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Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Nonstandard Analysis, in Terms of the Green's Function

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In a preceding paper, solution is given without using the Green's function, on the basis of nonstandard analysis, for a restricted class of inhomogeneous terms. In the present paper, the corresponding solutions are given in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

Keywords: Green's function; differential equations with polynomial coefficients; nonstandard analysis; Kummer's differential equation; hypergeometric differential equation.

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1 Introduction

In the present paper, we treat the problem of obtaining the particular solutions of a differential equation with polynomial coefficients in terms of the Green's function.

In a preceding paper [1], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In another paper [2], this problem is studied in the framework of nonstandard analysis, and it is applied to a simple fractional and a first-order ordinary differential equation.

In a recent paper [3], a compact recipe based on nonstandard analysis, is presented, and is applied to Kummer's differential equation.

In the preceding paper [4], solutions of an eqution which has a special class of inhomogeneous part, are given, without using the Green's functuion. It is applied to the hypergeometric differential equation, the differential equations treated in [2] and the Hermite differential equation.

In the present paper, we solve the problem considered in [4], but now the results are expressed in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

The presentation in this paper follows those in [1, 2, 3, 4], in Introduction and in many descriptions in the following sections.

We consider a fractional differential equation, which takes the form:

$$p_n(t, {}_RD_t)u(t) = \sum_{l=0}^{n} a_l(t) {}_RD_t^{\rho_l} u(t) = f(t), \tag{1}$$

where (i) $n \in \mathbb{Z}_{>-1}$, $t \in \mathbb{R}$, (ii) $a_l(t)$ for $l \in \mathbb{Z}_{[0,n]}$ are polynomials of t, (iii) $\rho_l \in \mathbb{C}$ for $l \in \mathbb{Z}_{[0,n]}$ satisfy $\mathrm{Re}\,\rho_0 > \mathrm{Re}\,\rho_1 \geq \cdots \geq \mathrm{Re}\,\rho_n$ and $\mathrm{Re}\,\rho_0 > 0$.

Here $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$.

We use Heaviside's step function $H(t)$, which is equal to 1 if $t > 0$ and, to 0 if $t \leq 0$. Here ${}_RD_t^{\rho_l}$ are the Riemann-Liouville fractional integrals and derivatives defined by the following definition; see [5, 6].

Definition 1.1. *Let $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, $u_0(t)$ be locally integrable on $\mathbb{R}_{>\tau}$, $u(t) = u_0(t)H(t-\tau)$, $\lambda \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>-1}$ and $\rho = n - \lambda$. Then ${}_RD_t^{-\lambda}u(t)$ is the Riemann-Liouville fractional integral defined by*

$$\begin{aligned} {}_RD_t^{-\lambda}u(t) &= \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{t} (t-x)^{\lambda-1} u_0(x) H(x-\tau) dx \\ &= \frac{1}{\Gamma(\lambda)} \int_{\tau}^{t} (t-x)^{\lambda-1} u_0(x) dx \cdot H(t-\tau), \end{aligned} \tag{2}$$

and ${}_RD_t^{-\lambda}u(t) = 0$ for $t \leq \tau$, where $\Gamma(\lambda)$ is the gamma function, ${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t)$ is the Riemann-Liouville fractional derivative defined by

$${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t) = \frac{d^n}{dt^n}[{}_RD_t^{-\lambda}u_0(t)] \cdot H(t-\tau), \tag{3}$$

when $n \geq \mathrm{Re}\,\lambda$, and ${}_RD_t^{n}u(t) = \frac{d^n}{dt^n}u_0(t) \cdot H(t-\tau)$ when $\rho = n \in \mathbb{Z}_{>-1}$.

In accordance with Definition 1.1, when $u_0(t) = \frac{1}{\Gamma(\nu)}(t-\tau)^{\nu-1}$, we adopt

$$ {}_R D_t^{\rho} \frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)} H(t-\tau) = \begin{cases} \frac{(t-\tau)^{\nu-\rho-1}}{\Gamma(\nu-\rho)} H(t-\tau), & \nu-\rho \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ 0, & \nu-\rho \in \mathbb{Z}_{<1}, \end{cases} \tag{4} $$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $\tau \in \mathbb{R}$. Here ${}_R D_t$ is used in place of usually used notation ${}_\tau D_R$, in order to show that the variable is t.

Remark 1.1. *Let $g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t)$ for $\nu \in \mathbb{C}$. Then $g_\nu(t) = 0$ if $\nu \in \mathbb{Z}_{<1}$, and Equation (4) shows that if $\nu \notin \mathbb{Z}_{<1}$, ${}_R D_t^{\rho} g_\nu(t) = g_{\nu-\rho}(t)$. As a consequence, we have ${}_R D_t^{\nu+n} g_\nu(t) = g_{-n}(t) = 0$ for $n \in \mathbb{Z}_{>-1}$.*

In distribution theory [1, 7, 8, 9], we use distribution $\tilde{H}(t)$, which corresponds to function $H(t)$, differential operator D, and distribution $\delta(t) = D\tilde{H}(t)$, which is called Dirac's delta function.

1.1 Preliminaries on Nonstandard analysis

In nonstandard analysis [10], where infinitesimal numbers appear. We denote the set of all infinitesimal real numbers by $\mathbb{R}^0$. We also use $\mathbb{R}^0_{>0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon > 0\}$, which is such that if $\epsilon \in \mathbb{R}^0_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ satisfying $\epsilon < \frac{1}{N}$. We use $\mathbb{R}^{ns}$, which has subsets $\mathbb{R}$ and $\mathbb{R}^0$. If $x \in \mathbb{R}^{ns}$ and $x \notin \mathbb{R}$, x is expressed as $x_1 + \epsilon$ by $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^0$, where x_1 may be $0 \in \mathbb{R}$. Equation $x \simeq y$ for $x \in \mathbb{R}^{ns}$ and $y \in \mathbb{R}^{ns}$, is used, when $x - y \in \mathbb{R}^0$. We denote the set of all infinitesimal complex numbers by $\mathbb{C}^0$, which is the set of complex numbers z which satisfy $|\mathrm{Re}\, z| + |\mathrm{Im}\, z| \in \mathbb{R}^0$. We use $\mathbb{C}^{ns}$, which has subsets $\mathbb{C}$ and $\mathbb{C}^0$. If $z \in \mathbb{C}^{ns}$ and $z \notin \mathbb{C}$, z is expressed as $z_1 + \epsilon$ by $z_1 \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^0$, where z_1 may be $0 \in \mathbb{C}$.

In place of (4), we now use

$$ {}_R D_t^{\rho} g_{\nu+\epsilon}(t) = {}_R D_t^{\rho} \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) = g_{\nu-\rho+\epsilon}(t) = \frac{1}{\Gamma(\nu-\rho+\epsilon)} t^{\nu-\rho-1+\epsilon} H(t), \tag{5} $$

for all $\rho \in \mathbb{C}$ and $\nu \in \mathbb{C}$, where $\epsilon \in \mathbb{R}^0_{>0}$.

Lemma 1.1. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{R}^0_{>0}$ and $g_{\nu+\epsilon}(t) = \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu+\epsilon-1} H(t)$. Then the index law:*

$$ {}_R D_t^{\rho_1}\, {}_R D_t^{\rho_2} g_{\nu+\epsilon}(t) = {}_R D_t^{\rho_1+\rho_2} g_{\nu+\epsilon}(t) = g_{\nu-\rho_1-\rho_2+\epsilon}(t), \tag{6} $$

always holds.

In the present study in nonstandard analysis, in place of $\tilde{H}(t)$ and $\delta(t)$ in distribution theory, $H_\epsilon(t)$ and $\delta_\epsilon(t)$ are used, which are given by

$$ H_\epsilon(t) = {}_R D_t^{-\epsilon} H(t) = g_{1+\epsilon}(t) = \frac{1}{\Gamma(\epsilon+1)} t^{\epsilon} H(t), \tag{7} $$

$$ \delta_\epsilon(t) = g_\epsilon(t) = \frac{d}{dt} H_\epsilon(t) = \frac{1}{\Gamma(\epsilon)} t^{\epsilon-1} H(t) = \frac{\epsilon}{\Gamma(\epsilon+1)} t^{\epsilon-1} H(t), \tag{8} $$

for $\epsilon \in \mathbb{R}^0_{>0}$. We note that they tend to $H(t)$ and 0, respectively, in the limit of $\epsilon \to 0$.

Lemma 1.2. *In the notation in Remark 1.1, $H_\epsilon(t) = g_{1+\epsilon}(t)$, $\delta_\epsilon(t) = g_\epsilon(t)$, and*

$$ {}_R D_t^{\epsilon} H_\epsilon(t) = {}_R D_t^{\epsilon} g_{1+\epsilon}(t) = g_1(t) = H(t), \quad {}_R D_t^{\epsilon} \delta_\epsilon(t) = {}_R D_t^{\epsilon} g_\epsilon(t) = g_0(t) = 0. \tag{9} $$

1.2 Summary of the following sections

In solving Equation (1) in nonstandard analysis, we consider the solution of the following equation for $\tilde{u}(t) = {}_RD_t^{-\epsilon}u(t)$:

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t)\tilde{u}(t) = \tilde{f}(t), \tag{10}$$

where $\epsilon \in \mathbb{R}^0_{>0}$ and

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t) := {}_RD_t^{-\epsilon}p_n(t, {}_RD_t){}_RD_t^{\epsilon}. \tag{11}$$

In [4], Conditions 1.2 and 1.1 on p. 52, are adopted. It is

Condition 1.1. *Let* $\epsilon \in \mathbb{R}^0_{>0}$ *and* $\beta \in \mathbb{C}$.

(i) $\tilde{f}(t) = \delta_\epsilon(t)$ *and* $f(t) = 0$.

(ii) $\tilde{f}(t) = {}_RD_t^{\beta}\delta_\epsilon(t) = g_{\epsilon-\beta}(t)$. *When* $\beta \notin \mathbb{Z}_{>-1}$, $f(t) = {}_RD_t^{\beta+1}H(t)$, *and when* $\beta \in \mathbb{Z}_{>-1}$, $f(t) = 0$.

(iii) $\tilde{f}(t)$ *and* $f(t)$ *are expressed as follows:*

$$\tilde{f}(t) = \sum_{l=1}^{\infty} c_l \cdot {}_RD_t^{\beta_l}\delta_\epsilon(t) = \sum_{l=1}^{\infty} c_l \cdot g_{\Gamma(\epsilon-\beta_l)}(t), \quad f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_RD_t^{\beta_l}\delta_\epsilon(t). \tag{12}$$

respectively, where $c_l \in \mathbb{C}$ *are constants,* $\beta_l \in \mathbb{C}$ *satisfy* $-\mathrm{Re}\,\beta_l \geq -\mathrm{Re}\,\beta_1 \in \mathbb{R}$, *for all* $l \in \mathbb{Z}_{>0}$, *and* $d_l = c_l$ *if* $\beta_l \notin \mathbb{Z}_{>-1}$, *and* $d_l = 0$ *if* $\beta_l \in \mathbb{Z}_{>-1}$.

Remark 1.2. *Condition 1.1(i) is satisfied, when Condition 1.1(ii) is satisfied and* $\beta = 0$.

In Sections 2 and 3, full expressions of the Green's functions and the solutions, are derived for Kummer's and the hypergeometric differential equation, respectively.

Section 4 is for Conclusion.

2 Solution of Kummer's Differential Equation

Kummer's differential equation is described by

$$p_K(t, {}_RD_t)u(t) := [t\frac{d^2}{dt^2} + (c - bt)\frac{d}{dt} - ab]u(t) = f(t), \tag{13}$$

where a, b and c are constants satifying $a \neq 0$ and $b \neq 0$.

Lemma 2.1. *When Condition 1.1(ii) is satisfied, we construct the following transformed differential equations of Equation* (13), *for* $\tilde{u}(t) = {}_RD_t^{-\epsilon}u(t)$, $\tilde{w}(t) = {}_RD_t^{-\beta}\tilde{u}(t)$ *and* $w(t) = {}_RD_t^{\epsilon}\tilde{w}(t) = {}_RD_t^{-\beta}u(t)$:

$$\begin{aligned}\tilde{p}_{K,\epsilon}(t, {}_RD_t)\tilde{u}(t) &:= {}_RD_t^{-\epsilon}p_K(t, {}_RD_t){}_RD_t^{\epsilon}\tilde{u}(t)\\ &= [t\frac{d^2}{dt^2} + (c - \epsilon - bt)\frac{d}{dt} - (a - \epsilon)b]\tilde{u}(t) = {}_RD_t^{\beta}\delta_\epsilon(t) = g_{\epsilon-\beta}(t), \qquad (14)\\ \tilde{p}_{K,\beta+\epsilon}(t, {}_RD_t)\tilde{w}(t) &:= {}_RD_t^{-\beta}p_{K,\epsilon}(t, {}_RD_t){}_RD_t^{\beta}\tilde{w}(t)\\ &= [t\frac{d^2}{dt^2} + (c - \beta - \epsilon - bt)\frac{d}{dt} - (a - \beta - \epsilon)b]\tilde{w}(t) = \delta_\epsilon(t), \qquad (15)\\ \tilde{p}_{K,\beta}(t, {}_RD_t)w(t) &:= {}_RD_t^{-\beta}p_K(t, {}_RD_t){}_RD_t^{\beta}w(t)\\ &= [t\frac{d^2}{dt^2} + (c - \beta - bt)\frac{d}{dt} - (a - \beta)b]w(t) = 0. \qquad (16)\end{aligned}$$

In obtaining these equations from (13), we use (17) in the following lemma, which is given in [4].

Lemma 2.2. *Let* $\lambda \in \mathbb{C}_+$*,* $m \in \mathbb{Z}_{>-1}$ *and* $\rho = m - \lambda$*. Then*

$$ {}_R D_t^{\rho}[tu(t)] = t \cdot {}_R D_t^{\rho} u(t) + \rho \cdot {}_R D_t^{\rho-1} u(t), \tag{17}$$

$$ {}_R D_t^{\rho}[t^2 u(t)] = t^2 \cdot {}_R D_t^{\rho} u(t) + 2\rho t \cdot {}_R D_t^{\rho-1} u(t) + \rho(\rho-1) \cdot {}_R D_t^{\rho-2} u(t). \tag{18}$$

Lemma 2.3. *Two complementary solutions of Equation* (16)*, which are given by*

$$K_{\beta,1}(t) = {}_1F_1(a-\beta; c-\beta; bt) = \sum_{k=0}^{\infty} \frac{(a-\beta)_k b^k}{k!(c-\beta)_k} t^k, \quad t > 0, \tag{19}$$

$$\begin{aligned} K_{\beta,2}(t) &= \frac{1}{\Gamma(2-c+\beta)} t^{1-c+\beta} \cdot {}_1F_1(a-c+1; 2-c+\beta; bt) \\ &= \sum_{k=0}^{\infty} \frac{(a-c+1)_k b^k}{k!\Gamma(2-c+\beta+k)} t^{1-c+\beta+k} = {}_R D_t^{-\beta} K_{0,2}(t) H(t), \quad t > 0, \end{aligned} \tag{20}$$

exist, when $c - \beta \notin \mathbb{Z}_{<1}$ *and when* $c - \beta \notin \mathbb{Z}_{>1}$*, respectively.*

Following (60) in [3], we define the Green's function $G_{K,\beta,\epsilon}(t,\tau)$, which satisfies

$$\tilde{p}_{K,\beta+\epsilon}(t, {}_R D_t) G_{K,\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{21}$$

for $\tau \in \mathbb{R}$.

In the present section, formulas are derived with the aid of two complementary solutions given by (19) and (20), and hence they hold when $c - \beta \notin \mathbb{Z}_{<1}$.

Lemma 2.4. *Let* $K_{\beta,1}(t)$ *be given by Equation* (19)*. Then Lemmas 2.1 and 2.3 show that* $G_{K,\beta,\epsilon}(t,0)$ *and* $G_{K,\beta,0}(t,0)$*, given by*

$$\tilde{w}_{\beta,\epsilon}(t) = G_{K,\beta,\epsilon}(t,0) = {}_R D_t^{-\epsilon} G_{K,\beta,0}(t,0), \tag{22}$$

$$w_{\beta,0}(t) = G_{K,\beta,0}(t,0) = \frac{1}{-1+c-\beta} K_{\beta,1}(t) H(t), \tag{23}$$

are a particular solution of Equation (21) *for* $\tau = 0$*, and also of* (15)*, and a complementary solution of Equation* (16)*, respectively.*

Remark 2.1. *The derivation of* (22) *with* (23) *for* $\beta = 0$*, with the aid of Frobenius method, is given in Section 3.1 of [3].*

Theorem 2.1. *Let Condition 1.1(ii) be satisfied, and* $G_{K,\beta,\epsilon}(t,0)$ *and* $G_{K,\beta,0}(t,0)$ *be given by* (22) *with* (23)*. Then Lemmas 2.1, 2.4 and 2.3 show that*

(i) *when* $\beta \in \mathbb{C}$ *and* $\tilde{f}(t) = g_{\epsilon-\beta}(t)$*,* $\tilde{u}_{\epsilon-\beta}(t) = {}_R D_t^{\beta} \tilde{w}_{\beta,\epsilon}(t)$*, given by*

$$\tilde{u}_{\epsilon-\beta}(t) = {}_R D_t^{\beta} G_{K,\beta,\epsilon}(t,0) = \frac{1}{-1+c-\beta} \sum_{k=0}^{\infty} \frac{(a-\beta)_k b^k}{(c-\beta)_k \Gamma(k-\beta+1+\epsilon)} t^{k-\beta+\epsilon} H(t), \tag{24}$$

which is a particular solution of Equation (14)*,*

(ii) *if $\beta = n \in \mathbb{Z}_{>-1}$, by using* (24)*, we obtain $u_{-n}(t) = {}_R D_t^{\epsilon} \tilde{u}_{\epsilon-n}(t)$, expressed by*

$$u_{-n}(t) = \sum_{k=n}^{\infty} \frac{(a-n)_k b^k}{(-1+c-n)_{k+1}} \frac{1}{(k-n)!} t^{k-n} H(t)$$
$$= \sum_{l=0}^{\infty} \frac{(a-n)_{n+l} b^{n+l}}{(-1+c-n)_{n+l+1}} \frac{1}{l!} t^l H(t) = C_n G_{K,0,0}(t,0), \tag{25}$$

which is a complementary solution of (13)*, where $C_n = \frac{(a-n)_n}{(-1+c-n)_{n+1}} b^n = \frac{\Gamma(a)\Gamma(c-n-1)}{\Gamma(a-n)\Gamma(c)} b^n$, and*

(iii) *if $\beta \notin \mathbb{Z}_{>-1}$, by using* (24)*, we obtain $u_{-\beta}(t)$, given by $u_{-\beta}(t) = {}_R D_t^{\epsilon} \tilde{u}_{\epsilon-\beta}(t) = {}_R D_t^{\beta} G_{K,\beta,0}(t,0)$, which is a particular solution of* (13).

Theorem 2.1 shows that if $\tilde{f}(t) = {}_R D_t^{\beta} \delta_{\epsilon}(t)$, the particular solution of (14) is given by (24). As a consequence, we have the following theorem.

Theorem 2.2. *Let $\tilde{f}(t)$ and $f(t)$ satisfy Condition 1.1(iii), so that they are given by* (12)*, and*

$${}_R D_t^{\beta_l} G_{K,\beta_l,\epsilon}(t,0) = \frac{1}{-1+c-\beta_l} \sum_{k=0}^{\infty} \frac{(a-\beta_l)_k b^k}{(c-\beta_l)_k \Gamma(k-\beta_l+1+\epsilon)} t^{k-\beta_l+\epsilon} H(t). \tag{26}$$

Then $\tilde{u}_f(t)$ and $u_f(t)$, given by

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot {}_R D_t^{\beta_l} G_{K,\beta_l,\epsilon}(t,0), = \sum_{l=1}^{\infty} d_l \cdot {}_R D_t^{\beta_l} G_{K,\beta_l,0}(t,0), \tag{27}$$

are particular solutions of (14) *and* (13)*, respectively. Condition $c-\beta \notin \mathbb{Z}_{<1}$ in Lemma 2.3 requires the condition $c-\beta_l \notin \mathbb{Z}_{<1}$ for all $l \in \mathbb{Z}_{>0}$, in the present case.*

Lemma 2.5. *Lemmas 2.3 and 2.1 show that $w_c(t)$ and $u_c(t)$, given by*

$$w_c(t) := K_{\beta,2}(t)H(t) = {}_R D_t^{-\beta} K_{0,2}(t)H(t), \quad u_c(t) = {}_R D_t^{\beta} w_c(t) = K_{0,2}(t)H(t), \tag{28}$$

are a complementary solution of Equations (16) *and* (13)*, respectively.*

Remark 2.2. *We now give a derivation of* (28) *for $\beta = 0$, by modifying the above mentioned proof of* (45) *given in [3]. We assume that the solution of* (13) *is expressed by* (52-c)*, which is obtained from* (52) *in [3] by replacing $\tilde{u}$ by u. Then* (13) *is expressed by* (53-c)*, which is obtained from* (53) *in [3] by replacing $\tilde{u}$ by u, ϵ by* 0*, and $\tilde{f}$ by f. We then note that when $f(t) = 0$,* (53-c) *is satisfied by* (55-c)*, which is obtained from* (55) *in [3] by replacing ϵ by* 0*. By using these in* (52-c) *and putting $u(t) = p_0 u_c(t)$, we obtain* (46) *in [3], which gives* (28) *for $\beta = 0$.*

3 Solution of the Hypergeometric Differential Equation

As stated in Introduction, solutions of the hypergeometric differential equation, are given in [4] without the Green's function. We now give them in terms of the Green's function.

The hypergeometric differential equation is described by

$$p_H(t, {}_R D_t)u(t) = [t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab]u(t) = f(t), \tag{29}$$

where a, b and c are constants satifying $a \neq 0$ and $b \neq 0$.

Lemma 3.1. *When Condition 1.1(ii) is satisfied, we construct the following transformed differential equations of Equation* (29)*, for* $\tilde{u}(t) = {}_RD_t^{-\epsilon}u(t)$*,* $\tilde{w}(t) = {}_RD_t^{-\beta}\tilde{u}(t)$ *and* $w(t) = {}_RD_t^{\epsilon}\tilde{w}(t) = {}_RD_t^{-\beta}u(t)$*:*

$$\tilde{p}_{H,\epsilon}(t, {}_RD_t)\tilde{u}(t) = {}_RD_t^{\beta}\delta_\epsilon(t) = g_{\epsilon-\beta}(t), \tag{30}$$

$$\tilde{p}_{H,\beta+\epsilon}(t, {}_RD_t)\tilde{w}(t) = \delta_\epsilon(t), \tag{31}$$

$$\tilde{p}_{H,\beta}(t, {}_RD_t)w(t) = 0. \tag{32}$$

where the left hand side of (30) *is given by*

$$\tilde{p}_{H,\epsilon}(t, {}_RD_t) = t(1-t)\frac{d^2}{dt^2} + (c - \epsilon - (a+b+1-2\epsilon)t)\frac{d}{dt} - (a-\epsilon)(b-\epsilon), \tag{33}$$

and then those of (31) *and* (32) *are obtained from* (33) *by replacing* ϵ *by* $\beta+\epsilon$ *and* β*, respectively.*

In obtaining these equations from (29), we use Lemma 2.2, which is given in [4].

Lemma 3.2. *Let* $c - \beta \notin \mathbb{Z}_{<1}$*. Then there exist two complementary solutions of Equation* (32)*, which are given by*

$$H_{\beta,1}(t) = {}_2F_1(a-\beta, b-\beta; c-\beta; t) = \sum_{k=0}^{\infty} \frac{(a-\beta)_k(b-\beta)_k}{k!(c-\beta)_k}t^k, \quad t > 0, \tag{34}$$

$$\begin{aligned} H_{\beta,2}(t) &= \frac{1}{\Gamma(2-c+\beta)}t^{1-c+\beta} \cdot {}_2F_1(1+a-c, 1+b-c; 2-c+\beta; t) \\ &= {}_RD_t^{-\beta}H_{0,2}(t)H(t), \quad t > 0. \end{aligned} \tag{35}$$

Following (2.20) in [4], we define the Green's function $G_{H,\beta,\epsilon}(t,\tau)$, which satisfies

$$\tilde{p}_{H,\beta+\epsilon}(t, {}_RD_t)G_{H,\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{36}$$

for $\tau \in \mathbb{R}$.

In the present section, formulas are derived with the aid of two complementary solutions given by (34) and (35), and hence they hold when $c - \beta \notin \mathbb{Z}_{<1}$.

Lemma 3.3. *Let* $H_{\beta,1}(t)$ *be given by Equation* (34)*. Then Lemmas 3.1 and 3.2 show that* $G_{H,\beta,\epsilon}(t,0)$ *and* $G_{H,\beta,0}(t,0)$*, given by*

$$\tilde{w}_{\beta,\epsilon}(t) = G_{H,\beta,\epsilon}(t,0) = {}_RD_t^{-\epsilon}G_{H,\beta,0}(t,0), \tag{37}$$

$$w_{\beta,0}(t) = G_{H,\beta,0}(t,0) = \frac{1}{-1+c-\beta}H_{\beta,1}(t)H(t), \tag{38}$$

are a particular solution of Equation (36) *for* $\tau = 0$*, and also of* (31)*, and a complementary solution of Equation* (32)*, respectively.*

Remark 3.1. *The derivation of* (37) *with* (38) *for* $\beta = 0$*, with the aid of Frobenius method, is given in p. 53 of [4].*

Theorem 3.1. *Let Condition 1.1(ii) be satisfied, and $G_{H,\beta,\epsilon}(t,0)$ and $G_{H,\beta,0}(t,0)$ be given by (37) with (38). Then Lemmas 3.3 and 3.1 show that*

(i) *when $\beta \in \mathbb{C}$ and $\tilde{f}(t) = g_{\epsilon-\beta}(t)$, $\tilde{u}_{\epsilon-\beta}(t)$, given by*

$$\tilde{u}_{\epsilon-\beta}(t) = {}_R D_t^{\beta} G_{H,\beta,\epsilon}(t,0) = \frac{1}{-1+c-\beta} \sum_{k=0}^{\infty} \frac{(a-\beta)_k (b-\beta)_k}{(c-\beta)_k \Gamma(k-\beta+\epsilon+1)} t^{k-\beta+\epsilon} H(t), \tag{39}$$

is a particular solution of Equation (30),

(ii) *if $\beta = n \in \mathbb{Z}_{>-1}$, $u_{-n}(t) = {}_R D_t^{\epsilon} \tilde{u}_{\epsilon-n}(t)$, expressed by*

$$\begin{aligned} u_{-n}(t) &= \sum_{k=n}^{\infty} \frac{(a-n)_k (b-n)_k}{(-1+c-n)_{k+1}} \frac{1}{(k-n)!} t^{k-n} H(t) \\ &= \sum_{l=0}^{\infty} \frac{(a-n)_{n+l} (b-n)_{n+l}}{(-1+c-n)_{n+l+1}} \frac{1}{l!} t^{l} H(t) = C_n G_{H,0,0}(t,0), \end{aligned} \tag{40}$$

is a complementary solution of Equation (29), where $C_n = \frac{(a-n)_n (b-n)_n}{(-1+c-n)_{n+1}} = \frac{\Gamma(a)\Gamma(b)\Gamma(c-n-1)}{\Gamma(a-n)\Gamma(b-n)\Gamma(c)}$, and

(iii) *if $\beta \notin \mathbb{Z}_{>-1}$, $u_{-\beta}(t)$, given by $u_{-\beta}(t) = {}_R D_t^{\epsilon} \tilde{u}_{\epsilon-\beta}(t) = {}_R D_t^{\beta} G_{H,\beta,0}(t,0)$ with the aid of (39), is a particular solution of (29).*

Theorem 3.2. *Let $\tilde{f}(t)$ and $f(t)$ satisfy Condition 1.1(iii), so that they are given by (12), and*

$${}_R D_t^{\beta_l} G_{H,\beta_l,\epsilon}(t,0) = \frac{1}{-1+c-\beta_l} \sum_{k=0}^{\infty} \frac{(a-\beta_l)_k (b-\beta_l)_k}{(c-\beta_l)_k \Gamma(k-\beta_l+1+\epsilon)} t^{k-\beta_l+\epsilon} H(t). \tag{41}$$

Then $\tilde{u}_f(t)$ and $u_f(t)$, given by

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot {}_R D_t^{\beta_l} G_{H,\beta_l,\epsilon}(t,0), \quad u_f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_R D_t^{\beta_l} G_{H,\beta_l,0}(t,0), \tag{42}$$

are particular solutions of Equations (30) and (29), respectively. The condition, which corresponds to Condition $c-\beta \notin \mathbb{Z}_{<1}$ in Lemma 2.3, requires the condition $c-\beta_l \notin \mathbb{Z}_{<1}$ for all $l \in \mathbb{Z}_{>0}$, in the present case.

Lemma 3.4. *Lemmas 3.1 and 3.2 show that $u_c(t)$, given by*

$$u_c(t) = H_{0,2}(t) H(t), \tag{43}$$

is a complementary solution of Equation (29).

4 Conclusion

In the preceding paper [4], we consider the problem of solving the equation which has a special class of inhomogeneous part, and we adopt a recipe without the Green's functuion, which is applied to the hypergeometric differential equation, the differential equations treated in [2] and the Hermite differential equation.

In the present paper, we solve the problem considered in [4], but now the results are expressed in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

In Section 2, the results for Kummer's differential equation, in which Condition 1.1(ii) is satisfied, are given in Theorem 2.1 and Lemma 2.5, and the results, in which Condition 1.1(iii) is satisfied, are given in Theorem 2.2. Here Condition 1.1(i) is treated as a special case of Condition 1.1(ii).

In Section 3, the corresponding results for the hypergeometric differential equation, are given in Theorems 3.1 and 3.2 and Lemma 3.4.

Competing Interests

Author has declared that no competing interests exist.

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