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Application of *q*-Calculus to the Solution of Partial *q*-Differential Equations

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Abstract

We introduce the concept of q-calculus in quantum geometry. This involves the q-differential and q-integral operators. With these, we study the basic rules governing q-calculus as compared with the classical Newton-Leibnitz calculus, and obtain some important results. We introduce the reduced q-differential transform method (RqDTM) for solving partial q-differential equations. The solution is computed in the form of a convergent power series with easily computable coefficients. With the help of some test examples, we discover the effectiveness and performance of the proposed method and employing MathCAD 14 software for computation. It turns out that when $q = 1$, the solution coincides with that for the classical version of the given initial value problem. The results demonstrate that the RqDTM approach is quite efficient and convenient.

Keywords

q-Differential, q-Integral Operators, RqDTM Method, Initial Value Problem, Partial Differential Equation, MathCAD 14

1. Introduction

Every physical theory is formulated in terms of mathematical objects. It is thus necessary to establish a set of rules to map physical concepts and objects into mathematical objects that we use to represent them. Often times this mapping is evident, as in classical mechanics, while for other theories, such as quantum mechanics, the mathematical objects are not so intuitive. The background of our study is based on the dynamics of closed quantum systems, with intrinsic components such as states, observables, measurements and evolution. Quantum geometry on the other hand can be traced to the early days of quantum mechanics. Specifically, if we consider Heisenberg's commutation relations [1] [2]

$$[x, p] = i\hbar \tag{1}$$

it becomes obvious that the geometry of classical phase space is completely lost. When coordinates such as x (position) and p (momentum) on a phase space cease to commute, then there can be no such space! Moreover, we discover that this operator algebra [3] [4] [5] forms some kind of noncommutative geometric space. This is in contrast to algebraic geometry [6] [7], which is built on a correspondence between *spaces* and *commutative algebras*. This correspondence in particular associates with any given space, the algebra of functions on it, and geometric notions are then expressed in a purely algebraic format. This principle turns out to be the most logical starting point for a generalized geometry such as quantum geometry. While for algebraic geometry, the spaces are affine schemes, a correspondence that is closer to differential geometry is given by the Gelfand-Naimark theorem [8]. In this case, the spaces are topological spaces and the algebras are commutative C^*-algebras.

In 2014, Maliki *et al.* [9], discussed the notion of q-deformed calculus in quantum geometry. Here they showed that the mathematical study of noncommutative geometry is intimately related to the so-called q-calculus, which is a generalization of the Newton-Leibnitz classical calculus.

In this work, we summarize some important q-calculus results which will enable us to study non-commutative differential equations, specifically we shall employ the reduced q-differential transform method (RqDTM) to solve partial q-differential equations.

1.1. The *q*-Differential Operator

For $1 \neq q \in \mathbb{C}$, we define the q-differential operator D_q as;

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \tag{2}$$

Note that $D_q \to \mathrm{d}/\mathrm{d}x \equiv D$ as $q \to 1$. We make the following remarks.

1) At the beginning of the 20th century, F.H. Jackson [10] studied this modified derivative and many of its consequences.

2) This q-derivative can be applied to function not containing 0 in their domain of definition. Then it reduces to the ordinary derivative when q goes to 1.

3) One can easily check that the q-derivative operator is linear, *i.e.*,

$$\text{(a)}\quad D_q(f+g) = D_q f + D_q g \quad \text{(b)}\quad D_q(\lambda f) = \lambda D_q f \tag{3}$$

1.2. Basic Notions of *q*-Calculus

The mathematical study of non-commutative geometry is intimately related to the so-called q-calculus. We begin with the q-differentials of some standard functions, then we give a brief introduction to q-numbers, q-factorials and q-integrals.

1.2.1. q-Derivative of Some Standard Functions

Following the procedure for computing the q-derivative from first principles, we now obtain important results for the q-derivative of the following standard functions such $\sin x$, and e^x.

1.2.2. q-Derivative of the Function $f(x)=\sin x$

By definition, we have

$$\begin{aligned}D_q(\sin x)&=\frac{\sin x-\sin qx}{(1-q)x}\\&=\frac{\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right)-\left(qx-\frac{1}{3!}q^3x^3+\frac{1}{5!}q^5x^5-\frac{1}{7!}q^7x^7+\cdots\right)}{(1-q)x}\\&=1-\frac{1}{3!}x^2\frac{(1-q^3)}{(1-q)}+\frac{1}{5!}x^4\frac{(1-q^5)}{(1-q)}-\frac{1}{7!}x^6\frac{(1-q^7)}{(1-q)}+\cdots\end{aligned} \tag{4}$$

**Note: for typesetting use $D_q(\sin x)$ below, similarly for $D(\sin x)$.

Using the fact that $\frac{1-q^r}{1-q}=1+q+q^2+\cdots+q^{r-1}$

$$\begin{aligned}\therefore\ D_q\sin x=1&-\frac{1}{3!}x^2\left(1+q+q^2\right)+\frac{1}{5!}x^4\left(1+q+q^2+q^3+q^4\right)\\&-\frac{1}{7!}x^6\left(1+q+q^2+q^3+q^4+q^5+q^6\right)+\cdots\end{aligned}$$

When $q=1$, we obtain the classical derivative of $\sin x$ function, *i.e.*

$$\begin{aligned}D\sin x&=1-\frac{1}{3!}x^2(3)+\frac{1}{5!}x^4(4)-\frac{1}{7!}x^6(6)+\cdots\\&=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots=\cos x\end{aligned}$$

1.2.3. q-Derivative of the Exponential Function $f(x)=e^x$

We have by definition;

$$D_q e^x=\frac{e^x-e^{qx}}{(1-q)x}=\frac{e^x\left(1-e^{-x(1-q)}\right)}{(1-q)x} \tag{5}$$

Since $e^u=\sum_{n=0}^{\infty}\frac{u^n}{n!}$, letting $u=(1-q)x$ we have;

$$\begin{aligned}e^{-x(1-q)}&=\sum_{n=0}^{\infty}\frac{(-x(1-q))^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\frac{(1-q)^n x^n}{n!}\\&=1-(1-q)x+(1-q)^2\frac{x^2}{2!}+\cdots+(-1)^n(1-q)^n\frac{x^n}{n!}+\cdots\end{aligned}$$

$$\begin{aligned}\therefore\quad 1-e^{-x(1-q)}&=(1-q)x-(1-q)^2\frac{x^2}{2!}+(1-q)^3\frac{x^3}{3!}-\cdots\\&+(-1)^{n+1}(1-q)^n\frac{x^n}{n!}+\cdots\end{aligned}$$

$$\Rightarrow \quad D_q \mathrm{e}^x = \frac{\mathrm{e}^x\left(1-\mathrm{e}^{-x(1-q)}\right)}{(1-q)x}$$
$$= \mathrm{e}^x\left[1-(1-q)\frac{x}{2!}+(1-q)^2\frac{x^2}{3!}-\cdots+(-1)^n(1-q)^{n-1}\frac{x^{n-1}}{n!}+\cdots\right] \tag{6}$$

From the above expression it is clear that when $q=1$, we obtain the classical exponential function. We remark that the q-derivative of other standard functions can be obtained similarly.

1.2.4. q-Numbers and q-Factorials

We adopt basically the notations in [11]. Thus the set of positive integers is denoted by $\mathbb{N}$. Furthermore, throughout this paper K denotes a field of characteristic 0 and $K(q)$ denotes the field of rational functions in one parameter q over $K(q)$. $K(q)$ is our base field in the q-deformed setting, while K is the base field in the classical setting. We define respectively the q-integers, q-factorials and q-binomials as follows:

1) $$[\![n]\!]_q := \sum_{i=0}^{n-1} q^i = \frac{1-q^n}{1-q} \tag{7}$$

2) $$[\![n]\!]_q ! := [\![1]\!]_q [\![2]\!]_q \cdots [\![n]\!]_q ;\ [\![0]\!]_q ! = 1 \tag{8}$$

3) $$\begin{Bmatrix} n \\ m \end{Bmatrix}_q := \frac{[\![n]\!]_q !}{[\![m]\!]_q ! [\![n-m]\!]_q !},\ \forall n,m \in \mathbb{N}\cup\{0\},\ n \geq m \tag{9}$$

For example, given $f(x)=x^n$ then $D_q f(x) = [\![n]\!] x^{n-1}$.

1.2.5. Remark

The properties of the q-integers, q-factorials and q-binomials and their proofs are presented in [9]. We also have the following important results on the q-differential operator. For $1 \neq q \in \mathbb{C}$, and with D_q as defined previously;

1) $$D_q f(x) = \sum_{r=0}^{\infty} \frac{(q-1)^r}{(r+1)!} x^r \frac{\mathrm{d}^{r+1}}{\mathrm{d}x^{r+1}} f(x) \tag{10}$$

2) $$D_q^n f(x) = \frac{x^{-n} q^{-n(n-1)/2}}{(q-1)^n} \sum_{j=0}^{n} \begin{Bmatrix} n \\ j \end{Bmatrix}_q (-1)^j q^{j(j-1)/2} f\left(q^{n-j}x\right) \tag{11}$$

3) $$D_q\{u(x)v(x)\} = v(x)D_q u(x) + u(qx)D_q v(x) \tag{12}$$

4) $$D_q\{u(x)/v(x)\} = \frac{v(qx)D_q u(x) - u(qx)D_q v(x)}{v(x)v(qx)} \tag{13}$$

1.2.6. The q-Integral Operator

A function $F(x)$ is a q-antiderivative of $f(x)$ if $D_q F(x) = f(x)$. It is denoted by $F(x) = \int f(x) d_q x$ and called the Jackson integral [10]. We make the following remarks.

- Similar to classical integral calculus, any given function has multiple q-antiderivatives.

- Though the q-antiderivative of a function might not be unique, it can prove that if $q \in (0,1)$, a function has up to an additive constant one q-antiderivative which is continuous at $x = 0$ [12].
- In [8] it is shown that;

$$F(x) = \int f(x) d_q x = (1-q) x \sum_{k=0}^{\infty} q^k f(q^k x) \tag{14}$$

1.2.7. Properties of the q-Integral

1) The Jackson integral gives a q-antiderivative $F(x)$ which is continuous at $x = 0$, and is unique up to additive constant.

2) Generally, given $g(x)$ is another function and $D_q g(x)$ denotes its q-derivative, we have formally

$$\begin{aligned}\int f(x) D_q g(x) d_q x &= (1-q) x \sum_{r=0}^{\infty} q^r f(q^r x) D_q g(q^r x)\\ &= (1-q) x \sum_{r=0}^{\infty} q^r f(q^r x) \frac{g(q^r x) - g(q^{r+1} x)}{(1-q) q^r x}\end{aligned}$$

$$\Rightarrow \quad \int f(x) d_q g(x) = \sum_{r=0}^{\infty} f(q^r x)\left(g(q^r x) - g(q^{r+1} x)\right) \tag{15}$$

giving a q-analogue of the Riemann–Stieltjes integral [12]. As an example of the foregoing, let $f(x) = x^n$, $n \in \mathbb{Z}^+$. We have

$$\int x^n d_q x = (1-q) x \sum_{j=0}^{\infty} q^j q^{jn} x^n = (1-q) x^{n+1} \sum_{j=0}^{\infty} q^{j(n+1)} = \frac{(1-q)}{1-q^{n+1}} x^{n+1} = \frac{x^{n+1}}{[\![n+1]\!]_q} \tag{16}$$

3) The integration by parts formula of Newton-Leibnitz calculus is interpreted in the present non-commutative context as;

$$\int f_2(x) D_q f_1(x) d_q x = f_1(x) f_2(x) - \int f_1(qx) D_q f_2(x) d_q x \tag{17}$$

1.3. Partial q-Derivative of a Multivariable Function

We define the partial q-derivative of a multivariable real continuous function $f(x_1, x_2, \cdots, x_n)$ with respect to a variable x_i by;

$$D_{q,x_i} f(\boldsymbol{x}) = \frac{(\varepsilon_{q,i} f)(\boldsymbol{x}) - f(\boldsymbol{x})}{(q-1) x_i}, \quad x_i \neq 0,\ q \in (0,1) \tag{18}$$

$$\left[D_{q,x_i} f(\boldsymbol{x})\right]_{x_i=0} = \lim_{x_i \to 0} D_{q,x_i} f(\boldsymbol{x}) \tag{19}$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and, $(\varepsilon_{q,i} f)(\boldsymbol{x}) = f(x_1, x_2, \cdots, qx_i, \cdots, x_n)$.

We adopt subsequently the identity $D_{q,x}^k \equiv \dfrac{\partial_q^k}{\partial_q x^k}$ for the k^{th} order q-derivative with respect to x^k.

Our objective in this section is to solve partial q-differential equations using a novel q-differential transform method introduced in [13].

1) Reduced q-Differential Transform Method (RqDTM)

Given that all q-differentials of $u(x,t)$ exist in some neighborhood of $t=a$, then let

$$U_k(x)=\frac{1}{[\![k]\!]_q!}\left[\frac{\partial_q^k}{\partial_k t^k}u(x,t)\right]_{t=a} \tag{20}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. Subsequently, the lowercase $u(x,t)$ represents the original function while the uppercase $U_k(x)$ stands for the transformed function. We have the following important definition.

2) Definition

The q-differential inverse transform of $U_k(x)$ is defined by;

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)(t-a)^{(k)} \tag{21}$$

Substituting Equations (20) in (21) we obtain

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{[\![k]\!]_q!}\left[\frac{\partial_q^k}{\partial_k t^k}u(x,t)\right]_{t=a}(t-a)^{(k)} \tag{22}$$

In the subsequent theorems, we set $a=0$ so that $(t-a)^{(k)}=(t-0)^{(k)}=t^k$. From the linearity of the q-derivative, we can establish the following fact; given $w(x,t)=\alpha u(x,t)\pm v(x,t)$ then $W_k(x)=\alpha U_k(x)\pm V_k(x)$, α being a constant. We have the following important theorem.

3) Theorem: Given $w(x,t)=x^m t^n$ then $W_k(x)=x^m\delta(k-n)$ where

$$\delta(k)=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases} \tag{23}$$

Proof: By definition (20), we have;

$$\begin{aligned}W_k(x)&=\frac{1}{[\![k]\!]_q!}\left[\frac{\partial_q^k\left(x^m t^n\right)}{\partial_q t^k}u(x,t)\right]_{t=0}=\frac{x^m}{[\![k]\!]_q!}\left[\frac{\partial_q^k\left(x^m t^n\right)}{\partial_q t^k}u(x,t)\right]_{t=0}\\&=\begin{cases}x^m\cdot\dfrac{[\![k]\!]_q!}{[\![k]\!]_q!}=x^m, & k=n\\ x^m\cdot\dfrac{[\![n_q]\!]\cdot[\![(n-1)]\!]_q\cdots[\![(n-k+1)]\!]_q}{[\![k]\!]_q!}t^{n-k}\Big|_{t=0}, & k\neq 0\\ x^m\cdot 0=0, & k>n\end{cases}\\&=x^m\delta(k-n)\end{aligned} \tag{24}$$

4) Theorem: Given $w(x,t)=\frac{\partial_q}{\partial_q x}u(x,t)$ then $W_k(x)=\frac{\partial_q}{\partial_q x}U(x)$.

Proof

$$\begin{aligned}W_k(x)&=\frac{1}{[\![k]\!]_q!}\left[\frac{\partial_q^k}{\partial_q t^k}\left(\frac{\partial_q}{\partial_q x}u(x,t)\right)\right]=\frac{1}{[\![k]\!]_q!}\left[\frac{\partial_q}{\partial_q x}\left(\frac{\partial_q^k}{\partial_q t^k}u(x,t)\right)\right]_{t=0}\\&=\frac{\partial_q}{\partial_q x}\left[\frac{1}{[\![k]\!]_q!}\frac{\partial_q^k}{\partial_q t^k}u(x,t)\right]_{t=0}=\frac{\partial_q}{\partial_q x}U_k(x)\end{aligned} \tag{25}$$

5) Theorem: Given $w(x,t)=\frac{\partial_q^r}{\partial_q x}u(x,t)$ then

$$W_k(x)=[\![k+1]\!]_q [\![k+2]\!]_q \cdots [\![k+r]\!]_q U_{k+r}(x) \tag{26}$$

Proof

$$\begin{aligned} W_k(x) &= \frac{1}{[\![k]\!]_q!}\left[\frac{\partial_q^k}{\partial_q t^k}\left(\frac{\partial_q^r}{\partial_q t^r}u(x,t)\right)\right] \\ &= \frac{[\![k+r]\!]_q!}{[\![k]\!]_q!}\frac{1}{[\![k+r]\!]_q!}\left(\frac{\partial_q^{k+r}}{\partial_q t^{k+r}}u(x,t)\right)_{t=0} \\ &= [\![k+1]\!]_q [\![k+2]\!]_q \cdots [\![k+r]\!]_q U_{k+r}(x) \end{aligned}$$

6) Example

$$\frac{\partial_q}{\partial_q t}u(x,t)=u^2(x,t)+\frac{\partial_q}{\partial_q x}u(x,t),\quad u(x,0)=1+3x \tag{27}$$

Taking the RqDTM of the given partial q-differential equation, we have

$$[\![k+1]\!]_q U_{k+1}(x)=\sum_{n=0}^{k}U_{k-n}(x)U_k(x)+\frac{\partial_q}{\partial_q x}U_k(x),\quad k=0,1,2,\cdots \tag{28}$$

The initial condition becomes

$$U_0(x)=u(x,0)=1+3x.$$

Starting with $k=0$, the values of $U_k(x)$ are computed successively as follows;

$$\begin{aligned} [\![1]\!]_q U_1(x) &= U_0(x)U_0(x)+\frac{\partial_q}{\partial_q x}U_0(x) \\ &= (1+3x)^2+\frac{\partial_q}{\partial_q x}(1+3x)=(1+3x)^2+3 \end{aligned} \tag{29}$$

$$\therefore\quad U_1(x)=4+6x+9x^2$$

When $k=1$, we have

$$\begin{aligned} [\![2]\!]_q U_2(x) &= 2U_0(x)U_1(x)+\frac{\partial_q}{\partial_q x}U_1(x) \\ &= 2(1+3x)(4+6x+9x^2)+\frac{\partial_q}{\partial_q x}(4+6x+9x^2) \\ &= 2(4+6x+9x^2+12x+18x^2+27x^3)+(6+9x(1+q)) \end{aligned}$$

$$\therefore\quad U_2(x)=\frac{14+9(5+q)x+54x^2+54x^3}{1+q} \tag{30}$$

Following the same procedure, it is easy to compute an expression for $k=2$.

Formally, we have the required solution of the partial q-differential equation to be

$$u(x,t)=1+3x+(4+6x+9x^2)t+\left(\frac{14+9(5+q)x+54x^2+54x^3}{1+q}\right)t^2+\cdots \tag{31}$$

Let us now consider the classical version of the given partial q-differential equation, namely

$$\frac{\partial}{\partial t}u(x,t)=u^2(x,t)+\frac{\partial}{\partial x}u(x,t),\quad u(x,0)=1+3x \tag{32}$$

The above is a first order quasilinear partial differential equation easily solved by the method of characteristics. The associated auxiliary equations are;

$$\frac{\mathrm{d}t}{1}=-\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}u}{u^2} \tag{33}$$

These provide us with two possible integrals given by;

$$t+x=c_1 \quad \text{and} \quad t+\frac{1}{u}=c_2 \tag{34}$$

c_1, c_2 being arbitrary constants of integration. Using the initial condition, we have at $t=0, u=1+3x$. Hence, $x=c_1$ and $\frac{1}{1+3x}=c_2$. It then follows that $\frac{1}{1+3c_1}=c_2$.

Hence the required solution is

$$t+\frac{1}{u}=\frac{1}{1+3(t+x)} \quad \text{or} \quad u(t,x)=\frac{1+3(t+x)}{1-t(1+3x)-3t^2} \tag{35}$$

Using MathCAD14, a numerical algebra software [14] to expand the expression for u in terms of powers of t, we obtain

$$u(t,x)=(1+3x)+\left(4+6x+9x^2\right)t+\left(7+27x^2+27x^3\right)t^2+\cdots \tag{36}$$

We make the interesting observation that when we set $q=1$ in (31) we obtain exactly the solution to the classical PDE.

7) Example. As an example of a second order partial q-differential equation we consider the q-diffusion Cauchy problem

$$\frac{\partial_q}{\partial_q t}u(x,t)=\frac{\partial_q^2}{\partial_q x^2}u(x,t),\quad u(x,0)=e_q^x \tag{37}$$

Again we employ the RqDTM transform. Here e_q^x in the initial data, is the q-exponential function defined by $e_q^x=\sum_{i=0}^{\infty}\frac{x^i}{[\![i]\!]_q!}$, with q-derivative e_q^x. The RqDTM transform of the q-diffusion equation, gives the following recursion;

$$[\![k+1]\!]_q U_{k+1}(x)=\frac{\partial_q^2}{\partial_q x^2}U_k(x),\quad k=0,1,2,\cdots \tag{38}$$

The initial data given is then written

$$U_0(x)=u(x,0)=e_q^x \tag{39}$$

Now, substituting (38) into (37), we obtain the following $U_k(x)$ values successively

$$U_1(x) = \frac{1}{[\![1]\!]_q!} e_q^x,\ U_2(x) = \frac{1}{[\![2]\!]_q!} U_1(x) = \frac{1}{[\![1]\!]_q![\![2]\!]_q!} e_q^x = \frac{1}{[\![2]\!]_q!} e_q^x \tag{40}$$

$$U_3(x) = \frac{1}{[\![3]\!]_q!} e_q^x, \cdots, U_k(x) = \frac{1}{[\![k]\!]_q!} e_q^x \tag{41}$$

In view of (28), the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k t^k = \sum_{k=0}^{\infty} \frac{1}{[\![k]\!]_q!} e_q^x t^k = e_q^x e_q^t \tag{42}$$

which is the analytic solution of the problem (36). Now consider the classical diffusion equation is written

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t),\quad u(x,0) = \mathrm{e}^x \tag{43}$$

By the method of separation of variables, it is easy to show that the solution is given by

$$u(x,t) = X(x)T(t) = \mathrm{e}^x \mathrm{e}^t = \mathrm{e}^{x+t} \tag{44}$$

Comparing with the q-solution $u(x,t) = e_q^x e_q^t$, we see that the two solutions are in perfect agreement when $q = 1$, and satisfying their appropriate initial conditions.

1.4. Conclusion

In this research article, we introduced the concept of q-calculus in quantum geometry. This involves the study of the basic rules governing q-calculus as compared with the classical Newton-Leibnitz calculus. Our main objective is to employ the results obtained to solve partial q-differential equations. To this end we introduced the reduced q-differential transform method which provides the solution in the form of a convergent power series with easily computable components. With the help of a few examples, we were able to show that the proposed iteration technique is very effective and convenient. It turns out that when $q = 1$, the solution coincides with the classical version of the given initial value problem. In conclusion, q-calculus is a non-commutative calculus that generalizes the Newton-Leibnitz classical calculus. The Reduced Differential Transform method for solving differential equations was introduced and extended in this work to solve partial q-differential equations, which represent some form of dynamics in non-commutative spaces.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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