quad f = f(x), F = F(x) \quad (19)$$

where $F(x)$ is a given function that may be expanded as a power series in the region of interest, and A is assumed constant (~ 2 m). They solve the homogeneous equation

$$Af_h + \partial f_h = 0 \Rightarrow A dx + \frac{\partial f_h}{f_h} = A dx + d(\ln f_h). \quad (20)$$

Therefore $\int d(\ln f_h) = \int A dx \Rightarrow f_h = Ce^{-Ax}$ where C is an arbitrary constant. The general solution is the homogeneous solution f_h plus any particular solution f_p , where

$$f_h = (A + \partial)^{-1} F \quad (21)$$

The binomial theorem is used to rewrite this

$$(1+x)^{-1} = \frac{1}{1+x} = (1-x+x^2+\dots) \quad (22)$$

so we have

$$(A + \partial)^{-1} = \left[A \left(1 + \frac{\partial}{A} \right) \right]^{-1} = \frac{1}{A} \left(1 + \frac{\partial}{A} \right)^{-1} \equiv \frac{1}{A} \left(1 - \frac{\partial}{A} + \frac{\partial^2}{A^2} - \dots \right) \quad (23)$$

From Equation (21) we obtain

$$(A + \partial)^{-1} F = \frac{1}{A} \left(F - \frac{\partial F}{A} + \frac{\partial^2 F}{A^2} - \dots \right) \quad (24)$$

thus, if $(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})$ commutes with this operation, we obtain

$$i\partial_t \psi = (2m + i\partial_t)^{-1} (\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 \psi, \quad (25)$$

Since $\boldsymbol{\pi} = (\mathbf{p} + e\mathbf{A})$ where \mathbf{A} is the gauge field $\mathbf{B} = (\nabla \times \mathbf{A})$ and we have specified a free particle $\mathbf{B} \equiv 0$ then $\boldsymbol{\sigma} \cdot \boldsymbol{\pi} = \boldsymbol{\sigma} \cdot \mathbf{p}$ and $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$ we obtain $(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 = \mathbf{p}^2$. Therefore,

$$\partial_t \psi = (2m + i\partial_t)^{-1} \mathbf{p}^2 \psi \quad (26)$$

for the non-relativistic particles of interest to Adler, we restore the potential V and obtain

$$i\partial_t \Psi = V\Psi + \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} \Psi \quad (27)$$

which is the Schrödinger equation for spin 1/2 particles, also known as the Pauli equation. Our immediate goal was to exhibit the above inverse differential operator treatment for $(A + \partial)^{-1}$. The original symbolic Equation (16) is an exact equation for Ψ “*although it is of infinite order in the time derivative*”.

4. Physics Is Mathematical Formula plus Ontological Formulation

In recent papers I have focused on the physics of reality as consisting of two primary aspects, the *mathematical* formalism, and the *ontology* of the theory. It is worth noting that, although the inverse operator mathematics developed by Adler is correct as presented, the Dirac equation is more complicated than is indicated above. In [4] Dirac notes that the relativistic Hamiltonian provided by classical mechanics of the free particle leads to the wave equation:

$$\left\{ p_0 - (m^2 c^2 + p_1^2 + p_2^2 + p_3^2)^{1/2} \right\} \psi = 0. \quad (28)$$

where p 's are interpreted as operators. He explicitly notes that this is unsatisfactory from the point of view of relativity theory, “*because is very unsymmetrical between p_0 and the other p 's*”, and therefore we must look for a new wave equation. In other words, if $p_0 \sim i\partial_t$ and $\mathbf{p} \sim i\nabla$, then Schrödinger's equation

$$\left(-i \frac{\partial}{\partial t} = \nabla^2 \right) \psi \quad (29)$$

treats time linearly and space quadratically. Dirac sets out to fix this, since he claims that “*the theory cannot display the symmetry between space and time required by relativity*”. In place of Einstein's *space-time physics*, from which Di-

rac drew his assumption, one can use *energy-time* physics [5]. In Einstein's *space-time* theory, the Minkowski relation is viewed as *governing space-time*; in *energy-time* theory, the Minkowski relation is viewed as an *invariance relation*. Both approaches end up with the energy equation

$$E^2 = p^2 + m^2 \quad (30)$$

which Adler, *et al.* claim is their fundamental assumption. Ontology is a strong indicator of which physics is correct. For instance, Dirac's equation, solved for velocity, yields the result $v = \sqrt{3}c$. *I.e.*, the velocity is constant and significantly exceeds the speed of light. Although mentioned in only a few texts, this nonsense prediction of Dirac is simply ignored by almost all physicists.

Another severe problem is pointed out by Messiah [6]:

“Due to the coupling between the positive and negative components of the 4-component Dirac wave equation, is, properly speaking, no longer an eigenvalue equation.”

This motivates the Foldy-Wouthuysen transformation, allowing one to approximate the 4-component Dirac theory by a 2-component theory to any order in v/c . In the Dirac representation, the orbital angular momentum $\mathbf{r} \times \mathbf{p}$ and the spin angular momentum $\boldsymbol{\sigma}/2$ are not separately constants of the motion, although their sum is. After the Foldy-Wouthuysen transformation these are decoupled and are separately constants of the motion. At this point the transformed operators are in one-to-one correspondence with the operators of the Pauli theory. But as Trigg notes [7]: the Foldy-Wouthuysen is not a point transformation but an integral transformation, *i.e.*, for an integral transformation in coordinate space, the transformed state vector involves contributions from an extended region in the original description.

“The particle described by the transformed Hamiltonian is therefore ‘smeared out’ and interacts not only with the potential at the mean position, but with the average of the potential over the region it ‘occupies.’”

Finally, physicists assume that Dirac describes a particle with “spin up” or “spin down”, but the Hamiltonian operator $H = c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2$ commutes with the Hermitian operator

$$s(\mathbf{p}) = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \quad \text{where } \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad (31)$$

so, for Dirac particles the spin matrices are not an eigenstate of $\boldsymbol{\Sigma}$. In fact, the Dirac spin has two possible values relative to the momentum; this is the only “spin projection” fact that is derived from Dirac's equation. In other words, *Dirac predicts not spin eigenstates but helicity eigenstates.*

Given these results that conflict with physicist's expectations, one might ask how the Dirac equation yields anything worthwhile. In actually solving the Dirac equation, one must perform a multiplication that *restores* the quadratic nature of the momentum $\left(\frac{p^2}{2m}\right)$ despite that Dirac believed he was correcting this

“fault”.

This section, while departing from the mathematical presentation of inverse differential operators, is intended to emphasize that, merely solving the operator problem mathematically does not imply in any way that the physics has been solved. Physics is not purely mathematical, which is based on logic, but is also ontological, that is, based on physical reality.

5. Alternate Approaches to Inverse Spatial Differential Operators

A recently confirmed ontological fact is that electromagnetic waves and gravitomagnetic waves travel at the finite speed of light—there’s always a time delay before a change in the field conditions initiated at a point in space can produce an effect at any other point in space. This time delay is discussed in Jackson [8] and other texts, and Jefimenko has written a book, “*Retardation and Relativity*” [9] as a sequel to his “*Electricity and Magnetism*”. Aside from the ontological significance of this fact, the theory of electromagnetic retardation leads to, and duplicates, many relations customarily considered consequences of relativistic electrodynamics. It does not, for example, predict “length contraction”, which has never been measured, and almost certainly never will be.

Jefimenko begins with inhomogeneous or general wave equation for a field vector \mathbf{V} :

$$\nabla \times \nabla \times \mathbf{V} + \frac{1}{c^2} \frac{\partial^2 \mathbf{V}}{\partial t^2} = \mathbf{J}(x, y, z, t) \quad (32)$$

where \mathbf{J} is some vector function of space and time, assumed to be zero outside some finite region. His basic *wave field theorem* is represented by the retardation integral:

$$\mathbf{V} = -\frac{1}{4\pi} \int \frac{[\nabla'(\nabla' \cdot \mathbf{V}) - \mathbf{J}]}{r} dV'. \quad (33)$$

The retardation bracket $[f] = f(x', y', z', t - r/c)$ indicates space and time dependence where t is the time for which the retarded integrals are evaluated, and the value of the function is that which it had at some earlier time $t' = t - r/c$. The function is said to be *retarded*, representing the fact that time r/c must elapse before some event at point x', y', z' can produce an effect at x, y, z a distance r away. Further, primed operator ∇' operates on primed coordinates, ∇ on unprimed. Jefimenko treats the Maxwell-Heaviside equations in detail in formulating their retarded representations but observes that the retardation effects can frequently be neglected, and the equations handled with ordinary functions. We make that assumption in what follows.

The general wave equation holds for any vector \mathbf{V} and some function $\mathbf{J}(x, y, z, t)$, and can be manipulated using vector identities. Ontologically, we know of four physical fields that undoubtedly exist: $\mathbf{E}, \mathbf{B}, \mathbf{G}, \mathbf{C}$. Thus, we can perform substitutions based on the Maxwell-Heaviside equations governing these fields

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \quad (34)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \rho \mathbf{v} + \frac{\partial \mathbf{E}}{\partial t} \quad (35)$$

and gravity-based equivalent. Therefore, it is useful to replace general vector \mathbf{V} by $\mathbf{E}, \mathbf{B}, \mathbf{G}$ or \mathbf{C} . When this is done the static source \mathbf{J} becomes a function of density (charge or mass) and the dynamic source is density in motion, $\rho \mathbf{v}$ where \mathbf{v} is the velocity of the charge or mass.

Based on the above we apply the vector identity:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} . \quad (36)$$

to the general wave equation and obtain

$$\nabla^2 \mathbf{V} - \frac{1}{c^2} \frac{\partial^2 \mathbf{V}}{\partial t^2} = \nabla (\nabla \cdot \mathbf{V}) - \mathbf{J} . \quad (37)$$

While this equation is generally true everywhere, physicists typically consider two cases, the *near* field case where changes in the source are immediately reflected in field behavior, and the *far* field case in which the sources are effectively zero. The intermediate case is the region governed by retarded theory.

Consider the far field (source free) case in which the left side of Equation (36) is the wave equation

$$\nabla^2 \mathbf{V} - \frac{1}{c^2} \frac{\partial^2 \mathbf{V}}{\partial t^2} = 0 . \quad (38)$$

This describes the behavior of fields in free space, far from the source, and is very well understood.

From Heaviside's equations Jefimenko obtains the general gravitomagnetic wave equation

$$\nabla \times \nabla \times \mathbf{C} + \frac{1}{c^2} \frac{\partial^2 \mathbf{C}}{\partial t^2} = -\nabla \times \mathbf{p} \quad (39)$$

Applying the vector wave field theorem, we can write

$$\mathbf{C} = -\frac{1}{4\pi} \int \frac{[\nabla'(\nabla' \cdot \mathbf{C}) - \nabla' \times \mathbf{p}]}{r} dV' \quad (40)$$

and a similar approach leads to the gravitational field

$$\mathbf{G} = -\frac{1}{4\pi} \int \frac{[\nabla'(\nabla' \cdot \mathbf{G}) + \mu_0 \frac{\partial \mathbf{p}}{\partial t}]}{r} dV' \quad (41)$$

6. A New Exact Inverse Operator for Field Equations

In [10] I proved the existence of an *exact inverse operator on field equations*. Specifically, I showed that the inverse of the curl operation $(\nabla \times)$ has the following form:

$$(\nabla \times)^{-1} = (\mathbf{r} \times) \quad (42)$$

while

$$(\mathbf{r} \times)^{-1} = (\nabla \times). \quad (43)$$

I now apply this to Equation (39) after transforming the second order time derivative as follows

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{C}}{\partial t^2} = \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{C}}{\partial t} \right) = \frac{1}{c^2} \frac{\partial}{\partial t} (-\nabla \times \mathbf{G}) \quad (44)$$

The equation of interest (39) becomes

$$\nabla \times \nabla \times \mathbf{C} - \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{G}) = -\nabla \times \mathbf{p} \quad (45)$$

The spatial derivative commutes with the time derivative, so we rewrite this as

$$\nabla \times \nabla \times \mathbf{C} - \frac{1}{c^2} \left(\nabla \times \frac{\partial \mathbf{G}}{\partial t} \right) = -\nabla \times \mathbf{p} \quad (46)$$

this is now in optimal form to apply my inverse differential operator:

$$(\mathbf{r} \times)(\nabla \times)(\nabla \times) \mathbf{C} - \frac{1}{c^2} (\mathbf{r} \times)(\nabla \times) \frac{\partial \mathbf{G}}{\partial t} = -(\mathbf{r} \times)(\nabla \times) \mathbf{p} \quad (47)$$

which yields

$$\nabla \times \mathbf{C} = -\mathbf{p} + \frac{1}{c^2} \frac{\partial \mathbf{G}}{\partial t}, \quad (48)$$

exactly the Heaviside relation for the circulation of the C-field induced by momentum density \mathbf{p} . The momentum density is a local entity, and at any moment it experiences a constant gravitational acceleration \mathbf{G} , therefore we ignore $\frac{\partial \mathbf{G}}{\partial t}$ to obtain $\nabla \times \mathbf{C} = -\mathbf{p}$. Another application of the exact inverse operator yields

$$(\mathbf{r} \times)(\nabla \times) \mathbf{C} = -(\mathbf{r} \times) \mathbf{p} \quad (49)$$

from which we obtain the local C-field

$$\mathbf{C} = \mathbf{r} \times \mathbf{p}. \quad (50)$$

Compare this to the Green's function equivalent (40).

7. Summary and Conclusion

The Green's function solution to Poisson's equation is well known, but not particularly well understood unless one has worked through the delta-function approach. In every case one is left with what may be a very nasty integral to evaluate over a three-space volume. In addition to the fact that the basic integral is complicated, addition of the initial physical conditions and/or boundary conditions adds considerable complexity. Thus, what Green's functions approaches do is to provide a starting point to arrive at a meaningful formulation in analytic terms, but solution of the terms is not guaranteed.

We reviewed a general approach to this gradient-based spatial operation, then addressed a problem that presents an inverse time-based differential operator,

applied to the Dirac equation by Adler, *et al.* Instead of an integral over space, one ends up with a potentially infinite series of increasing powers of the time derivative. The mathematics of this approach appears faultless, but there are ontological problems associated with Dirac's equation, so the next section summarizes the issue.

Jefimenko's presentation of the general wave equation for a field vector, again obtains an integral formulation as necessary to calculate the field involved, based on the driving or source function. After developing the formula for calculating the field at position \mathbf{r} based on an event at \mathbf{r}' , it is applied to gravitational field \mathbf{G} and gravitomagnetic field \mathbf{C} . The result again yields a potentially nasty calculation, with no guarantees that the integrals can be solved for arbitrary sources.

Next, I apply a recently discovered exact inverse operator to the same Heaviside field equations for which we derived the integral formulation of the fields. The result is easier to derive, it is easier to understand, and it results in much easier calculations, which are almost guaranteed to be computable. The approach differs significantly from the Green's function and similar approaches, which provide one expression in the form of an integral which may or may not be computable. It is particularly well adapted to real physical fields, that is, $\mathbf{E}, \mathbf{B}, \mathbf{G}$ and \mathbf{C} . The expressions that I derive from the fields are exact and agree with well-known relations in physics.

Generally speaking, one wishes to calculate values of the field of interest over specific paths or boundaries, and this may involve a number of matrix-like calculations, most of which are simple and solvable. This is easily done by a computer, and I have used the approach in a number of different problems, with excellent success [Kasner], [particle creation], [Tajmar].

Finally, examination of Equations (48) to (50) implies that the matrix-operator inverse relations are "built-in" to the field equations in a manner entirely different from the Green's function integral-based operators that are imposed on the field theory from outside the field equations.

I hope these insights will prove useful to you the reader.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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