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The Local Antimagic Chromatic Numbers of Some Join Graphs

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Abstract: Let $G = (V(G), E(G))$ be a connected graph with n vertices and m edges. A bijection $f : E(G) \to \{1, 2, \cdots, m\}$ is an edge labeling of G. For any vertex x of G, we define $\omega(x) = \sum_{e \in E(x)} f(e)$ as the vertex label or weight of x, where $E(x)$ is the set of edges incident to x, and f is called a local antimagic labeling of G, if $\omega(u) \neq \omega(v)$ for any two adjacent vertices $u, v \in V(G)$. It is clear that any local antimagic labelling of G induces a proper vertex coloring of G by assigning the vertex label $\omega(x)$ to any vertex x of G. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of different vertex labels taken over all colorings induced by local antimagic labelings of G. In this paper, we present explicit local antimagic chromatic numbers of $F_n \vee \overline{K_2}$ and $F_n - v$, where F_n is the friendship graph with n triangles and v is any vertex of F_n. Moreover, we explicitly construct an infinite class of connected graphs G such that $\chi_{la}(G) = \chi_{la}(G \vee \overline{K_2})$, where $G \vee \overline{K_2}$ is the join graph of G and the complement graph of complete graph K_2. This fact leads to a counterexample to a theorem of Arumugam et al. in 2017, and our result also provides a partial solution to Problem 3.19 in Lau et al. in 2021.

Keywords: local antimagic labeling; local antimagic chromatic number; join graph; friendship graph

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1. Introduction

Throughout, we only consider undirected connected simple graphs. Let $G = (V(G), E(G))$ be a connected graph with n vertices and m edges. A bijection $f : E(G) \to \{1, 2, \cdots, m\}$ is an edge labeling of G. For any vertex x of G, we define $\omega(x) = \sum_{e \in E(x)} f(e)$ as the vertex label or weight of x, where $E(x)$ is the set of edges incident to x, and f is called an antimagic labeling of G, if $\omega(u) \neq \omega(v)$ for any two distinct vertices $u, v \in V(G)$. A graph G is called antimagic if G has an antimagic labeling.

The antimagic labeling of a graph was initially introduced by Hartsfield and Ringel [1] in 1990. They conjectured that every connected graph except K_2 admits such an antimagic labeling, which remains open till today.

Recently, based on the concept of antimagic labeling, Arumugam et al. [2] and Bensmail et al. [3] independently introduced the notation local antimagic labeling of graphs in 2017, which is weaker than antimagic labeling of graphs. Let $G = (V(G), E(G))$ be a connected graph of order n and size m. A bijection $f : E(G) \to \{1, 2, \cdots, m\}$ is called a local antimagic labeling of G if any two adjacent vertices u and v in G satisfy $\omega(u) \neq \omega(v)$. It is clear that assigning $\omega(x)$ to x for each $x \in V(G)$ naturally induced a proper vertex coloring of G, which is called a local antimagic vertex coloring of G. A graph G is called local antimagic if G has a local antimagic labeling. Haslegrave [4] showed that every connected graph with at least three vertices is local antimagic. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of different vertex labels taken over all colorings of G induced by local antimagic labelings of G. If f is a local antimagic labeling of G, the number of distinct induced vertex labels under f, denoted by $c(f)$, is called *the color number of f*.

A friendship graph, denoted by F_n, is a simple graph in which any two vertices have exactly one common neighbour, which consists of n triangles with a common vertex. In [2], Arumugam et al. gave the exact value of the local antimagic chromatic numbers of special graphs, such as P_n, C_n, F_n, $K_{m,n}$, $K_{2,n}$, W_n, and $L(n)$, where P_n and C_n are path and cycle with n vertices, respectively, $K_{m,n}$ is the complete bipartite graph ($m \equiv n \pmod 2$), W_n is the wheel graph ($n \not\equiv 0 \pmod 4$), and $L(n)$ is the graph obtained by inserting a vertex to each edge of the star S_n. Ref. [5] was used in [2] to determine local antimagic chromatic numbers of complete bipartite graphs. When the graph is the wheel graph for $n \equiv 0 \pmod 4$ or the join graph $G \vee \overline{K_2}$ for $|V(G)| \geq 4$, where $\overline{K_2}$ is the complement graph of complete graph K_2, they also provided the lower and upper bounds of the local antimagic chromatic numbers of these graphs.

In 2018, Lau et al. [6] gave counterexamples to the lower bound of $\chi_{la}(G \vee \overline{K_2})$ that was obtained in [2]. Another counterexample was independently found by Shaebani [7]. A sharp lower bound of $\chi_{la}(G \vee \overline{K_n})$ and sufficient conditions for the given lower bound were obtained. Moreover, they gave affirmative solutions on Problem 3.3 of [2] and settled Theorem 2.15 of [2]. They also completely determined the local antimagic chromatic number of complete bipartite graphs.

In [8], Lau et al. provided several sufficient conditions for $\chi_{la}(H) \leq \chi_{la}(G)$, where H is obtained from G with a certain number of edge-deleted or -added operations. They then determined the exact values of the local antimagic chromatic numbers of many cycle-related join graphs.

In 2019, Lau et al. [9] gave the sharp lower bound of the local antimagic chromatic number of a graph with cut-vertices given by pendant edges and then solved Problem 3.3 in [2] affirmatively. In Section 2 of [9], Lau et al. gave sufficient conditions for the one-point union of cycles with $\chi_{la}(G) = 2$. In Section 3 of [9], they determined the exact values of the local antimagic chromatic numbers of many families of graphs with pendant edges. Finally, in Section 4, they obtained a few families of graphs with $\chi_{la}(G) = n$. This partially answered Problem 3.1 in [2].

Based on some known results, in this paper, we present the exact local antimagic chromatic numbers of $F_n \vee \overline{K_2}$ and $F_n - v$, where v is any vertex of F_n. Moreover, we explicitly construct an infinite class of connected graphs G such that $\chi_{la}(G) = \chi_{la}(G \vee \overline{K_2}) = 3$, where $G \vee \overline{K_2}$ is the join graph of G and the complement graph of K_2. This fact leads to a counterexample to a theorem of [2], and our result also provides a partial solution to Problem 3.19 in [8].

2. Main Results

In [2], the authors gave the local antimagic chromatic number of the friendship graph as shown in the following lemma.

Lemma 1 ([2]). *Let F_n be a friendship graph, then we have $\chi_{la}(F_n) = 3$.*

For two vertex disjoint graphs F_n and $\overline{K_2}$, let $F_n \vee \overline{K_2}$ denote the join graph obtained by joining every vertex of F_n with every vertex of $\overline{K_2}$. In the proof of the local antimagic chromatic number of $F_n \vee \overline{K_2}$, we write $i \equiv t \pmod s$ $(0 \leq t < s)$ as $i \overset{s}{\equiv} t$ in the following formula. The following theorem gives an exact value of the local antimagic chromatic number of $F_n \vee \overline{K_2}$.

Theorem 1. *Let H be the join graph $F_n \vee \overline{K_2}$, then we have $\chi_{la}(H) = 4$.*

Proof. Let $\{v, v_1, v_2, \cdots, v_{2n}\}$ be the vertex set of the friendship graph F_n, where v is its central vertex, and let x, y be the two vertices of $\overline{K_2}$. It is clear that there are $7n + 2$ edges in H, namely, $\{v_i v_{i+1} : 1 \leq i \leq 2n \text{ and } i \equiv 1 \pmod 2\} \cup \{vv_i, xv_i, yv_i : 1 \leq i \leq 2n\} \cup \{xv, yv\}$. Since K_4 is an induced subgraph of H, we have $\chi_{la}(H) \geq \chi(H) \geq 4$. In order to prove

$\chi_{la}(H) = 4$, it suffices to provide a local antimagic labeling of H that induces a local antimagic vertex coloring using exactly four colors.

We suppose that there is a local antimagic labeling $f : E(H) \rightarrow \{1, 2, 3, \cdots, 7n + 2\}$, such that $c(f) = 4$. It means that $\omega(v_1) = \omega(v_3) = \cdots = \omega(v_{2n-1})$, $\omega(v_2) = \omega(v_4) = \cdots = \omega(v_{2n})$, and $\omega(x) = \omega(y)$, which are distinct with $\omega(v)$. In this regard, we first assign $f(xv_i) = i$ or $4n + 1 - i$ and $f(yv_i) = 4n + 1 - i$ or i, for each $i \in \{1, 2, \cdots, 2n\}$, then determine the exact value of remaining edges of H. Let us consider the following four cases.

Case 1. $n \equiv 1 \pmod 4$

For $n = 1$, the graph $H = F_1 \vee \overline{K_2}$ admits a local antimagic labeling f with $c(f) = 4$ as shown in Figure 1, which shows that $\chi_{la}(H) \leq 4$, and so $\chi_{la}(H) = 4$.

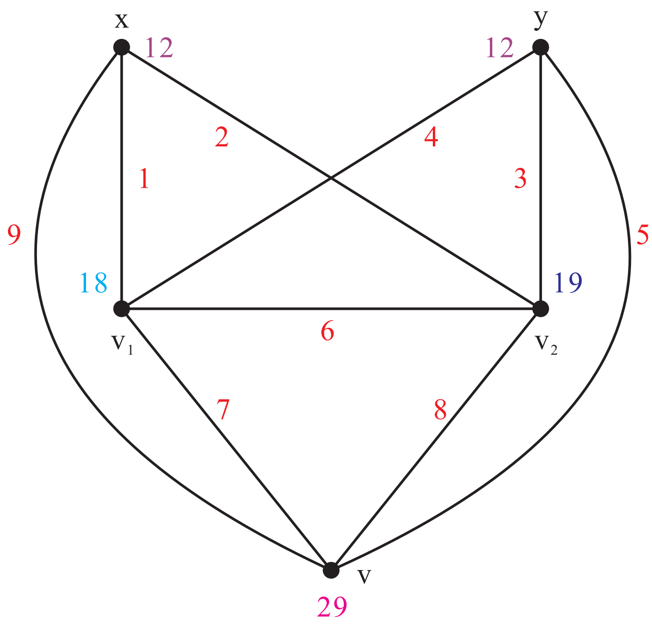

Figure 1. $F_1 \vee \overline{K_2}$.

For $n = 5$, we give the exact value of every edge label for the graph $H = F_5 \vee \overline{K_2}$ as shown in Figure 2.

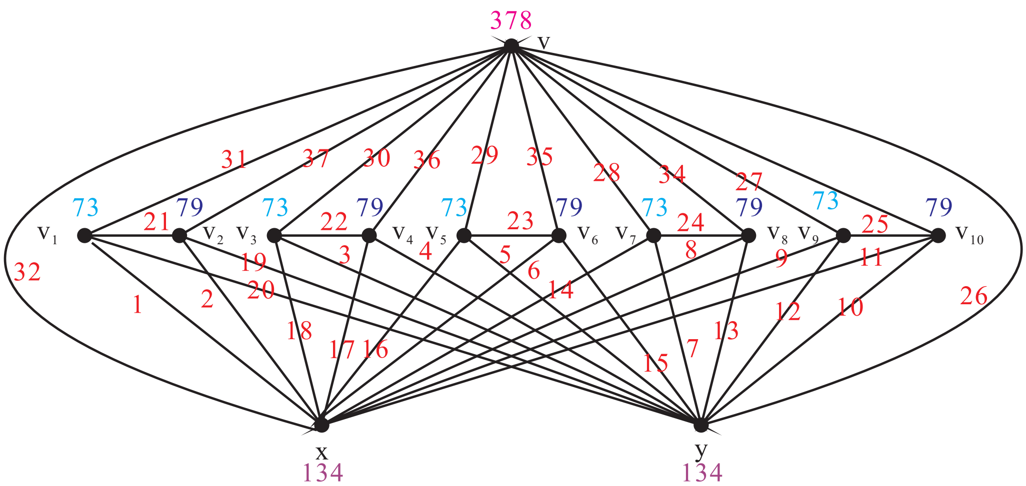

Figure 2. $F_5 \vee \overline{K_2}$.

It is obvious that

$$\begin{aligned}
&\omega(x) = \omega(y) = 134,\\
&\omega(v_1) = \omega(v_3) = \omega(v_5) = \omega(v_7) = \omega(v_9) = 73,\\
&\omega(v_2) = \omega(v_4) = \omega(v_6) = \omega(v_8) = \omega(v_{10}) = 79,\\
&\omega(v) = 378.
\end{aligned}$$

From the above labeling, f is a local antimagic labeling of H that induces a local antimagic vertex coloring using exactly four colors. It means that $\chi_{la}(H) \leq 4$, and so $\chi_{la}(H) = 4$.

For $n \geq 9$, define $f : E(H) \to \{1, 2, \cdots, 7n + 2\}$ in the following way:

Let $f(xv) = 6n + 2,\ f(yv) = 5n + 1$, and determine the values of $f(xv_i)$ and $f(yv_i)$ for each $i \in \{1, 2, 3, 4, 5, 2n - 4, 2n - 3, 2n - 2, 2n - 1, 2n\}$ as follows.

$$f(xv_i) = \begin{cases} i, & \text{if } i \in \{1, 2, 2n-4, 2n-2, 2n-1\},\\ 4n+1-i, & \text{if } i \in \{3, 4, 5, 2n-3, 2n\}. \end{cases}$$

$$f(yv_i) = \begin{cases} 4n+1-i, & \text{if } i \in \{1, 2, 2n-4, 2n-2, 2n-1\},\\ i, & \text{if } i \in \{3, 4, 5, 2n-3, 2n\}. \end{cases}$$

Then label the edges xv_i and yv_i for $6 \leq i \leq 2n - 5$, respectively.

$$f(xv_i) = \begin{cases} i, & \text{if } i \overset{8}{\equiv} 6,\ i \overset{8}{\equiv} 0,\ i \overset{8}{\equiv} 1 \text{ or } i \overset{8}{\equiv} 2,\\ 4n+1-i, & \text{if } i \overset{8}{\equiv} 7,\ i \overset{8}{\equiv} 3,\ i \overset{8}{\equiv} 4 \text{ or } i \overset{8}{\equiv} 5. \end{cases}$$

$$f(yv_i) = \begin{cases} 4n+1-i, & \text{if } i \overset{8}{\equiv} 6,\ i \overset{8}{\equiv} 0,\ i \overset{8}{\equiv} 1 \text{ or } i \overset{8}{\equiv} 2,\\ i, & \text{if } i \overset{8}{\equiv} 7,\ i \overset{8}{\equiv} 3,\ i \overset{8}{\equiv} 4 \text{ or } i \overset{8}{\equiv} 5. \end{cases}$$

Finally, we give the exact value of the remaining edges as follows.

$$\begin{aligned}
&f(v_i v_{i+1}) = 4n + \tfrac{i+1}{2}, \quad i \overset{2}{\equiv} 1 \text{ and } 1 \leq i \leq 2n,\\
&f(vv_i) = 6n + 2 - \tfrac{i+1}{2}, \quad i \overset{2}{\equiv} 1,\\
&f(vv_i) = 7n + 3 - \tfrac{i}{2}, \quad i \overset{2}{\equiv} 0.
\end{aligned}$$

Since $n \equiv 1 \pmod 4$ and $n \geq 9$, we have $2n \equiv 2 \pmod 8$, and so the number of vertices in $\{v_i | 6 \leq i \leq 2n - 5\}$ is divisible by 8.

If $\{i, i + 1, i + 2, \cdots, i + 7\} \subseteq \{6, 7, \cdots, 2n - 5\}$ and $i \equiv 6 \pmod 8$, then

$$\sum_{j=i}^{i+7} f(xv_j) = 16n - 6, \quad \sum_{j=i}^{i+7} f(yv_j) = 16n + 14.$$

Accordingly, we have

$$\sum_{i=6}^{2n-5} f(xv_i) = 4n^2 - \frac{43n}{2} + \frac{15}{2},$$

$$\sum_{i=6}^{2n-5} f(yv_i) = 4n^2 - \frac{33n}{2} - \frac{35}{2}.$$

Since

$$\sum_{i=1}^{5} f(xv_i) = 12n - 6, \quad \sum_{i=2n-4}^{2n} f(xv_i) = 10n - 2,$$

$$\sum_{i=1}^{5} f(yv_i) = 8n + 11, \qquad \sum_{i=2n-4}^{2n} f(xv_i) = 10n + 7.$$

It is clear that f is a local antimagic labeling of H and

$$\omega(x) = \omega(y) = 4n^2 + \tfrac{13n}{2} + \tfrac{3}{2},$$
$$\omega(v_i) = 14n + 3, \; i \overset{2}{\equiv} 1,$$
$$\omega(v_i) = 15n + 4, \; i \overset{2}{\equiv} 0,$$
$$\omega(v) = 12n^2 + 15n + 3.$$

Hence, $\chi_{la}(H) \leq 4$. The local antimagic labeling of the graph $F_9 \vee \overline{K_2}$ is shown in Figure 3.

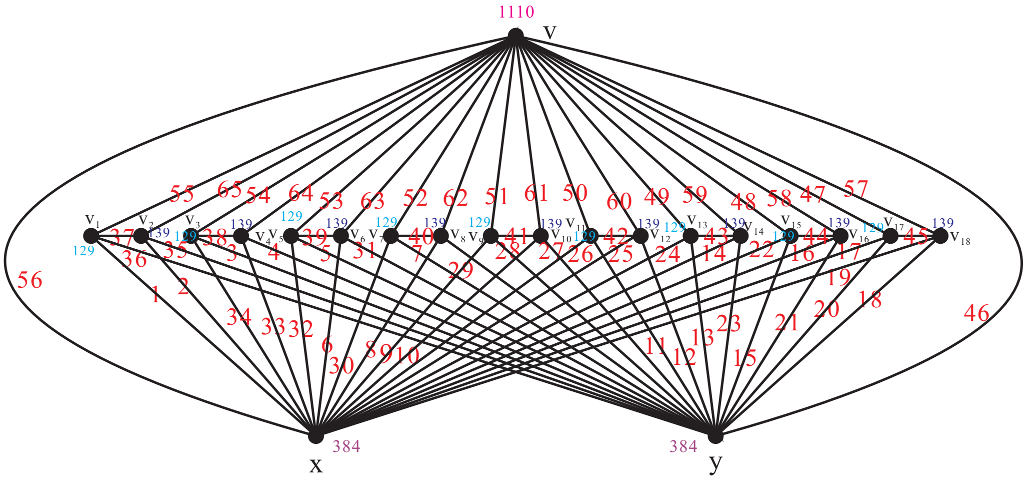

Figure 3. $F_9 \vee \overline{K_2}$.

Case 2. $n \equiv 3 \pmod 4$

For $n = 3$ as shown in Figure 4, we obtain a local antimagic labeling of $F_3 \vee \overline{K_2}$ with $c(f) = 4$.

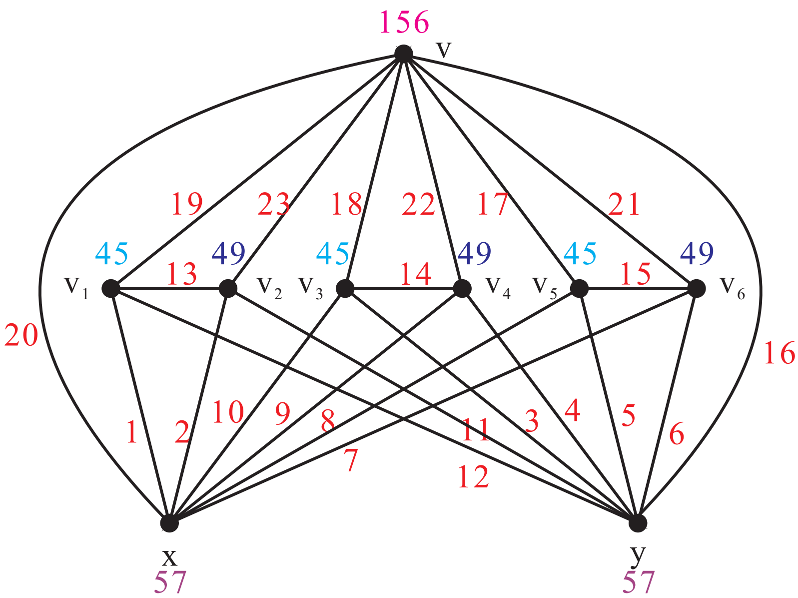

Figure 4. $F_3 \vee \overline{K_2}$.

For $n \geq 7$, define $f : E(H) \to \{1, 2, \cdots, 7n + 2\}$ by the following

$$f(xv) = 6n + 2,\ f(yv) = 5n + 1.$$

Firstly, we set the following assignments of xv_i and yv_i for some special i, respectively.

$$f(xv_i) = \begin{cases} i, & \text{if } i \in \{1, 2\}, \\ 4n + 1 - i, & \text{if } i \in \{3, 4, 5, 2n\}. \end{cases}$$

$$f(yv_i) = \begin{cases} 4n + 1 - i, & \text{if } i \in \{1, 2\}, \\ i, & \text{if } i \in \{3, 4, 5, 2n\}. \end{cases}$$

Secondly, considering the following assignments of the edges xv_i and yv_i for $6 \leq i \leq 2n - 1$,

$$f(xv_i) = \begin{cases} i, & \text{if } i \overset{8}{\equiv} 6,\ i \overset{8}{\equiv} 0,\ i \overset{8}{\equiv} 1 \text{ or } i \overset{8}{\equiv} 2, \\ 4n + 1 - i, & \text{if } i \overset{8}{\equiv} 7,\ i \overset{8}{\equiv} 3,\ i \overset{8}{\equiv} 4 \text{ or } i \overset{8}{\equiv} 5. \end{cases}$$

$$f(yv_i) = \begin{cases} 4n + 1 - i, & \text{if } i \overset{8}{\equiv} 6,\ i \overset{8}{\equiv} 0,\ i \overset{8}{\equiv} 1 \text{ or } i \overset{8}{\equiv} 2, \\ i, & \text{if } i \overset{8}{\equiv} 7,\ i \overset{8}{\equiv} 3,\ i \overset{8}{\equiv} 4 \text{ or } i \overset{8}{\equiv} 5. \end{cases}$$

Finally, label the remaining edges as follows

$$\begin{aligned} &f(v_i v_{i+1}) = 4n + \tfrac{i+1}{2}, && i \overset{2}{\equiv} 1 \text{ and } 1 \leq i \leq 2n, \\ &f(vv_i) = 6n + 2 - \tfrac{i+1}{2}, && i \overset{2}{\equiv} 1, \\ &f(vv_i) = 7n + 3 - \tfrac{i}{2}, && i \overset{2}{\equiv} 0. \end{aligned}$$

Because $n \equiv 3 \pmod 4$ and $n \geq 7$, we have $2n \equiv 6 \pmod 8$, and so the number of vertices in $\{v_i | 6 \leq i \leq 2n - 1\}$ is divisible by 8.

If $\{i, i + 1, i + 2, \cdots, i + 7\} \subseteq \{6, 7, \cdots, 2n - 1\}$ and $i \equiv 6 \pmod 8$, then

$$\sum_{j=i}^{i+7} f(xv_j) = 16n - 6, \quad \sum_{j=i}^{i+7} f(yv_j) = 16n + 14.$$

We can obtain that

$$\sum_{i=6}^{2n-1} f(xv_i) = 4n^2 - \frac{27n}{2} + \frac{9}{2},$$

$$\sum_{i=6}^{2n-1} f(yv_i) = 4n^2 - \frac{17n}{2} - \frac{21}{2}.$$

Since

$$\sum_{i=1}^{5} f(xv_i) = 12n - 6, \quad \sum_{i=1}^{5} f(yv_i) = 8n + 11,$$

$$f(xv_{2n}) = 2n + 1, \quad f(yv_i) = 2n.$$

For the vertex weights we have

$$\begin{aligned} &\omega(x) = \omega(y) = 4n^2 + \tfrac{13n}{2} + \tfrac{3}{2}, \\ &\omega(v_i) = 14n + 3,\ \ i \overset{2}{\equiv} 1, \\ &\omega(v_i) = 15n + 4,\ \ i \overset{2}{\equiv} 0, \\ &\omega(v) = 12n^2 + 15n + 3. \end{aligned}$$

Hence, we can obtain that $\chi_{la}(H) = 4$. For $n = 7$, the exact values of each edge label of the graph $F_7 \vee \overline{K_2}$ are given in Figure 5.

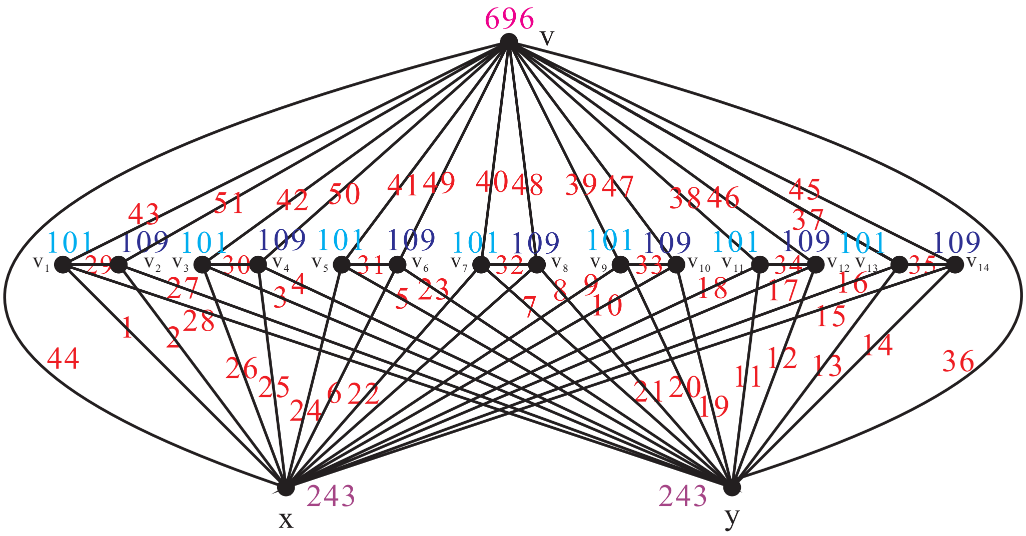

Figure 5. $F_7 \vee \overline{K_2}$.

Case 3. $n \equiv 2 \pmod 4$

In this case, we consider $n \equiv 2 \pmod 8$ and $n \equiv 6 \pmod 8$, respectively.

Subcase 3.1. $n \equiv 2 \pmod 8$

For $n = 2$, there is a local antimagic labeling of the graph $H = F_2 \vee \overline{K_2}$ in Figure 6. Hence, we have $\chi_{la}(H) = 4$.

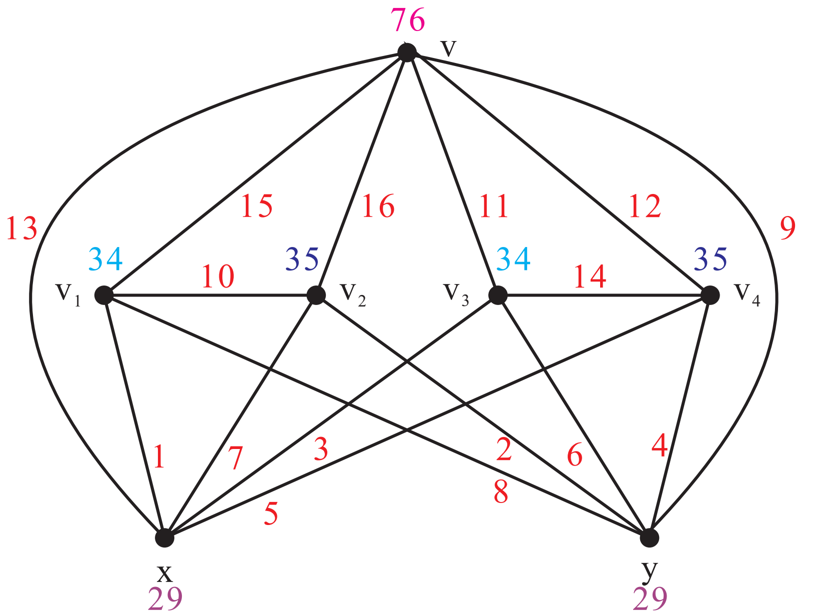

Figure 6. $F_2 \vee \overline{K_2}$.

For $n \geq 10$, define the edge labeling $f : E(H) \to \{1, 2, \cdots, 7n + 2\}$ as follows:

$$f(xv) = \frac{11n}{2} + 2, \quad f(yv) = 4n + 1.$$

Assume that $n = 8k + 2$, $k = 1, 2, \cdots$, then we give the following exact values of $f(xv_i)$ and $f(yv_i)$ for $1 \leq i \leq 2n$.

$$f(xv_i) = \begin{cases} 4n+1-i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 1, \\ i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 0, \\ i, & \text{if } 2k+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 1, \\ 4n+1-i, & \text{if } 2k+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

$$f(yv_i) = \begin{cases} i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 1, \\ 4n+1-i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 0, \\ 4n+1-i, & \text{if } 2k+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 1, \\ i, & \text{if } 2k+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

Then label the remaining edges as follows:

$$f(v_iv_{i+1}) = \begin{cases} 4n+1+\frac{i+1}{2}, & \text{if } 1 \le i \le n \text{ and } i \overset{2}{\equiv} 1, \\ 5n+2+\frac{i+1}{2}, & \text{if } n+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 1. \end{cases}$$

$$f(vv_i) = \begin{cases} \frac{13n+4}{2} - \frac{i-1}{2}, & \text{if } 1 \le i \le n \text{ and } i \overset{2}{\equiv} 1, \\ 7n+3-\frac{i}{2}, & \text{if } 1 \le i \le n \text{ and } i \overset{2}{\equiv} 0, \\ \frac{11n+2}{2} - \frac{i-1}{2}, & \text{if } n+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 1, \\ 6n+2-\frac{i}{2}, & \text{if } n+1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

It is clear that f is a local antimagic labeling of H, and we have

$$\begin{aligned} \omega(x) = \omega(y) &= 4n^2 + \tfrac{23n}{4} + \tfrac{3}{2}, \\ \omega(v_i) &= \tfrac{29n}{2} + 5, \ i \overset{2}{\equiv} 1, \\ \omega(v_i) &= 15n + 5, \ i \overset{2}{\equiv} 0, \\ \omega(v) &= \tfrac{23n^2}{2} + \tfrac{27n}{2} + 3. \end{aligned}$$

So, we have $\chi_{la}(F_n \vee \overline{K_2}) = 4$ for $n \overset{8}{\equiv} 2$. The local antimagic labeling of the graph $F_{10} \vee \overline{K_2}$ is shown in Figure 7.

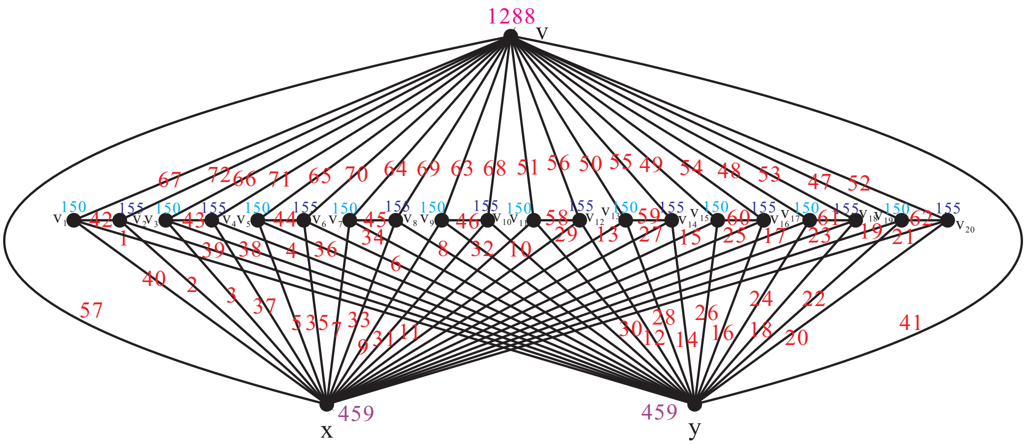

Figure 7. $F_{10} \vee \overline{K_2}$.

Subcase 3.2. $n \equiv 6 \pmod 8$

For $n \ge 6$, label the edges of H by the labeling $f : E(H) \to \{1, 2, \cdots, 7n+2\}$ such that

$$f(xv) = \frac{11n}{2} + 2, \ \ f(yv) = 4n+1.$$

Assume $n = 8k - 2,\ k = 1, 2, \cdots$, then we label $f(xv_i)$ and $f(yv_i)$ for each i such that $1 \le i \le 2n - 1$.

$$f(xv_i) = \begin{cases} 4n + 1 - i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 1, \\ i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 0, \\ i, & \text{if } 2k + 1 \le i \le 2n - 1 \text{ and } i \overset{2}{\equiv} 1, \\ 4n + 1 - i, & \text{if } 2k + 1 \le i \le 2n - 1 \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

$$f(yv_i) = \begin{cases} i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 1, \\ 4n + 1 - i, & \text{if } 1 \le i \le 2k \text{ and } i \overset{2}{\equiv} 0, \\ 4n + 1 - i, & \text{if } 2k + 1 \le i \le 2n - 1 \text{ and } i \overset{2}{\equiv} 1, \\ i, & \text{if } 2k + 1 \le i \le 2n - 1 \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

For the last vertex v_{2n},

$$f(xv_{2n}) = 2n,\ f(yv_{2n}) = 2n + 1.$$

Now, determine the exact value of $f(vv_i)$ for each i such that $1 \le i \le 2n$.

$$f(vv_i) = \begin{cases} \frac{13n+4}{2} - \frac{i-1}{2}, & \text{if } 1 \le i \le n \text{ and } i \overset{2}{\equiv} 1, \\ 7n + 3 - \frac{i}{2}, & \text{if } 1 \le i \le n \text{ and } i \overset{2}{\equiv} 0, \\ \frac{11n+2}{2} - \frac{i-1}{2}, & \text{if } n + 1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 1, \\ 6n + 2 - \frac{i}{2}, & \text{if } n + 1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

When i is odd for $1 \le i \le 2n$, we can label $f(v_i v_{i+1})$ as follows.

$$f(v_i v_{i+1}) = \begin{cases} 4n + 1 + \frac{i+1}{2}, & \text{if } 1 \le i \le n \text{ and } i \overset{2}{\equiv} 1, \\ 5n + 2 + \frac{i+1}{2}, & \text{if } n + 1 \le i \le 2n \text{ and } i \overset{2}{\equiv} 1. \end{cases}$$

For the vertex weights under the labeling f, we have

$$\begin{aligned} \omega(x) &= \omega(y) = 4n^2 + \tfrac{23n}{4} + \tfrac{3}{2}, \\ \omega(v_i) &= \tfrac{29n}{2} + 5,\ i \overset{2}{\equiv} 1, \\ \omega(v_i) &= 15n + 5,\ i \overset{2}{\equiv} 0, \\ \omega(v) &= \tfrac{23n^2}{2} + \tfrac{27n}{2} + 3. \end{aligned}$$

This implies that $\chi_{la}(H) = 4$. For $n = 6$, we obtain the local antimagic labeling of the graph $F_6 \vee \overline{K_2}$ under f as shown in Figure 8.

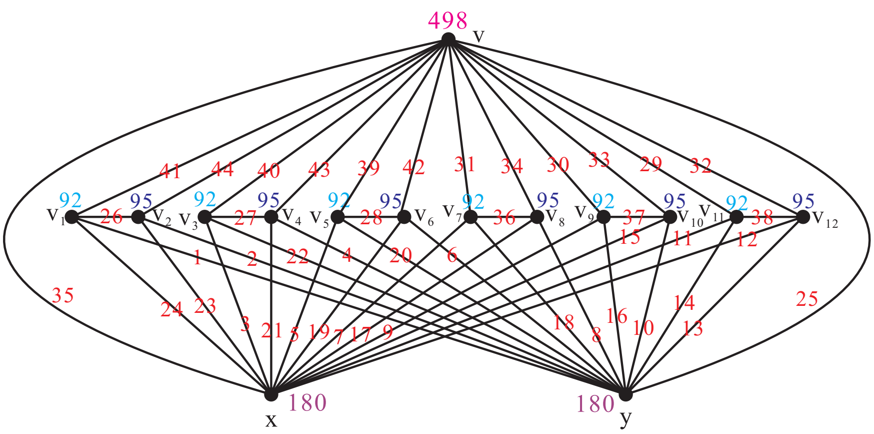

Figure 8. $F_6 \vee \overline{K_2}$.

Case 4. $n \equiv 0 \pmod 4$

We define $f : E(H) \rightarrow \{1, 2, \cdots, 7n+2\}$ as follows:

$$f(xv) = 4n+3, \quad f(yv) = 4n+1.$$

The following labeling has the desired properties:

$$f(xv_i) = \begin{cases} 4n+1-i, & \text{if } i \overset{4}{\equiv} 1 \text{ or } i \overset{4}{\equiv} 0, \text{ and } i \neq 2n, \\ i, & \text{if } i \overset{4}{\equiv} 3 \text{ or } i \overset{4}{\equiv} 2, \text{ or } i = 2n. \end{cases}$$

$$f(yv_i) = \begin{cases} i, & \text{if } i \overset{4}{\equiv} 1 \text{ or } i \overset{4}{\equiv} 0, \text{ and } i \neq 2n, \\ 4n+1-i, & \text{if } i \overset{4}{\equiv} 3 \text{ or } i \overset{4}{\equiv} 2, \text{ or } i = 2n. \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 5n+2+i, & \text{if } 1 \leq i \leq n+1 \text{ and } i \overset{2}{\equiv} 1, \\ 7n+3-i, & \text{if } n+3 \leq i \leq 2n \text{ and } i \overset{2}{\equiv} 1. \end{cases}$$

$$f(vv_i) = \begin{cases} 5n+3-i, & \text{if } 1 \leq i \leq n+2 \ \text{ and } i \overset{2}{\equiv} 1, \\ 7n+4-i, & \text{if } 1 \leq i \leq n+2 \ \text{ and } i \overset{2}{\equiv} 0, \\ 3n+2+i, & \text{if } n+3 \leq i \leq 2n \ \text{ and } i \overset{2}{\equiv} 1, \\ 5n+1+i, & \text{if } n+3 \leq i \leq 2n \ \text{ and } i \overset{2}{\equiv} 0. \end{cases}$$

For the vertex weights under the labeling f, we have

$$\begin{aligned} &\omega(x) = \omega(y) = 4n^2 + 5n + 2, \\ &\omega(v_i) = 14n + 6, \ \ i \overset{2}{\equiv} 1, \\ &\omega(v_i) = 16n + 6, \ \ i \overset{2}{\equiv} 0. \\ &\omega(v) = 11n^2 + 13n + 2. \end{aligned}$$

The above arguments indicate that f is a local antimagic labeling of H with four colors, and so $\chi_{la}(H) = 4$. The exact values of each edge label of the graph $F_4 \vee \overline{K_2}$ are given in Figure 9. The proof is completed. □

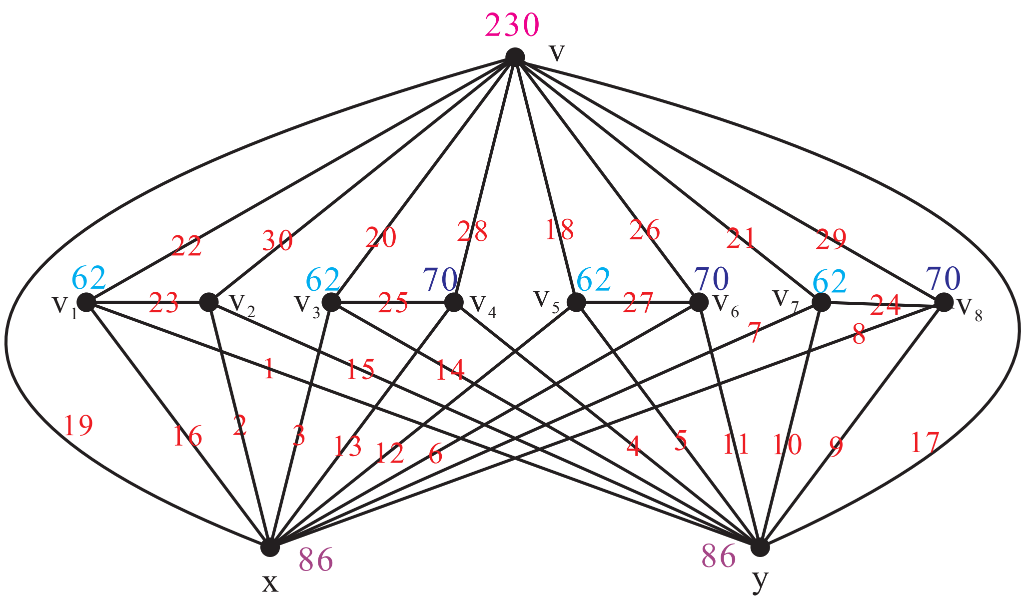

Figure 9. $F_4 \vee \overline{K_2}$.

Let $H = F_n - v$ be a graph obtained from the friendship graph $F_n(n \geq 2)$ by deleting any vertex v of F_n. If the deleted vertex is its central vertex, then H does not have a local antimagic labeling. Thus, we only consider that the deleted vertex is a vertex with degree 2.

Theorem 2. *Let H be the graph $F_n - v$, where v is any vertex of $F_n(n \geq 2)$ with degree 2, then we have $\chi_{la}(H) = 3$.*

Proof. Let $V(F_n) = \{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{x\}$ and $E(F_n) = \{u_i v_i : 1 \leq i \leq n\} \bigcup \{xu_i : 1 \leq i \leq n\} \bigcup \{xv_i : 1 \leq i \leq n\}$. Without loss of generality, we assume that the deleted vertex is $v_n \in V(F_n)$, then define $h : E(H) \rightarrow \{1, 2, \cdots, 3n - 2\}$ by

$$\begin{aligned} &h(u_i v_i) = i, && 1 \leq i \leq n-1, \\ &h(xu_i) = 3n - 2 - i, && 1 \leq i \leq n-1, \\ &h(xv_i) = 2n - 1 - i, && 1 \leq i \leq n-1, \\ &h(xu_n) = 3n - 2. \end{aligned}$$

Clearly, h is a local antimagic labeling of H and we have

$$\begin{aligned} \omega(v_i) &= 2n - 1, \text{ where } 1 \leq i \leq n-1, \\ \omega(u_i) &= 3n - 2, \text{ where } 1 \leq i \leq n, \\ \omega(x) &= 4n^2 - 4n + 1. \end{aligned}$$

Thus, $\chi_{la}(H) \leq 3$. Since $\chi_{la}(H) \geq \chi(H) = 3$; it follows that $\chi_{la}(H) = 3$. □

Theorem 2.16 of [2] asserts that if a graph G has at least four vertices, then $\chi_{la}(G) + 1 = \chi_{la}(G \vee \overline{K_2})$, when G is of even order n. In this section, we explicitly construct an infinite class of connected graphs G such that $\chi_{la}(G) = 3$ and $\chi_{la}(G \vee \overline{K_2}) = 3$. Our procedure is to consider path P_n that satisfies $\chi_{la}(P_n) = 3$ for each positive integer $n \geq 3$. We show that if n is even, then $\chi_{la}(P_n \vee \overline{K_2}) = 3$. Our result provides partial solution to Problem 3.19 in [8].

Theorem 3. *If P_n is a path of order n, then we have $\chi_{la}(P_n \vee \overline{K_2}) = 3$ for even n.*

Proof. The lower bound of the local antimagic chromatic number of the join graph $P_n \vee \overline{K_2}$ even for n is clearly obtained. We have $\chi_{la}(P_n \vee \overline{K_2}) \geq \chi(P_n \vee \overline{K_2}) = 3$ since K_3 is a induced subgraph of $P_n \vee \overline{K_2}$. We show that the upper bound of the chromatic number $\chi_{la}(P_n \vee \overline{K_2})$ is attainable.

Let $\{u_i : 1 \leq i \leq n\}$ and $\{x, y\}$ be the vertex set of the path P_n and the complement graph of K_2, respectively. Then $E(P_n \vee \overline{K_2}) = \{xu_i, yu_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$, and $|E(P_n \vee \overline{K_2})| = 3n - 1$.

Label the edges $u_i u_{i+1}$ as follows:

$$f(u_i u_{i+1}) = \begin{cases} n - \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

Then, label the edges xu_i as follows:

$$f(xu_i) = \begin{cases} n + \frac{i-1}{2}, & \text{if } i \text{ is odd,} \\ 3n - \frac{i+2}{2}, & \text{if } i \text{ is even,} i \neq n, \\ 3n - 1, & i = n. \end{cases}$$

Finally, label the edges yu_i as follows:

$$f(yu_i) = \begin{cases} \frac{5n}{2} - \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{3n}{2} + \frac{i-2}{2}, & \text{if } i \text{ is even.} \end{cases}$$

We can conclude that

$$\omega(u_i) = \frac{9n}{2} - 2, \quad \text{if } i \text{ is odd;}$$
$$\omega(u_i) = \frac{11n}{2} - 2, \quad \text{if } i \text{ is even;}$$
$$\omega(x) = \omega(y) = 2n^2 - \frac{n}{2}.$$

Therefore, f is a local antimagic labeling of $P_n \vee \overline{K_2}$ that induces a local antimagic vertex coloring using exactly three colors. The local antimagic labeling of the graph $P_6 \vee \overline{K_2}$ as an example is shown in Figure 10. □

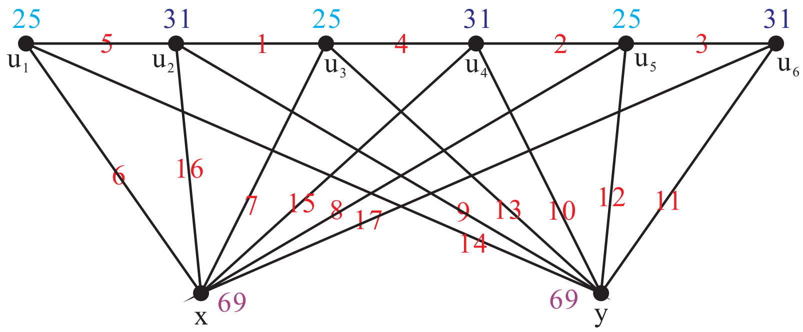

Figure 10. $P_6 \vee \overline{K_2}$.

3. Conclusions and Scope

In this paper, we obtain the exact values of the local antimagic chromatic number of the join graphs $F_n \vee \overline{K_2}$, $P_n \vee \overline{K_2}$ and the graph $F_n - v$. Hence, the following problem arises naturally.

Problem 1. *Find the local antimagic chromatic number of the cartesian product of simple graphs G and H.*

Problem 2. *Find the local antimagic chromatic number of other operations of graphs.*

Problem 3. *Characterize the class of a graph G for which* $\chi_{la}(G \vee \overline{K_2}) = \chi_{la}(G)$.

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References

1. Hartsfield, N.; Ringel, G. *Pearls in Graph Theory*; Academic Press Inc.: Boston, MA, USA, 1994.
2. Arumugam, S.; Premalatha, K.; Bacă, M.; Semaničová-Feňovčíková, A. Local antimagic vertex coloring of a graph. *Graphs Comb.* **2017**, *33*, 275–285. [CrossRef]

3. Bensmail, J.; Senhaji, M.; Lyngsie, K.S. On a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture. *Discret. Math. Theor. Comput. Sci.* **2017**, *19*, 1–17.
4. Haslegrave, J. Proof of a local antimagic conjecture. *Discr. Math. Theor. Comp. Sci.* **2018**, *20*, 1–14.
5. Hagedorn, T.R. Magic rectangles revisited. *Discrete Math.* **1990**, *207*, 65–72. [CrossRef]
6. Lau, G.C.; Shiu, W.C.; Ng, H.K. Affirmative solutions on local antimagic chromatic number. *Graphs Comb.* **2020**, *5*, 69–78. [CrossRef]
7. Shaebani, S. On local antimagic chromatic number of graphs. *Algebr. Syst.* **2020**, *7*, 245–256.
8. Lau, G.C.; Shiu, W.C.; Ng, H.K. On local antimagic chromatic number of cycle-related join graphs. *Discuss. Math. Graph Theory* **2021**, *41*, 133–152. [CrossRef]
9. Lau, G.C.; Shiu, W.C.; Ng, H.K. On local antimagic chromatic number of graphs with cut-vertices. *arXiv* **2018**, arXiv:1805.04801v6.