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New Modified Burr III Distribution, Properties and Applications

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Abstract: In this article, Burr III distribution is proposed with a significantly improved functional form. This new modification has enhanced the flexibility of the classical distribution with the ability to model all shapes of hazard rate function including increasing, decreasing, bathtub, upside-down bathtub, and nearly constant. Some of its elementary properties, such as rth moments, sth incomplete moments, moment generating function, skewness, kurtosis, mode, ith order statistics, and stochastic ordering, are presented in a clear and concise manner. The well-established technique of maximum likelihood is employed to estimate model parameters. Middle-censoring is considered as a modern general scheme of censoring. The efficacy of the proposed model is asserted through three applications consisting of complete and censored samples.

Keywords: Burr III distribution; stochastic ordering; middle-censoring; order statistics

MSC: 60E05; 62N05; 62F10

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1. Introduction

Burr devised a dynamic family of probability distributions based on the Pearson differential equations. The Burr XII (BXII) and Burr III (BIII) distributions are widely used models from the system of Burr distributions. On the contrary, according to [1], the Burr X (BX) model has also gained much attention from applied statisticians along with the BXII and BIII models. The prime reason is that these densities exists in simpler forms and can yield a range of shapes to model a variety of scenarios in diverse scientific fields. The authors in [2] are of the view that the most adaptable of these three is BIII, especially in environmental, reliability, and survival sciences. The BIII distribution is also called the Dagum distribution in studies of income, wage, and wealth distribution [3]. In the actuarial literature, it is known as the inverse Burr distribution [4] and the kappa distribution in the meteorological literature [5]. As per [4], it is a prime case of the four-parameter generalised Beta-II distribution. In order to follow the ambit regarding the scope of this provision, we now shift our attention to the BIII distribution. For a random variable X defined on a positive real line, the cumulative distribution function (cdf) and probability density function (pdf) of two-parameter BIII distribution, respectively, are given below:

$$F(x;c,k) = \left(1 + x^{-c}\right)^{-k} \tag{1}$$

and

$$f(x; c, k) = c\,k\,x^{-c-1}\left(1 + x^{-c}\right)^{-k-1}, \tag{2}$$

where $c, k > 0$ are the shape parameters.

The shape parameter plays a significant role in yielding the hazard rate of BIII distribution, which can be decreasing or unimodal. Thus, it cannot be used to model lifetime data with a bathtub-shaped hazard function, such as human mortality and deterioration modelling. For the last few decades, statisticians have been developing various extensions and modifications in Weibull distribution due to its simple functional form. The two-parameter flexible Weibull extension of [6] has a hazard function that can be increasing, decreasing, or bathtub shaped. Zhang and Xie [7] studied the characteristics and application of the truncated Weibull distribution, which has a bathtub-shaped hazard function. A three-parameter model, called exponentiated Weibull distribution, was introduced by [8]. Another three-parameter model is referred to as the extended Weibull distribution by [9]. Xie et al. [10] proposed a three-parameter modified Weibull extension with a bathtub-shaped hazard function. A new modified Weibull distribution by the authors in [11] has been presented with increasing and a bathtub-shaped hazard function.

Various extensions of BIII distribution have been studied in the literature. In reference [12], the authors studied low-flow frequency analysis in hydrology with three-parameter-modified BIII distribution with supreme interest in the lower tail of a distribution. Çankaya et al. [13] extended the BIII model by adding a skew parameter with an epsilon skew extension approach. Modi and Gill [14] introduced the unit BIII model. Haq et al. [15] introduced the unit-modified BIII model. Ali et al. [16] re-parameterized BIII distribution and proposed the modified BIII (MBIII) distribution with the following cdf:

$$F(x) = \left(1 + \mu x^{-c}\right)^{\frac{-k}{\mu}} \qquad x > 0, \tag{3}$$

where c, k, and μ are the shape parameters. The authors claimed that the newly structured model is a limiting case of generalized inverse Weibull, BIII, and log-logistic distribution. Still, the density of the improved model can only model positively skewed data, which greatly dented the proposition of the model in the first place. Other extensions are mostly based on the generalized families of distributions that sare complex in nature. Some of them are mentioned as: Beta Dagum by [17], Modified BIII by [18], Marshall Olkin BIII by [19], Gamma BIII by [20], and Gamma BIII by [21]. However, we feel that a flexible model with computationally simpler functional forms is still presently needed. Motivated by a lack of availability of literature related to the modified BIII distribution, we present a much more flexible new modification of BIII distribution. The cdf of the new, modified BIII (NMBIII) distribution is defined as

$$F(x; c, k, \lambda) = \left(1 + x^{-c}\mathrm{e}^{-\lambda x}\right)^{-k} \qquad x > 0, \tag{4}$$

where the $\mathrm{e}^{-\lambda x}$ is the additional factor, with λ as the rate parameter and c, k are power parameters of the baseline model.

It is worth mentioning that when we use the additional term to add flexibility in the model, we specifically refer to the ability of the proposed model to fit a diverse range of real life phenomena. Additionally, flexibility may also be associated with the instantaneous failure rate or hazard rate, and is more commonly known as risk function. By selecting precise values for the shape parameters, the hazard rate function of the NMBIII distribution can take on a variety of appealing shapes. Generally speaking, the classical models deal with normal extreme observations. A new modification of BIII distribution will also enable us to observe the tail behaviour of the distribution, which is skewed in nature. Further, the BIII distribution has a monotonic decreasing and unimodal hazard rate function, but due to its modification, NMBIII has monotonic, decreasing, increasing, unimodal, bathtub, and approximately constant hazard-rate shapes. Moreover, many standard distributions are nested models or limiting cases of the Burr system of distributions, which include the

Weibull, exponential, logistic, generalised logistic, Gompertz, normal, extreme value, and uniform distributions. The NMBIII distribution outperforms most of these competitive existing models. When $\lambda = 0$, NMBIII distribution reduces to BIII distribution. When $\lambda = 0$ and $k = 1$, then NMBIII distribution gives us log-logistic distribution. When $k = 1$, then NMBIII distribution gives us modified log-logistic distribution (new). When $c = 0$ and $k = 1$, the NMBIII distribution reduces to logistic distribution. When $c = 1$, it reduces to modified skew logistic distribution (new). When $c = 0$ and $\lambda = 1$, it reduces to generalized logistic distribution type I or Burr type II, or this type has also been called the "skew-logistic" distribution (see [22]). In a nutshell, with the proposed NMBIII, we seek and hope to attract applied researchers from all scientific community to utilize it in the significant modelling of real-life scenarios.

The article is structured as follows: In Section 2, we focus our attention on the idea behind the new modification. {In Section 3, we acquaint the readers with some of the structural properties including the linear expansion, moments, mode, moment-generating functions, order statistics, and stochastic ordering of NMBIII distribution. In Section 4, model parameters are estimated by maximum likelihood method, and the Fisher information matrix is derived. Section 5 gives the simulation method based on complete and incomplete samples (middle censored). In Section 6, three data sets on complete and middle-censored data sets have been employed to established the authenticity of the proposed model to the readers. Section 7 consists of the concluding remarks and discussions.

2. The New Modified BIII Model

The modified Weibull (MW) distribution (see [23] has the cumulative survival function that is the product of the Weibull cumulative hazard function αx^{β} and $e^{\lambda x}$. Hence, the distribution function was found to be

$$F(x) = \left(1 - e^{-\alpha x^{\beta}} e^{\lambda x}\right),$$

which was later generalized to exponentiated form by [24] using Lehmann alternative-I.

In the same vein, Equation (4) has been modified. The pdf corresponding to (4) is given as:

$$f(x; c, k, \lambda) = \frac{k\left(\lambda + \frac{c}{x}\right)}{x^{c}\, e^{\lambda x}} \left(1 + x^{-c}\, e^{-\lambda x}\right)^{-k-1}. \tag{5}$$

The corresponding survival and hazard functions of NMBIII are, respectively, given by:

$$S(x; c, k, \lambda) = 1 - \left(1 + x^{-c}\, e^{-\lambda x}\right)^{-k} \tag{6}$$

and

$$h(x; c, k, \lambda) = \frac{k\left(\lambda + \frac{c}{x}\right)}{x^{c}\, e^{\lambda x}} \frac{\left(1 + x^{-c}\, e^{-\lambda x}\right)^{-k-1}}{1 - \left(1 + x^{-c}\, e^{-\lambda x}\right)^{-k}}. \tag{7}$$

If a new random variable y is defined as $y = \frac{1}{x}$ in Equation (4), then we obtain the following model, referred to as modified Burr XII distribution, with cdf and pdf, respectively, as under

$$G(y) = 1 - \left(1 + \frac{y^{c}}{e^{\frac{\lambda}{y}}}\right)^{-k} \tag{8}$$

and

$$g(y) = \frac{k\left(c + \frac{1}{y}\right)}{e^{\frac{\lambda}{y}}} y^{c-1} \left(1 + \frac{y^{c}}{e^{\frac{\lambda}{y}}}\right)^{-k-1}. \tag{9}$$

As far as we can tell, Equations (4) and (8) are first modifications of BIII distribution and BXII distributions, respectively. Thus, the proposed distribution in (4) is more flexible and has tractable tail properties than its parent BIII distribution as well as MBIII distributions. The shapes of pdf and hrf are presented in Figures 1 and 2, respectively.

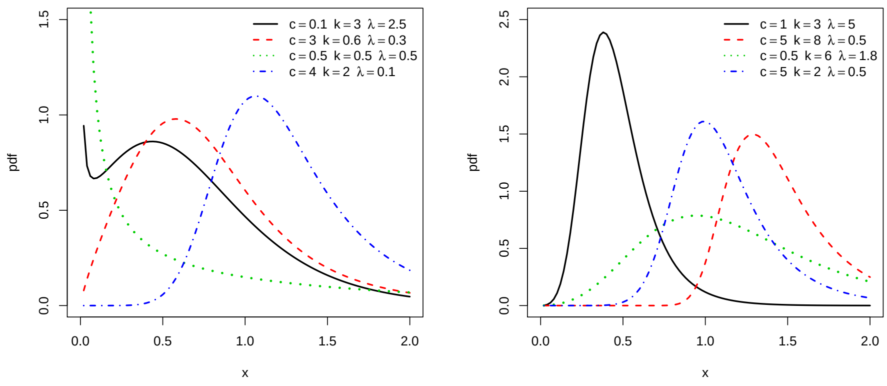

Figure 1. Density function of NMBIII distribution.

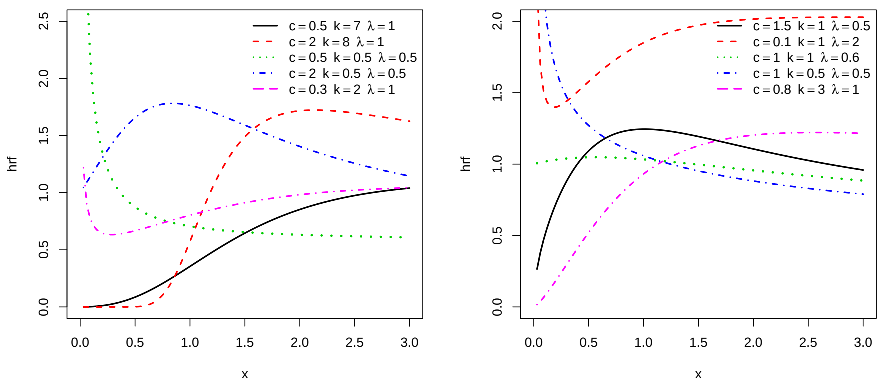

Figure 2. Hazard function for NMBIII distribution.

Figure 1 represents the different shapes of the proposed model, i.e., bimodal, reversed-J, right skewed, approximate left-skewed, and symmetrical shapes for different parameter values. Figure 2 reflects the different shapes of hazard function, which are increasing, decreasing, bathtub, upside-down bathtub, and nearly constant for different parameter values. The proposed distribution is more flexible and tractable than its parent BIII distribution, as well as MBIII distributions (see in Table 1).

Table 1. Sub models of NMBIII distributions.

Model	λ	c	k	$G(x)$	**Reference**
Burr III	0	-	-	$(1+x^{-c})^{-k}$	Standard
Log-Logistic	0	-	1	$\frac{x^c}{1+x^c}$	Standard
Modified Log-Logistic	-	-	1	$\frac{x^c e^{-\lambda x}}{1+x^c e^{-\lambda x}}$	New
Logistic	-	0	1	$\frac{e^{-\lambda x}}{1+e^{-\lambda x}}$	Standard
Modified skew logistic	-	1	-	$\frac{x e^{-\lambda x}}{1+x e^{-\lambda x}}$	New
Generalized logistic Type-I or Burr II or skew logistic	1	0	-	$(1+e^{-x})^{-k}$	Johnson et al. [22] and Aljouiee et al. [25]

3. Some Properties of NMBIII

In this section, we will provide some significant properties of the NMBIII distribution such as rth moment, sth incomplete moment, moment generating function, skewness, kurtosis, mode, and order statistics.

3.1. Useful Expansion

The generalized binomial theorem or power series is given by:

$$(1+z)^{-b-1} = \sum_{i=0}^{\infty} \binom{b+i}{i} (-1)^i z^i. \tag{10}$$

Using series expansion in (10), Equation (4) becomes

$$f(x;c,k,\lambda) = \sum_{i=0}^{\infty} \binom{k+i}{i} (-1)^i \frac{k\left(\lambda+\frac{c}{x}\right)}{x^{c(i+1)}\, \mathrm{e}^{\lambda\, x(i+1)}}. \tag{11}$$

This expression can be used to obtain the following properties of the NMBIII distribution.

3.2. Moments

The rth moment of NMBIII distribution is given by:

$$\begin{aligned}
m_r' = E(X^r) &= \int_0^{\infty} x^r f(x)\, dx \\
&= \sum_{i=0}^{\infty} \binom{k+i}{i} (-1)^i \int_0^{\infty} x^{r-c(i+1)} \left(\lambda+\frac{c}{x}\right) \mathrm{e}^{-\lambda\,(i+1)\,x}\, dx \\
&= \lambda \sum_{i=0}^{\infty} a_i \int_0^{\infty} x^{r-c(i+1)}\, \mathrm{e}^{-\lambda\,(i+1)\,x}\, dx + c \sum_{i=0}^{\infty} a_i \int_0^{\infty} x^{r-c(i+1)-1}\, \mathrm{e}^{-\lambda\,(i+1)\,x}\, dx \\
&= \lambda \sum_{i=0}^{\infty} a_i\, \Gamma(r-c(i+1)-1) \left[\frac{1}{\lambda(i+1)}\right]^{r-c(i+1)-1} \\
&\quad + c \sum_{i=0}^{\infty} a_i\, \Gamma(r-c(i+1)) \left(\frac{1}{\lambda(i+1)}\right)^{r-c(i+1)} \\
&= \lambda \sum_{i=0}^{\infty} a_i \frac{\Gamma(r-c(i+1)-1)}{(\lambda(i+1))^{r-c(i+1)}} \left(\frac{1}{i+1} + c(r-c(i+1)-1)\right),
\end{aligned} \tag{12}$$

where $a_i = \binom{k+i}{i} (-1)^i$ and $\Gamma(a)\, b^a = \int_0^{\infty} x^{a-1}\, e^{-b\,x}$ dx is gamma function.

Remark 1. *By submitting $r=1$ in Equation (13), one can find mean of the NMBIII distribution.*

The sth incomplete moment of NMBIII distribution is

$$\begin{aligned}
T_s'(x) = \lambda \sum_{i=0}^{\infty} a_i\, \gamma\!\left(r-c(i+1)-1, \frac{x}{\lambda(i+1)}\right) \left(\frac{1}{\lambda(i+1)}\right)^{r-c(i+1)-1} \\
+ c \sum_{i=0}^{\infty} a_i\, \gamma\!\left(r-c(i+1), \frac{x}{\lambda(i+1)}\right) \left(\frac{1}{\lambda(i+1)}\right)^{r-c(i+1)}.
\end{aligned} \tag{13}$$

The application of incomplete moment refers to the mean deviations and Bonferroni and Lorenz curves. These curves are useful in economics reliability, demography, insurance, and medicine, to mention few.

3.3. Moment-Generating Function

The moment-generating function of NMBIII distribution is given by:

$$
\begin{aligned}
M_0(t) = E(e^{t\,x}) &= \int_{i=0}^{\infty} e^{t\,x} f(x)\,dx \\
&= \sum_{i=0}^{\infty} \binom{k+i}{i} (-1)^i \int_{i=0}^{\infty} x^{-c(i+1)} \left(\lambda + \frac{c}{x}\right) e^{(t-\lambda\,(i+1))\,x}\,dx \\
&= \sum_{i=0}^{\infty} a_i \int_{i=0}^{\infty} x^{-c(i+1)} \left(\lambda + \frac{c}{x}\right) e^{(t-\lambda\,(i+1))\,x}\,dx \\
&= \sum_{i=0}^{\infty} a_i \left(\lambda \int_0^{\infty} x^{-c(i+1)}\, e^{(t-\lambda\,(i+1))\,x}\,dx + c \int_0^{\infty} x^{-c(i+1)-1}\, e^{(t-\lambda\,(i+1))\,x}\,dx\right) \\
&= \sum_{i=0}^{\infty} a_i \left(\lambda \frac{\Gamma(1-c(i+1))}{(\lambda(i+1)-t)^{1-c(i+1)}} + c\, \frac{\Gamma(-c(i+1))}{(\lambda(i+1)-t)^{-c(i+1)}}\right).
\end{aligned}
\tag{14}
$$

The skewness and kurtosis of the NMBIII distribution can be obtained numerically by the following expression.

$$
\alpha = \frac{m_3' - 3\,m_2'\,m_1' + 2\,m_1'}{\left\{m_2' - (m_2')^2\right\}^{3/2}} \tag{15}
$$

and

$$
\beta = \frac{m_4' - 4\,m_3' m_1' + 6\,m_2'\,(m_1')^2 - 3\,(m_1')^4}{\left\{m_2' - (m_2')^2\right\}^2}, \tag{16}
$$

where m_r' is the rth moment can be obtained form Equation (13).

Remark 2. *The mode of the NMBIII distribution can be obtained as follows: taking the* log *of Equation* (5)*, one obtains*

$$
\log f(x) = \log k + \log\left(\lambda + \frac{c}{x}\right) - c\,\log x - \lambda x - (k+1)\,\log\left(1 + x^{-c}\, e^{-\lambda\,x}\right), \tag{17}
$$

Taking derivative with respect to x, we get

$$
\frac{d}{d\,x}\log f(x) = \frac{-\frac{1}{x^2}}{\lambda + \frac{c}{x}} - \frac{c}{x} - \lambda + (k+1)\frac{x^{-c}\,e^{-\lambda\,x}\left(\lambda + \frac{c}{x}\right)}{1 + x^{-c}\,e^{-\lambda\,x}}, \tag{18}
$$

by setting the above expression equal to zero and solving for x, one can find the mode. The numerical values of the first four moments are given in Table 2.

Table 2. The numerical values of the first four moments $(m_r', r = 1,2,3,4)$, skewness (α) and kurtosis (β) of the NMBIII for some parameter values.

c, k, λ	m_1'	m_2'	m_3'	m_4'	α	β
(0.5, 0.5, 0.5)	0.6754	2.0695	10.4250	72.6365	3.3418	20.5484
(1.5, 0.5, 0.5)	0.6662	1.0760	3.1293	14.2983	3.1239	27.1548
(1.5, 1.5, 0.5)	1.2849	2.6939	8.7612	41.8399	2.4599	23.6830
(1.5, 1.5, 1.5)	0.8024	0.8745	1.2564	2.3394	1.6650	70.6890
(2.0, 0.5, 0.5)	0.6814	0.9031	2.0319	7.3171	2.8155	31.7670
(2.0, 2.0, 0.5)	1.3695	2.6073	7.0943	27.8595	2.4280	39.1836
(2.0, 2.0, 2.0)	0.8041	0.7682	0.8775	1.2098	1.5101	226.2743

3.4. Order Statistics

The density function $f_{i:n}(x)$ of the i-th order statistic, for $i = 1, \ldots, n$, from i.i.d. random variables $X_1, \ldots, X_2$ following MBIII distribution is simply given by:

$$F_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{j+i} F(x)^{j+i}. \tag{19}$$

The corresponding pdf is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x)\, F(x)^{j+i-1}. \tag{20}$$

Using the pdf and cdf of NMBIII in Equations (4) and (5), we obtain

$$F_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{j+i} \left[1 + x^{-c}\, \mathrm{e}^{-\lambda\, x}\right]^{-k(j+i)}. \tag{21}$$

Using series expansion in (10), we obtain

$$F_{i:n}(x) = \sum_{j=0}^{n-i} b_j \sum_{l=0}^{\infty} \binom{k(j+i)+l}{l} (-1)^l\, x^{-c\, l}\, \mathrm{e}^{-\lambda\, l\, x}, \tag{22}$$

where $b_j = \frac{n!}{(i-1)!(n-i)!} \binom{n-i}{j} \frac{(-1)^j}{j+i}$. Similarly, following the above algebra, we have

$$f_{i:n}(x) = \sum_{j=0}^{n-i} a_j \sum_{l=0}^{\infty} \binom{j+i+l}{l} (-1)^l\, x^{-c\,(l+1)} \left(\lambda + \frac{c}{x}\right) \mathrm{e}^{-\lambda\,(l+1)\,x}, \tag{23}$$

where $a_j = k\, \frac{n!}{(i-1)!(n-i)!} \binom{n-i}{j} (-1)^j$.

3.5. Stochastic Ordering

The concept of stochastic ordering is frequently used to show the ordering mechanism in life-time distributions. For more details about stochastic ordering, see [26]. A random variable is said to be stochastically greater ($X \leq_{st} Y$) than Y if $F_X(x) \leq F_Y(x)$ for all x. In the similar way, X is said to be stochastically lower ($X \leq_{st} Y$) than Y in the

1. Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x.
2. Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x.
3. Mean residual order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x.
4. Likelihood ratio order ($X \leq_{hr} Y$) if $f_X(x) \geq f_Y(x)$ for all x.
5. Reversed hazard rate order ($X \leq_{rhr} Y$) if $\frac{F_X(x)}{F_Y(x)}$ is decreasing for all x.

The stochastic orders defined above are related to each other, as the following implications.

$$X \leq_{rhr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y. \tag{24}$$

Let $X_1 \sim NMBIII(c_1, k_1, \lambda_1)$ and $X_2 \sim NMBIII(c_2, k_2, \lambda_2)$. Then, according to the definition of likelihood ratio ordering $\left[\frac{f(x)}{g(x)}\right]$,

$$f(x) = \frac{k_1 \left(\lambda_1 + \frac{c_1}{x}\right)}{x^{c_1}\, e^{\lambda_1 x}} \left(1 + x^{-c_1}\, e^{-\lambda_1 x}\right)^{-k_1 - 1}, \tag{25}$$

$$g(x) = \frac{k_2 \left(\lambda_2 + \frac{c_2}{x}\right)}{x^{c_2}\, e^{\lambda_2 x}} \left(1 + x^{-c_2} e^{-\lambda_2 x}\right)^{-k_2 - 1}, \tag{26}$$

and

$$\frac{f(x)}{g(x)} = \frac{k_1}{k_2} \frac{\left(\lambda_1 + \frac{c_1}{x}\right)}{\left(\lambda_2 + \frac{c_2}{x}\right)} \frac{x^{c_1}\, e^{\lambda_1 x}}{x^{c_1}\, e^{\lambda_1 x}} \frac{\left(1 + x^{-c_1}\, e^{-\lambda_1 x}\right)^{-k_1 - 1}}{\left(1 + x^{-c_2}\, e^{-\lambda_2 x}\right)^{-k_2 - 1}}. \tag{27}$$

Taking log on both sides and taking the derivative with respect to x, we obtain

$$\begin{aligned} \frac{d}{d\,x}\left(\frac{f(x)}{g(x)}\right) = {} & \frac{c_1}{x_i^2 \left(\lambda_1 + \frac{c_1}{x}\right)} - \frac{c_2}{x_i^2 \left(\lambda_2 + \frac{c_2}{x}\right)} + \frac{c_2 - c_1}{x_i} + (\lambda_2 - \lambda_1) \\ & + (k_2 + 1) \frac{x^{-c_2}\, e^{\lambda_2 x} \left(\lambda_2 + \frac{c_2}{x}\right)}{1 + x^{-c_2}\, e^{\lambda_2 x}} - (k_1 + 1) \frac{x^{-c_1}\, e^{\lambda_1 x} \left(\lambda_1 + \frac{c_1}{x}\right)}{1 + x^{-c_1}\, e^{\lambda_1 x}}, \end{aligned} \tag{28}$$

if $c_1 = c_2 = c$ and $\lambda_1 = \lambda_2 = \lambda$, then $\frac{d}{d\,x} \frac{f(x)}{g(x)} < 0$ if $(k_2 < k_1)$ and then $X <_{lr} Y$.

4. Maximum Likelihood Estimation

In this section, we will use the maximum-likelihood method to estimate the unknown parameters of the proposed model from complete samples only. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the NMBIII family given in Equation (4) distribution. The log-likelihood function for the vector of parameter $\Theta = (c, k, \lambda)^T$ can be expressed as

$$\begin{aligned} l(\Theta) = {} & n \log k - c \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left(\lambda + \frac{c}{x}\right) \\ & - (k+1) \sum_{i=1}^{n} \log\left[1 + x^{-c}\, e^{-\lambda x}\right] \end{aligned}$$

Taking the derivative with respect to λ, c, k, respectively, we get

$$\begin{aligned} U_k &= \frac{\partial\, l(\Theta)}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \log\left(1 + x^{-c}\, e^{-\lambda x}\right) \\ U_\lambda &= \frac{\partial\, l(\Theta)}{\partial \lambda} = - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \left(\lambda + \frac{c}{x}\right)^{-1} + (k+1) \sum_{i=1}^{n} \left(\frac{x^{-c}\, e^{-\lambda x}\, x_i}{1 + x^{-c}\, e^{-\lambda x}}\right) \\ U_c &= \frac{\partial\, l(\Theta)}{\partial c} = - \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left(\frac{1}{x_i \left(\lambda + \frac{c}{x}\right)}\right) + (k+1) \sum_{i=1}^{n} \left(\frac{x^{-c}\, e^{-\lambda x} \log x_i}{1 + x^{-c}\, e^{-\lambda x}}\right) \end{aligned}$$

Setting U_k, U_λ, and U_k equal zero and solving these equations simultaneously yields the maximum likelihood estimates.

The observed information matrix for the parameter vector is given by

$$\begin{pmatrix} U_{k\,k} & U_{K\,\lambda} & U_{k\,c} \\ - & U_{\lambda\,\lambda} & U_{\lambda\,c} \\ - & - & U_{c\,c} \end{pmatrix}$$

whose elements are given below

$$
\begin{aligned}
U_{k\,k} &= -\frac{n}{k^2} \\
U_{k\,\lambda} &= \sum_{i=1}^{n}\left(\frac{x_i^{-c-1}\,e^{\lambda\,x_i}}{1+x^{-c}\,e^{-\lambda\,x_i}}\right) \\
U_{k\,c} &= -\sum_{i=1}^{n}\left(\frac{x_i^{-c}\,e^{\lambda\,x_i}\,\log x_i}{1+x^{-c}\,e^{-\lambda\,x_i}}\right) \\
U_{\lambda\,c} &= -\sum_{i=1}^{n}\frac{1}{x_i\left(\lambda+\frac{c}{x_i}\right)^2}+(k+1)\sum_{i=1}^{n}\left(\frac{x^{1-2c}\,e^{-2\,\lambda\,x_i}\log x_i}{\left(1+x^{-c}\,e^{-\lambda\,x_i}\right)^2}+\frac{e^{-\lambda\,x_i}\,x_i^{1-c}\,\log x_i}{1+x^{-c}\,e^{-\lambda\,x_i}}\right) \\
U_{\lambda\,\lambda} &= -\sum_{i=1}^{n}\frac{1}{\left(\lambda+\frac{c}{x_i}\right)^2}-(k+1)\sum_{i=1}^{n}\left(\frac{x^{2-2c}\,e^{-2\,\check{}\,x_i}}{\left(1+x^{-c}\,e^{-\lambda\,x_i}\right)^2}+\frac{e^{-\lambda\,x_i}\,x_i^{2-c}}{1+x^{-c}\,e^{-\lambda\,x_i}}\right) \\
U_{c\,c} &= -\sum_{i=1}^{n}\frac{1}{x_i^2\left(\lambda+\frac{c}{x_i}\right)^2}+(k+1)\sum_{i=1}^{n}\left(\frac{x^{-2c}\,e^{-2\,\lambda\,x_i}(\log x_i)^2}{\left(1+x^{-c}\,e^{-\lambda\,x_i}\right)^2}+\frac{e^{-\lambda\,x_i}\,x_i^{-c}\,\log x_i}{1+x^{-c}\,e^{-\lambda\,x_i}}\right)
\end{aligned}
$$

5. Middle-Censoring

The middle-censoring scheme is a non-parametric general censoring mechanism proposed by [27], where other censoring schemes can be obtained as special cases of this middle-censoring scheme (see [28]).

For n identical lifetimes $T_1,\ldots,T_n$ with a random censoring interval $(L_i \le R_i)$ at the ith item with some unknown bivariate distribution. Then, the exact value of T_i is observable only if $T_i \notin [L_i \le R_i]$; otherwise, the interval $(L_i \le R_i)$ is observed.

Middle-censoring had previously been applied to exponential and Burr XII lifetime distributions (see [28,29]). Furthermore, it was extended to parametric models with covariates [30], and its robustness was investigated by [31].

In this section, we analyse the NMBIII lifetime data when they are middle-censored. Assume that $T_1,\ldots,T_n$ are *i.i.d.* NMBIII (c,λ,k) random variable and let $Z_i = R_i - L_i$, $i = 1,\ldots,n$ be another random variable that defines the length of the censoring interval with exponential distribution with mean γ^{-1}, where the left-censoring point for each individual L_i is assumed to also be an exponential random variable with mean θ^{-1}. Moreover, the $T_i^{'}s$, $L_i^{'}s$, and $Z_i^{'}s$ are all independent of each other and the observed data, and $X_i^{'}s$ are given by $X_i = \begin{cases} T_i & if \quad T_i \notin (L_i \le R_i), \\ (L_i \le R_i) & otherwise. \end{cases}$

5.1. Estimation

For n randomly selected units from the NMBIII (c,λ,k) population, where c, λ, and k are unknown, were tested under middle-censoring scheme. In this setting, there are $n_1 > 0$ uncensored observations and $n_2 > 0$ censored observations. Then, by re-ordering the observed data into the uncensored and censored observations, we therefore have the following data

$$\{T_1,\ldots,T_{n_1},(L_{n_1+1},R_{n_1+1}),\ldots,(L_{n_1+n_2},R_{n_1+n_2})\},$$

where $n_1 + n_2 = n$.

The likelihood function of the observed data is given by:

$$
\begin{aligned}
L(c,\lambda,k|x) = \omega(k)^{n_1}\prod_{i=1}^{n_1}(\lambda+\frac{c}{x_i})\prod_{i=1}^{n_1}(x_i^{-c}e^{-\lambda x_i})\prod_{i=1}^{n_1}(1+x_i^{-c}e^{-\lambda x_i})^{-k-1} \\
\times\prod_{i=n_1+1}^{n_1+n_1}[(1+r_i^{-c}e^{-\lambda r_i})^{-k}-(1+l_i^{-c}e^{-\lambda l_i})^{-k}],
\end{aligned}
$$

where ω is a normalizing constant depending on γ and θ, and the estimation of them is not of interest and this is left as a constant. The log-likelihood function is given by

$$l(c,\lambda,k|x) = \log\omega + n_1\log k + \sum_{i=1}^{n_1}\log(\lambda+\frac{c}{x_i}) + n\sum_{i=1}^{n_1}\log(x_i^{-c}\mathrm{e}^{-\lambda x_i}) - (k+1)\sum_{i=1}^{n_1}\log(1+x_i^{-c}\mathrm{e}^{-\lambda x_i}) + \sum_{i=n_1+1}^{n_1+n_1}\log[(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k} - (1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k}].$$

The maximum-likelihood estimation (MLE) of c, λ, and k, denoted by $\widehat{c}_M$, $\widehat{\lambda}_M$, and $\widehat{k}_M$, can be derived by solving the following equations:

$$\frac{\partial l(c,\lambda,k|x)}{\partial c} = \sum_{i=1}^{n_1}(\lambda x_i+c)^{-1} - \sum_{i=1}^{n_1}\log x_i + (k+1)\sum_{i=1}^{n_1}\frac{(x_i^{-c}\mathrm{e}^{-\lambda x_i})\log x_i}{1+x_i^{-c}\mathrm{e}^{-\lambda x_i}} + \sum_{i=n_1+1}^{n_1+n_1}\frac{k(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k-1}(r_i^{-c}\mathrm{e}^{-\lambda r_i})\log(r_i) - k(1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k-1}(l_i^{-c}\mathrm{e}^{-\lambda l_i})\log(l_i)}{[(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k} - (1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k}]},$$

$$\frac{\partial l(c,\lambda,k|x)}{\partial \lambda} = \sum_{i=1}^{n_1}\frac{1}{\lambda+\frac{c}{x_i}} - \sum_{i=1}^{n_1}x_i - (k+1)\sum_{i=1}^{n_1}\frac{x_i^{-c+1}\mathrm{e}^{-\lambda x_i}}{1+x_i^{-c}\mathrm{e}^{-\lambda x_i}} - \sum_{i=n_1+1}^{n_1+n_1}\frac{k(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k-1}(r_i^{-c+1}\mathrm{e}^{-\lambda r_i}) - k(1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k-1}(l_i^{-c+1}\mathrm{e}^{-\lambda l_i})}{[(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k} - (1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k}]}$$

and

$$\frac{\partial l(c,\lambda,k|x)}{\partial k} = -\sum_{i=n_1+1}^{n_1+n_1}\frac{(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k}\log(1+r_i^{-c}\mathrm{e}^{-\lambda r_i}) - k(1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k}\log(1+l_i^{-c}\mathrm{e}^{-\lambda l_i})}{[(1+r_i^{-c}\mathrm{e}^{-\lambda r_i})^{-k} - (1+l_i^{-c}\mathrm{e}^{-\lambda l_i})^{-k}]} + \frac{n_1}{k} - \sum_{i=1}^{n_1}\log(1+x_i^{-c}\mathrm{e}^{-\lambda x_i}).$$

It is obvious that the MLE of c, λ, and k cannot be solved explicitly. Therefore, the solutions can be obtained using Newton–Raphson method or numerically using the solve systems of nonlinear equations "*nleqslv*" package in R.

Since the MLE is asymptotically normal, the approximate confidence intervals for the parameters c, λ and k can be computed as follows: $\hat{c}_M \pm z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_c^2}$, $\hat{\lambda}_M \pm z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_\lambda^2}$ and $\hat{k}_M \pm z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_k^2}$, where $\hat{\sigma}_{(.)}^2$ are the variances of the respective parameters c, k, and λ, and $z_{\frac{\alpha}{2}}$ is the value of the standard normal curve and α is the level of significance.

5.2. Simulation Results

We conducted Monte Carlo simulation studies to assess the finite sample behaviour of the MLEs of the parameters c, k and λ based on two settings; the first is the random variable generated from the NMBIII distribution, while the other considers the case where the NMBIII lifetime data were middle-censored.

The random samples for both settings were generated from distribution NMBIII(c,k,λ) based on accept-reject approach. Without loss of generality, random samples were used with five different sizes viz n = 10, 30, 50, 70, and 100 from NMBIII(c,k,λ) distribution with parameters $c=1, k=2,$ and $\lambda=0.5$.

The middle censoring settings considered three combinations of the censoring schemes $(\gamma^{-1},\theta^{-1}) = (0.25, 0.25)$, $(1, 0.75)$, and $(1.25, 0.5)$.

The results were obtained from 1000 Monte Carlo replications from simulations carried out using the software R, and the average estimates and the mean squared error (MSE) are obtained and reported in Table 3.

Results in Table 3 show that the ML estimates for both settings behave similarly. In general, there is a decreasing function between the sample size and the mean squared error, which verifies the consistency property of the derived estimators. The average estimates are insignificantly effected by the censoring status.

Table 3. Average MLE estimates and the corresponding MSE (within brackets).

Distribution	n	**Un-Censored**			**Middle-Censored**								
					(0.25, 0.25)			**(1, 0.75)**			**(1.25, 0.5)**		
(c,k,λ)		c	k	λ	c	k	λ	c	k	λ	c	k	λ
$(1,2,0.5)$	10	1.114 (0.130)	2.079 (0.102)	0.397 (0.122)	1.123 (0.141)	2.233 (0.163)	0.447 (0.096)	1.087 (0.111)	2.130 (0.159)	0.524 (0.108)	1.196 (0.121)	2.088 (0.099)	0.561 (0.125)
	30	1.039 (0.034)	2.036 (0.039)	0.464 (0.080)	1.082 (0.096)	2.170 (0.072)	0.452 (0.043)	1.072 (0.036)	2.080 (0.082)	0.519 (0.046)	1.127 (0.052)	2.080 (0.093)	0.547 (0.037)
	50	1.036 (0.03)	2.032 (0.031)	0.484 (0.029)	1.071 (0.033)	2.096 (0.031)	0.536 (0.032)	1.066 (0.028)	2.071 (0.032)	0.508 (0.028)	1.103 (0.022)	2.022 (0.025)	0.529 (0.027)
	70	1.015 (0.016)	1.984 (0.015)	0.511 (0.019)	1.035 (0.017)	2.018 (0.021)	0.510 (0.022)	1.042 (0.015)	2.053 (0.021)	0.496 (0.020)	1.042 (0.021)	1.985 (0.016)	0.476 (0.017)
	100	1.001 (0.012)	1.991 (0.013)	0.502 (0.011)	1.019 (0.013)	1.998 (0.015)	0.495 (0.014)	0.980 (0.015)	2.020 (0.016)	0.498 (0.017)	0.981 (0.016)	1.907 (0.013)	0.491 (0.013)
$(0.5,2,0.5)$	10	0.621 (0.052)	2.074 (0.040)	0.427 (0.151)	0.582 (0.127)	2.325 (0.086)	0.522 (0.063)	0.534 (0.056)	2.135 (0.084)	0.524 (0.088)	1.196 (0.080)	2.098 (0.105)	0.530 (0.096)
	30	0.613 (0.034)	2.057 (0.036)	0.464 (0.032)	0.531 (0.038)	2.264 (0.039)	0.513 (0.040)	0.529 (0.034)	2.104 (0.033)	0.516 (0.037)	1.127 (0.030)	2.087 (0.033)	0.521 (0.037)
	50	0.538 (0.026)	2.010 (0.012)	0.484 (0.044)	0.519 (0.094)	2.125 (0.067)	0.489 (0.032)	0.518 (0.019)	2.014 (0.064)	0.505 (0.037)	1.103 (0.031)	2.054 (0.016)	0.518 (0.031)
	70	0.5017 (0.012)	1.928 (0.009)	0.511 (0.041)	0.491 (0.057)	2.020 (0.037)	0.490 (0.021)	0.506 (0.012)	1.982 (0.035)	0.501 (0.013)	1.042 (0.017)	2.010 (0.012)	0.509 (0.014)
	100	0.492 (0.002)	2.003 (0.001)	0.502 (0.027)	0.504 (0.046)	2.003 (0.027)	0.507 (0.007)	0.492 (0.006)	2.004 (0.026)	0.499 (0.005)	0.981 (0.011)	1.923 (0.010)	0.495 (0.012)
$(2,2,2)$	10	2.212 (0.063)	2.452 (0.127)	2.517 (0.096)	2.298 (0.105)	2.571 (0.056)	2.322 (0.151)	2.331 (0.040)	2.280 (0.088)	2.371 (0.052)	2.102 (0.086)	2.493 (0.084)	2.256 (0.080)
	30	2.176 (0.043)	2.420 (0.096)	2.161 (0.037)	2.179 (0.093)	2.552 (0.036)	2.291 (0.080)	2.238 (0.039)	2.222 (0.046)	2.328 (0.034)	2.045 (0.072)	2.258 (0.082)	2.173 (0.052)
	50	1.962 (0.032)	2.013 (0.094)	2.008 (0.031)	2.057 (0.016)	2.150 (0.019)	2.171 (0.044)	2.061 (0.012)	2.064 (0.037)	2.091 (0.026)	1.959 (0.067)	2.041 (0.064)	1.901 (0.031)
	70	1.953 (0.021)	1.875 (0.057)	1.949 (0.014)	1.809 (0.012)	1.823 (0.012)	1.956 (0.041)	1.864 (0.009)	2.054 (0.013)	1.903 (0.012)	1.953 (0.037)	2.004 (0.035)	1.825 (0.017)
	100	2.045 (0.007)	2.113 (0.046)	2.160 (0.012)	2.070 (0.010)	2.503 (0.006)	2.207 (0.027)	2.183 (0.001)	2.145 (0.005)	2.143 (0.002)	2.026 (0.027)	2.125 (0.026)	2.144 (0.011)

6. Applications

This section provides three applications for complete data sets to show how the NMBIII distribution can be applied in practice. We compare NMBIII distribution to MBIII, BIII, Weibull (W), Gamma (Ga), Lognormal (LN), Generalized Weibull (EW), and Generalised Extreme value type-II (GEV-II) distributions. In these applications, the model parameters are estimated by the method of maximum likelihood. The Akaike information criterion (AIC), Bayesian information criterion (BIC), A*(Anderson Darling), and W*(Cramer–von Mises) are computed to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. Additionally, the asymptotic variance-covariance matrices of the NMBIII parameters are also provided. The plots of the fitted PDFs, CDFs, Probability–Probability (PP), and Quantile–Quantile (QQ) of NMBIII are displayed for visual comparison. The required computations are carried out in the R software.

The first data set consists of 119 observations on fracture toughness of Alumina (Al_2O_3) (in the units of MPa $m^{1/2}$. These data were studied by [32]. The second data set refers to the material thickness of hole (12 mm) and sheet (3.15 mm), comprising 50 observations, as reported by authors in [33]. The third data set was first analysed by [34] and represents the survival times, in weeks, of 33 patients suffering from Acute Myelogenous Leukaemia.

Tables 4–6 list the MLEs, standard errors, AIC, BIC, A*, and W* of the model for the data sets 1–3. The results in Tables 4–6 indicate that the NMBIII model provides the best fit as compared to all the other models. Figures 3–5 also support the results of Tables 4–6.

Table 4. Data set 1.

Model	Parameters	MLE	Standard Error	AIC	BIC	A*	W*
NMBII	c	2.543	0.507	362.159	370.497	1.888	0.296
	k	25.243	5.185				
	λ	1.703	0.179				
MBIII	c	1111.230	461.820	379.380	387.718	3.515	0.583
	k	4.943	0.281				
	μ	770.050	398.963				
BIII	c	3.058	0.180	423.535	429.094	7.658	1.365
	k	51.879	11.180				
W	α	0.002	0.0002	394.821	405.379	1.955	0.422
	β	3.984	0.0773				
Ga	α	15.521	1.991	385.737	374.295	2.745	0.457
	β	3.588	0.468				
LN	μ	1.432	0.025	428.845	434.403	3.374	0.568
	σ	0.269	0.0174				
EW	α	0.0114	0.006	374.644	386.981	1.945	0.315
	β	3.2126	0.278				
	θ	2.0077	0.388				
GEV-II	α	48.447	10.816	425.796	431.354	7.875	1.408
	β	3.022	0.185				

The variance–covariance matrix of the MLEs of the NMBIII distribution for data set 1 is

$$\begin{pmatrix} 0.25674663 & 0.2275027 & -0.08608337 \\ 0.22750269 & 26.8853417 & 0.18880701 \\ -0.08608337 & 0.1888070 & 0.03215706 \end{pmatrix}$$

Table 5. Data set 2.

Model	Parameters	MLE	Standard Error	AIC	BIC	A*	W*
NMBII	c	2.802	1.620	−106.358	−100.622	0.524	0.090
	k	0.317	0.219				
	λ	17.274	5.605				
MBIII	c	0.0020	0.0002	−99.778	−94.042	0.988	0.159
	k	3.466	0.205				
	μ	0.0039	0.0007				
BIII	c	7.788	26.572	−26.027	−22.202	1.056	0.177
	k	0.065	0.221				
W	α	36.141	14.390	−101.784	−93.960	0.644	0.105
	β	2.118	0.246				
Ga	α	3.029	0.576	−102.743	−98.919	1.636	0.279
	β	18.561	3.836				
LN	μ	1.987	0.095	105.700	109.524	1.922	0.331
	σ	0.670	0.067				
EW	α	819.305	2409.321	−106.069	−100.333	0.535	0.093
	β	4.982	2.636				
	θ	0.297	0.200				
GEV-II	α	0.054	0.020	−70.449	−66.625	3.567	0.634
	β	1.236	0.118				

The variance–covariance matrix of the MLEs of the NMBIII distribution for data set 2 is

$$\begin{pmatrix} 2.6257440 & 0.34637888 & 8.616552 \\ 0.3463789 & 0.04803978 & -1.104049 \\ 8.6165520 & -1.10404897 & 31.417567 \end{pmatrix}$$

Table 6. Data set 3.

Model	Parameters	MLE	Standard Error	AIC	BIC	A*	W*
NMBII	c	0.521	0.121	303.703	308.101	0.440	0.064
	k	4.734	1.065				
	λ	0.012	0.005				
MBIII	c	153.592	319.615	309.465	313.863	0.672	0.098
	k	1.494	0.464				
	μ	0.201	796.017				
BIII	c	0.755	0.092	309.714	312.645	0.919	0.151
	k	5.705	1.228				
W	α	0.057	0.028	304.302	307.234	0.552	0.079
	β	0.792	0.112				
Ga	α	0.706	0.150	304.357	309.288	0.459	0.085
	β	0.017	0.005				
LN	μ	2.884	0.266	320.9177	323.8491	0.648	0.102
	σ	1.504	0.188				
EW	α	0.0431	0.186	306.296	310.693	0.554	0.079
	β	0.844	0.794				
	θ	0.901	1.352				
GEV-II	α	4.259	0.933	310.463	313.395	0.983	0.160
	β	0.685	0.091				

The variance–covariance matrix of the MLEs of the NMBIII distribution for data set 3 is

$$\begin{pmatrix} 0.014574568 & 0.075708071 & -0.0003995590 \\ 0.075708071 & 1.134424017 & -0.001232737 \\ -0.000399559 & -0.001232737 & 0.00002838794 \end{pmatrix}$$

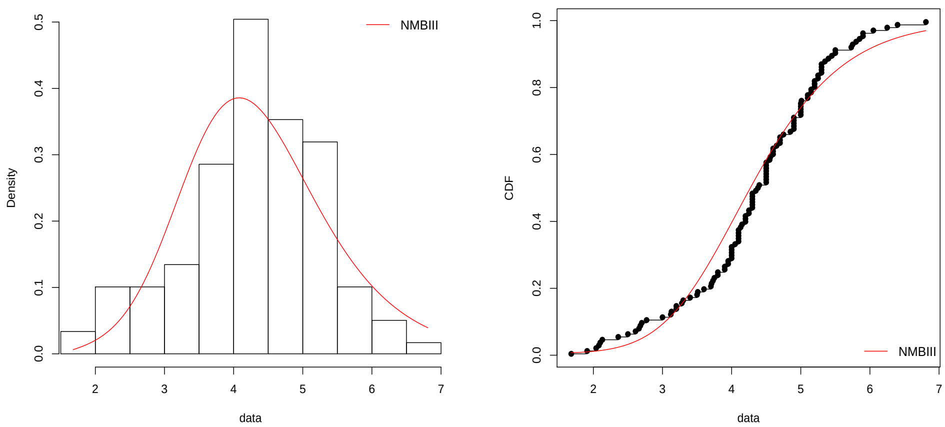

Figure 3. *Cont.*

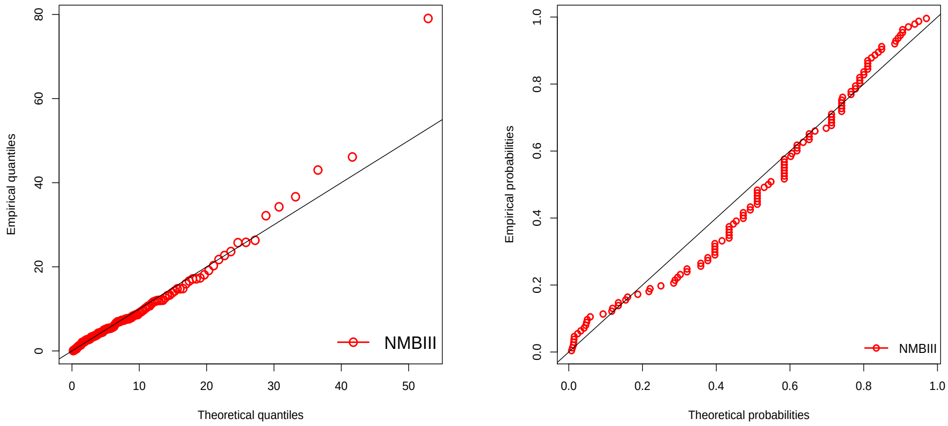

Figure 3. Estimated density (**top left**), cdf (**top right**), QQ-plot (**bottom left**), and PP-plot (**bottom right**) for data set 1.

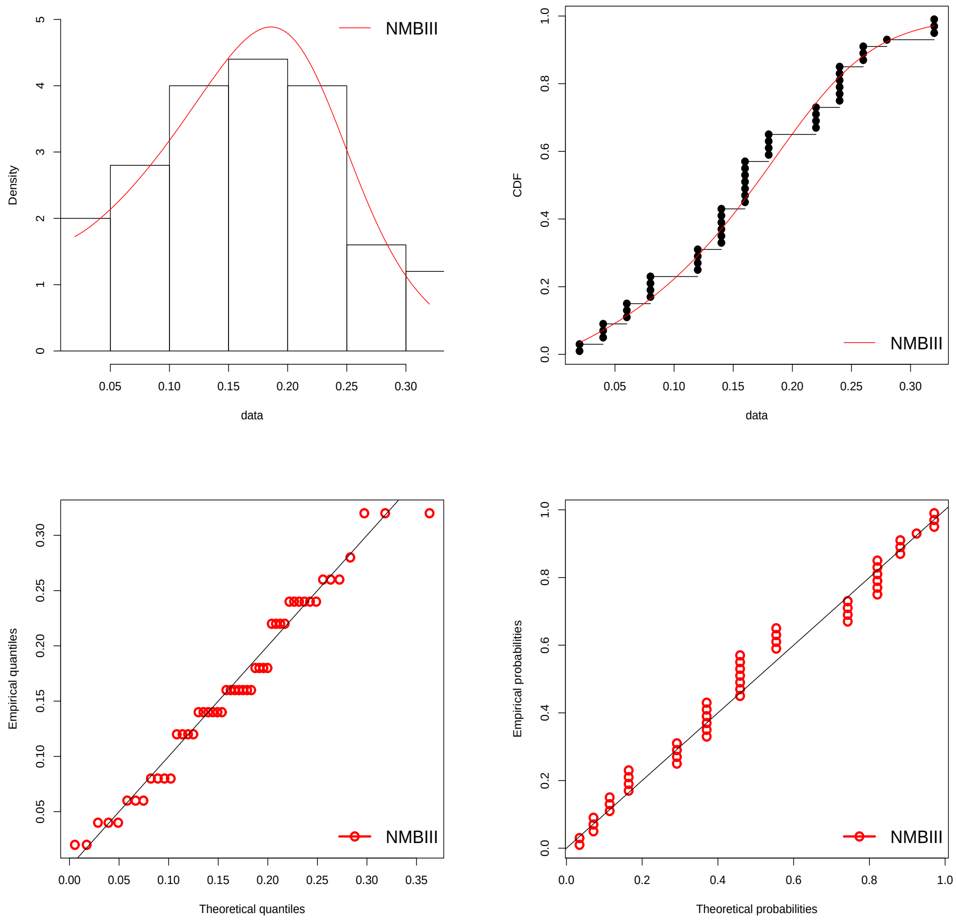

Figure 4. Estimated density (**top left**), cdf (**top right**), QQ-plot (**bottom left**) and PP-plot (**bottom right**) for data set 2.

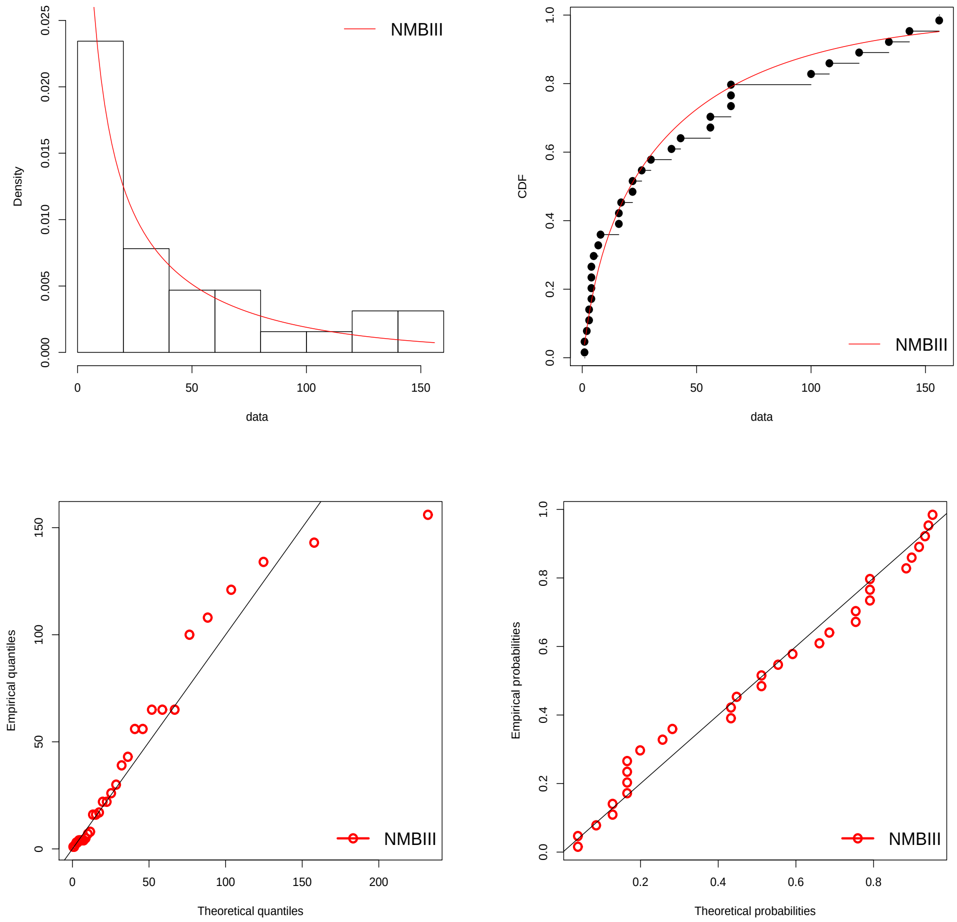

Figure 5. Estimated density (**top left**), cdf (**top right**), QQ-plot (**bottom left**), and PP-plot (**bottom right**) for data set 3.

7. Conclusions

A good theory should seek out the most concise explanation for the facts. With this in mind, a new modified form of BIII distribution has been introduced that can model well-specified forms of hazard rate shapes, including increasing, decreasing, bathtub, upside-down bathtub, and nearly constant. Some of its statistical properties, such as, rth moment, sth incomplete moment, moment generating function, skewness, kurtosis, mode, ith order statistics, and stochastic ordering have been derived. The maximum likelihood estimation is employed to estimate the model parameters. The usefulness of this model is demonstrated by applications on complete and censored samples. Simulation study is also performed. A future effort would include the contributions of new regression models, Bayesian parameter estimations, and research into diversified fields of data sets.

Author Contributions: F.J. gave the initial idea and proposed the new model. M.A.N. investigated mathematical properties. A.H.A. added the Middle-Censoring part of the paper. W.K.M. and M.H.T. wrote and structured the manuscript. S.K. investigated the numerical calculations applied to the data set and improved the overall presentation of the current draft. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: No conflict of interest was declared by the authors.

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