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Collisions, or Reflections and Rotations, Leading to the Digits of π

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Abstract

We analyze a problem of elastic collisions between elements of a system composed of two balls and a wall. Thanks to a change of variables, the problem is reduced to a sequence of reflections and rotations. Moreover, the total number of collisions is easily found. For specific ratios of ball weights, the number of collisions is related to the first successive digits of π.

Keywords

Dynamical System, Elastic Collisions, Reflection, Rotation, Digits of π

1. Introduction

The computation of the digits of π has interested several scientists in history [1]. There are several methods for computing them [2]. Some methods are fast and efficient in computing millions of digits in a short time, and others are ineffective but are remarkably creative. One of these latter methods was introduced by Galperin in 2003 [3]. The author proposes the computation of the first digits of π by counting the number of collisions of a system consisting of two balls and a wall under a condition on the ratio of the masses of the two balls. Since then, several articles have been published on the subject [4] [5] [6].

In this article, we present a simple and in-depth analysis of the problem of elastic collisions between two balls and a wall which include Galperin's analysis. Our analysis is based on a new and original decomposition of the velocity for given values of kinetic energy and momentum. Thanks to a useful transformation, the basic geometry of the problem is reduced to reflections and rotations on a circle. It follows that a sequence of collisions becomes a sequence of reflections. Then, for the two possible sequences of collisions, it is easy to determine the end of the process and the total number of collisions. We revisit Galperin's geometric method which is extended to find the time and the position of each collision. Finally, for particular ratios of the masses of the two balls, it is possible to link the number of collisions to the first successive digits of π.

2. Ideal Physical System

The system consists of two balls of point masses respectively M and m ($M \ge m$), noted ball$_M$ and ball$_m$, in position X and x placed to the right of the wall (positive coordinates), with $X > x > 0$. Ball$_m$ is between ball$_M$ and the wall, see Figure 1. Let V and v be the speeds of the ball$_M$ and ball$_m$.

Two quantities are important for this system: the momentum

$$Q = MV + mv,$$

and the kinetic energy

$$E_c = \frac{E}{2} = \frac{1}{2}\left(MV^2 + mv^2\right).$$

We introduce average velocities as follows

$$\bar{V} = \frac{MV + mv}{M+m} = \frac{Q}{M+m},$$

and

$$\overline{V^2} = \frac{MV^2 + mv^2}{M+m} = \frac{E}{M+m}.$$

Let the matrix G be defined by

$$G = \begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}.$$

we can write the momentum of the two balls as

$$Q = \begin{pmatrix} V & v \end{pmatrix} G \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (V, v) \bullet_G (1,1),$$

and velocities (V, v) having the same momentum are on a straight line with direction $(-m, M)$. Also the kinetic energy E_c of the two balls is

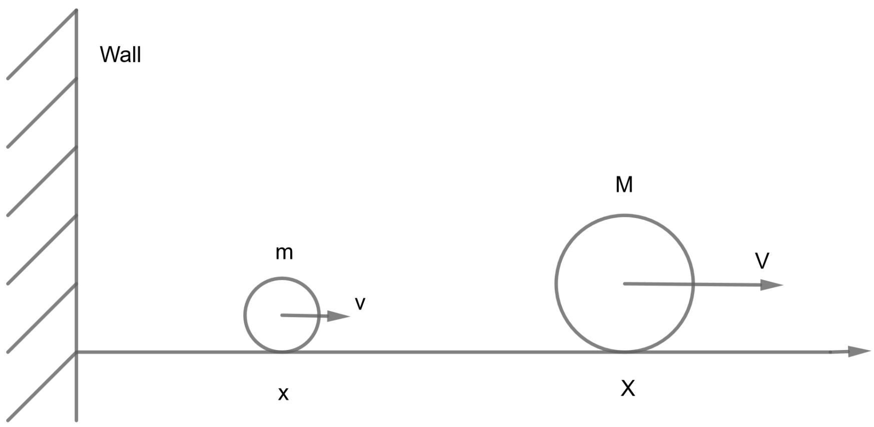

Figure 1. The ideal system.

$$E = (V,v) \bullet_G (V,v) = \|(V,v)\|_G^2,$$

and velocities (V,v) having the same kinetic energy are on an ellipse. Finally, using the Cauchy-Bunyakovski-Schwarz inequality, we get

$$|Q| = |(V,v) \bullet_G (1,1)| \le \|(V,v)\|_G \|(1,1)\|_G = \sqrt{E(M+m)},$$

which is equivalent to

$$\bar{V}^2 \le \overline{V^2}.$$

In the sequel, if G is the identity matrix I, G will be omitted and we will have the standard expressions

$$(Z_1,z_1) \bullet (Z_2,z_2) = Z_1 Z_2 + z_1 z_2, \quad \text{and} \quad \|(Z,z)\| = \sqrt{Z^2 + z^2},$$

for (Z_1,z_1), (Z_2,z_2), and $(Z,z) \in \mathbb{R}^2$.

For our problem, all collisions will be assumed to be elastic. Collisions between a ball and the wall produce sign changes of the velocity of the ball, so the momentum of the system changes while keeping constant its kinetic energy. On the other hand, for collisions between the two balls both momentum and kinetic energy remain constant. By following the dynamics of the two balls, we will look at the total number of collisions, counting collisions between the two balls and collisions between a ball and the wall. We will see that under certain conditions, the number of collisions corresponds to the first digits of π.

3. Direct Analysis: The Natural Coordinate System

In this section we analyze the system with respect to its natural coordinate systems, XOx for the position and VOv for the velocity.

3.1. Observations

In the XOx coordinate system, see **Figure 2**, the ball$_m$-wall collisions determine points on the line $x = 0$, the horizontal OX axis, while ball$_M$-ball$_m$ collisions are points on the line $X = x$. The direction of this line is $(1,1)$ and makes a π/4 angle with the OX axis. At any time, positions (X,x) are in the set

$$\{(X,x) \in \mathbb{R}^2 \mid X \ge x \ge 0\}.$$

Given (X,x) and (V,v), we can easily find the next collision point on $x = 0$ or on $X = x$ if there is a collision. Since the velocity is piecewise constant, it changes only at collisions, the trajectory is not only continuous but also piecewise linear. The next result follows from **Figure 2**.

Theorem 1. *Let the position* $(X,x) \in \{(X,x) \in \mathbb{R}^2 \mid X > x > 0\}$, *and let the velocity be given by*

$$(V,v) = \|(V,v)\|(\cos(\phi), \sin(\phi))$$

with $\|(V,v)\| = \sqrt{V^2 + v^2}$ and $\phi \in [0, 2\pi)$. Hence for

(a) $\phi \in [0, \pi/4]$, there will be no collision;

(b) $\phi \in (\pi/4, \pi]$, there will be a ball$_M$-ball$_m$ collision;

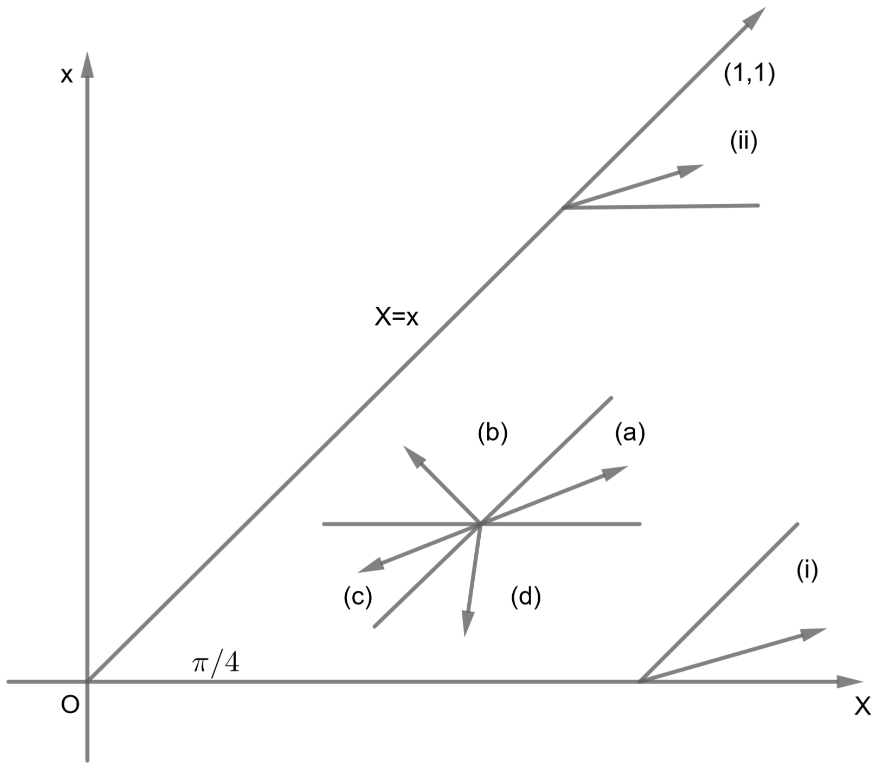

Figure 2. *XOx* coordinate system.

(c) $\phi \in (\pi, \pi+\pi/4)$ there will be either a ball$_M$-ball$_m$ or a ball$_m$-wall collision;

(d) $\phi \in [\pi+\pi/4, 2\pi)$ there will be a ball$_m$-wall collision.

3.2. Ball-Ball System

The velocity (V, v) of constant kinetic energy and constant momentum lends itself well to a decomposition foreseen in [7]. The possible velocities in this case are given by the intersection points of an ellipse (kinetic energy) and a line (momentum). There are no more than two intersection points. This decomposition will be helpful to explain the transformation of the velocity during a collision.

3.2.1. Decomposition of the Velocity

In the next theorems, we will break down the velocity (V, v) using an orthogonal basis.

Theorem 2. The set $\{(1,1), (-m, M)\}$ is an orthogonal basis with respect to the positive defined quadratic form used to define the ellipse of constant kinetic energy.

Proof. Indeed we have

$$(1,1) \bullet_G (-m, M) = 0.$$

Moreover

$$\|(1,1)\|_G^2 = (1,1) \bullet_G (1,1) = M + m,$$

and

$$\|(-m,M)\|_G^2=(-m,M)\bullet_G(-m,M)=Mm(M+m). \qquad \square$$

The decomposition now follows.

Theorem 3. The velocity (V,v) can be decomposed as

$$\begin{pmatrix}V\\v\end{pmatrix}=R\begin{pmatrix}1\\1\end{pmatrix}+S\begin{pmatrix}-m\\M\end{pmatrix}=\begin{pmatrix}1&-m\\1&M\end{pmatrix}\begin{pmatrix}R\\S\end{pmatrix},$$

where

$$R=\frac{(1,1)\bullet_G(V,v)}{(1,1)\bullet_G(1,1)}=\frac{MV+mv}{M+m}=\frac{Q}{M+m}=\overline{V},$$

and

$$S=\frac{(-m,M)\bullet_G(V,v)}{(-m,M)\bullet_G(-m,M)}=\frac{v-V}{M+m}.$$

Corollary 1. We have

$$V-v=-\frac{1}{Mm}(-m,M)\bullet_G(V,v),$$

so

$$V\begin{Bmatrix}>\\=\\<\end{Bmatrix}v \quad \text{if and only if} \quad (-m,M)\bullet_G(V,v)\begin{Bmatrix}<\\=\\>\end{Bmatrix}0.$$

3.2.2. Compatibility Condition

A compatibility condition on E and Q is given in the next theorem.

Theorem 4. The kinetic energy and the momentum of the ball$_M$-ball$_m$ system are related by the relation

$$E(M+m)=Q^2+Mm(v-V)^2,$$

so

(i) $|Q|$ is upper bounded, and $|Q|\le Q_{\max}=\sqrt{E(M+m)}$;

(ii) E is lower bounded, and $E\ge E_{\min}=\dfrac{Q^2}{M+m}$.

Proof. Using the decomposition of Theorem 3, to be on the ellipse (V,v) must satisfy

$$E=(V,v)\bullet_G(V,v)=\frac{(MV+mv)^2}{M+m}+\frac{Mm(v-V)^2}{M+m},$$

so the result follows.

Corollary 2. The velocities are related by the relation

$$\overline{V^2}=\overline{V}^2+\frac{Mm}{(M+m)^2}(v-V)^2.$$

For given compatible E and Q, possible values of the velocity are given in the next theorem.

Theorem 5. Under the compatibility condition

$$|Q| \le \sqrt{E(M+m)} \quad \text{or} \quad \bar{V}^2 \le \overline{V^2},$$

if

(j) $|Q| < \sqrt{E(M+m)}$, we have two possible velocities

$$\begin{pmatrix} V \\ v \end{pmatrix} = \frac{Q}{M+m}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \frac{1}{M+m}\left(\frac{E(M+m)-Q^2}{Mm}\right)^{1/2}\begin{pmatrix} -m \\ M \end{pmatrix}$$
$$= \bar{V}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \left(\frac{\overline{V^2}-\bar{V}^2}{Mm}\right)^{1/2}\begin{pmatrix} -m \\ M \end{pmatrix},$$

(jj) $|Q| = \sqrt{E(M+m)}$, we have only one possible velocity

$$\begin{pmatrix} V \\ v \end{pmatrix} = \frac{Q}{M+m}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \bar{V}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can also obtain a decomposition of the kinetic energy and the velocity of the system.

Theorem 6. The kinetic energy and the velocity of the system are decomposable as follows

$$E = (M+m)\bar{V}^2 + M\left(V-\bar{V}\right)^2 + m\left(v-\bar{V}\right)^2,$$

and

$$\overline{V^2} = \bar{V}^2 + \frac{M}{M+m}\left(V-\bar{V}\right)^2 + \frac{m}{M+m}\left(v-\bar{V}\right)^2.$$

3.2.3. Elastic Collision

In elastic collisions, momentum and kinetic energy remain constant. The velocity (V, v) is therefore one of the two points on the ellipse described in the preceding section.

Theorem 7. Let (V_-, v_-) be the velocity before the collision such that $V_- < v_-$ to eventually have a collision between the two balls (regardless the position of the wall). Let (V_+, v_+) be the velocity after the collision, then

$$\begin{pmatrix} V_+ \\ v_+ \end{pmatrix} = T\begin{pmatrix} V_- \\ v_- \end{pmatrix} \quad \text{where} \quad T = \begin{pmatrix} \dfrac{M-m}{M+m} & \dfrac{2m}{M+m} \\ \dfrac{2M}{M+m} & -\dfrac{M-m}{M+m} \end{pmatrix},$$

and

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} R_- \\ S_- \end{pmatrix}.$$

Also $V_+ > v_+$, so the two balls move away from each other.

Proof. From Theorem 5, (V_+, v_+) is the second point on the ellipse obtained by changing the sign of the coefficient *S*, so we have

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} R_- \\ -S_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} R_- \\ S_- \end{pmatrix}.$$

Using Theorem 3, we have

$$\begin{pmatrix} V_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} 1 & -m \\ 1 & M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ 1 & M \end{pmatrix}^{-1} \begin{pmatrix} V_- \\ v_- \end{pmatrix} = T \begin{pmatrix} V_- \\ v_- \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} V_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} V_- \\ v_- \end{pmatrix} - 2S_- \begin{pmatrix} -m \\ M \end{pmatrix}$$

so

$$(-m, M) \bullet_G (V_+, v_+) = -(-m, M) \bullet_G (V_-, v_-).$$

It follows that

$$V_+ - v_+ = -(V_- - v_-) > 0,$$

so $V_+ > v_+$. □

Corollary 3. $T^2 = I$.

Corollary 4. The eigenvalues of T are 1 and −1 and their corresponding eigenvectors are $(1,1)$ and $(-m, M)$.

On Figure 3, with the ellipse centered on the line $X = x$, the velocity moves from a point $P_- = (V_-, v_-)$ on the ellipse above the line $X = x$, or $V = v$, to a point $P_+ = (V_+, v_+)$ on the ellipse below the line $X = x$, or $V = v$, along the direction opposite to $(-m, M)$.

In the VOv coordinate system, incident angle, the angle between (V_-, v_-) and the line of direction $(1,1)$, and the reflection angle, the angle between the

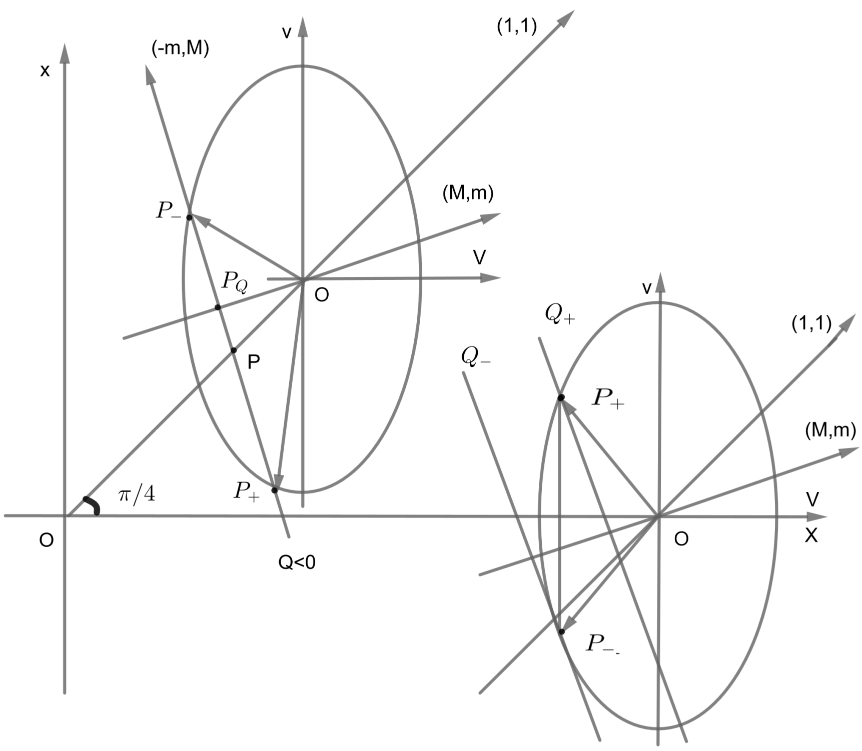

Figure 3. Effects of collision on the velocity (V, v).

line of direction $(1,1)$ and (V_+, v_+), are not equal except for $m = M$. Looking at Figure 3 (where $Q < 0$), we show that the incident angle $\angle POP_-$ is greater than the reflection angle $\angle POP_+$. Indeed, the two rectangular triangles $\Delta P_Q OP_-$ and $\Delta P_Q OP_+$ allow us to obtain $\angle OP_- P_Q > \angle OP_+ P_Q$ because the side $P_Q P_-$ is shorter than the side $P_Q P_+$. Since PP_- and PP_+ are of equal length, and the side OP is common to both triangles ΔPOP_- and ΔPOP_+, we get the result from the sine law. A similar analysis can be done for $Q > 0$, and it is obvious for $Q = 0$.

3.3. Ball-Wall System

The ball$_m$-wall collisions are easier to analyze.

Theorem 8. Let (V_-, v_-) be the velocity before the collision of ball$_m$ with the wall and (V_+, v_+) be the velocity after the collision. Let $v_- < 0$ to have a collision with the wall, so we have

$$\begin{pmatrix} V_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} V_- \\ v_- \end{pmatrix},$$

and

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \tilde{T} \begin{pmatrix} R_- \\ S_- \end{pmatrix} \quad \text{where} \quad \tilde{T} = \begin{pmatrix} \dfrac{M-m}{M+m} & -\dfrac{2Mm}{M+m} \\ -\dfrac{2}{M+m} & -\dfrac{M-m}{M+m} \end{pmatrix}.$$

Moreover the momentum increases at each collision with the wall, and we have

$$Q_+ = Q_- - 2mv_- > Q_-.$$

Proof. When the ball$_m$ hits the wall, it bounces with opposite velocity of the same magnitude, *i.e.* $v_+ = -v_-$. Since the ball$_M$ doesn't hit the wall, so $V_+ = V_-$, so we get first result. For the coefficients R and S, using Theorem 3, we have

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} 1 & -m \\ 1 & M \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ 1 & M \end{pmatrix} \begin{pmatrix} R_- \\ S_- \end{pmatrix} = \tilde{T} \begin{pmatrix} R_- \\ S_- \end{pmatrix}.$$

For the momentum we have

$$Q_+ = MV_+ + mv_+ = MV_- - mv_- = MV_- + mv_- - 2mv_- = Q_- - 2mv_-,$$

and since $v_- < 0$ for a collision, the momentum increases at each collision of the ball$_m$ with the wall.

Corollary 5. $\tilde{T}^2 = I$.

For the ellipse centered on the OX axis, see Figure 3, the velocity moves upward from P_- to P_+. Let us observe that for this kind of collisions, incident and reflection angles of the velocity with respect to the OX axis are equal.

3.4. Stopping Criterion and Trajectory on the Ellipse

Looking at Figure 2, we get conditions under which there will be no more collisions.

Theorem 9. Suppose that the velocity (V, v) in polar form is

$$(V,v)=\|(V,v)\|(\cos(\phi),\sin(\phi)),$$

with $\phi\in[0,2\pi)$. There will be no new collisions, if (V,v)

(i) is the initial condition and $\phi\in[0,\pi/4]$;

(ii) is the velocity after a ball$_m$-wall collision (after moving up vertically) and $\phi\in(0,\pi/4]$;

(iii) is the velocity after a ball$_M$-ball$_m$ collision (after moving down right) and $\phi\in[0,\pi/4)$.

These conditions say that the two balls are going away from the wall with velocity $0<v\le V$.

On **Figure 4**, the trajectory of the velocity (V,v) on the ellipse is given for the two possible ends of the process after at least one collision. We see that it moves successively to P_1, to P_2, to P_3, to P_4, and finally to (V,v) the final point.

4. A Useful Transformation

The standard parametrization of the ellipse suggests a way to transform the graph of the kinetic energy from its elliptic form to a circular form. The transform is defined by

$$\begin{pmatrix}Y\\y\end{pmatrix}=\begin{pmatrix}\sqrt{M}X\\\sqrt{m}x\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}W\\w\end{pmatrix}=\begin{pmatrix}\sqrt{M}V\\\sqrt{m}v\end{pmatrix}.$$

We will agree to call (Y,y) the position and (W,w) the velocity.

The direction $(\sqrt{M},\sqrt{m})$ will play a special role in the sequel. Let us note α the angle of this direction with a horizontal axis (OY or OW), so we have

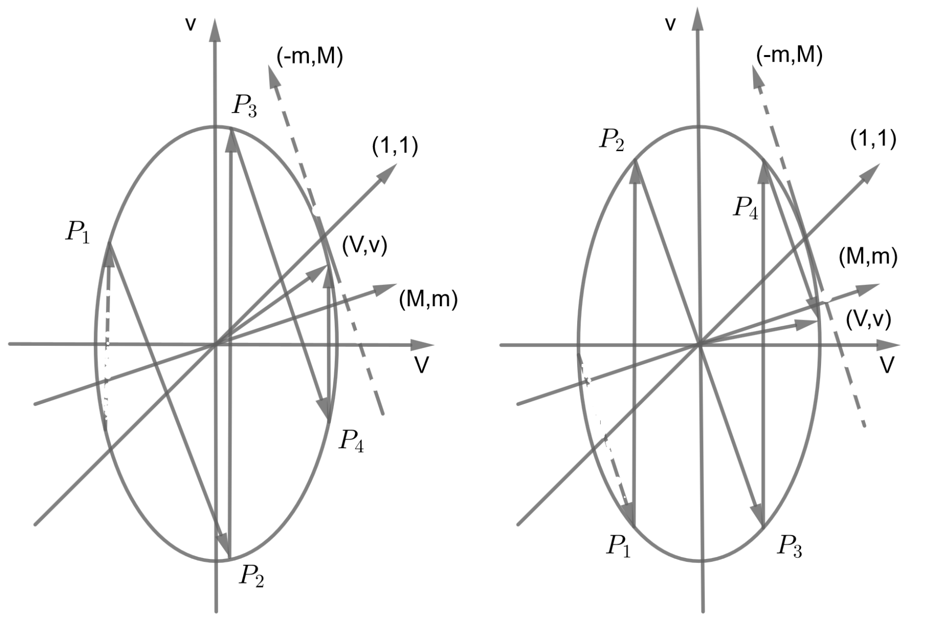

Figure 4. (V,v) trajectory on the ellipse and stopping region.

$$\begin{cases} \cos(\alpha) = \sqrt{\dfrac{M}{M+m}}, \\ \sin(\alpha) = \sqrt{\dfrac{m}{M+m}}. \end{cases}$$

Introducing the matrix Σ defined by

$$\Sigma = \begin{pmatrix} \sqrt{M} & 0 \\ 0 & \sqrt{m} \end{pmatrix} = \sqrt{M+m} \begin{pmatrix} \cos(\alpha) & 0 \\ 0 & \sin(\alpha) \end{pmatrix},$$

we can rewrite the transformation as

$$\begin{pmatrix} Y \\ y \end{pmatrix} = \Sigma \begin{pmatrix} X \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} W \\ w \end{pmatrix} = \Sigma \begin{pmatrix} V \\ v \end{pmatrix}.$$

Consequences of this change of variables are

1) to move the line $X = x$ of the ball$_M$-ball$_m$ collisions to

$$(-\sin(\alpha), \cos(\alpha)) \bullet (Y, y) = 0,$$

a line with direction $(\cos(\alpha), \sin(\alpha))$;

2) to leave fixed the line $x = 0$, the OX axis, for ball$_m$-wall collisions which becomes $y = 0$, the OY axis.

The expression for the kinetic energy becomes

$$E = W^2 + w^2 = (W, w) \bullet (W, w),$$

which is such that this quadratic form coincides now with the standard inner product in $\mathbb{R}^2$, the matrix G is now the identity matrix I. The momentum is now

$$Q = \sqrt{M} W + \sqrt{m} w = \sqrt{M+m} (\cos(\alpha), \sin(\alpha)) \bullet (W, w),$$

and the lines of constant momentum are of direction $(-\sin(\alpha), \cos(\alpha))$. They share the same normal vector $(\cos(\alpha), \sin(\alpha))$ which coincides with the direction of the line of ball$_M$-ball$_m$ collisions. As a consequence we will have the reflection property for (W, w) not only for the ball$_m$-wall collisions but also for the ball$_M$-ball$_m$ collisions.

Let us observe that

$$V - v = -\frac{1}{Mm} (-m, M) \bullet_G (V, v) = -\sqrt{\frac{M+m}{Mm}} (-\sin(\alpha), \cos(\alpha)) \bullet (W, w),$$

so

a) if $V > v$ the balls move away of each other and no ball$_M$-ball$_m$ collision will occur, this is equivalent to $(-\sin(\alpha), \cos(\alpha)) \bullet (W, w) < 0$;

b) if $V < v$ there will be eventually a ball$_M$-ball$_m$ collision, this is equivalent to $(-\sin(\alpha), \cos(\alpha)) \bullet (W, w) > 0$.

5. On Reflections and Rotations

Some useful results about rotations and reflections are now given. Let us consider any angle δ. For the rotation matrix $\operatorname{rot}(\beta)$ of an angle β, we have

$$\operatorname{rot}(\beta)\begin{pmatrix}\cos(\delta)\\ \sin(\delta)\end{pmatrix}=\begin{pmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{pmatrix}\begin{pmatrix}\cos(\delta)\\ \sin(\delta)\end{pmatrix}=\begin{pmatrix}\cos(\delta+\beta)\\ \sin(\delta+\beta)\end{pmatrix}.$$

For the reflection matrix $\operatorname{ref}(\gamma)$ which represents a reflection with respect to a line which makes an angle γ with the OY axis, we have

$$\operatorname{ref}(\gamma)=\begin{pmatrix}\cos(2\gamma) & \sin(2\gamma)\\ \sin(2\gamma) & -\cos(2\gamma)\end{pmatrix}$$

and

$$\operatorname{ref}(\gamma)\begin{pmatrix}\cos(\gamma+\delta)\\ \sin(\gamma+\delta)\end{pmatrix}=\begin{pmatrix}\cos(\gamma-\delta)\\ \sin(\gamma-\delta)\end{pmatrix}.$$

To complete this subsection let us present some identities whose proofs are simple and omitted.

Lemma 10. For any angle β

$$\operatorname{ref}(\beta)=\begin{cases}\operatorname{rot}(\beta)\operatorname{ref}(0)\operatorname{rot}(-\beta),\\ \operatorname{rot}(2\beta)\operatorname{ref}(0),\\ \operatorname{ref}(0)\operatorname{rot}(-2\beta).\end{cases}$$

Lemma 11. For any two angles β and γ, we have

a) $\operatorname{rot}(\beta)\operatorname{rot}(\gamma)=\operatorname{rot}(\beta+\gamma)$;

b) $\operatorname{rot}(\beta)\operatorname{ref}(\gamma)=\operatorname{rot}(2\gamma+\beta)\operatorname{ref}(0)$;

c) $\operatorname{ref}(\gamma)\operatorname{rot}(\beta)=\operatorname{rot}(2\gamma-\beta)\operatorname{ref}(0)$;

d) $\operatorname{ref}(\beta)\operatorname{ref}(\gamma)=\operatorname{rot}(2(\beta-\gamma))$.

6. Indirect Analysis: The Transformed Coordinate System

In this section we analyze the system with respect to the transformed coordinate systems, YOy for the position and WOw for the velocity.

6.1. Observations

The observations made in Theorem 1 in Section 2 can be transposed directly to the YOy coordinate system of **Figure 5**. Ball$_m$-wall collisions are points on the line $y=0$ while ball$_M$-ball$_m$ collisions are points on the line $(-\sin(\alpha),\cos(\alpha))\bullet(Y,y)=0$. At any time the position (Y,y) is in the set

$$\left\{(Y,y)\in\mathbb{R}^2 \mid y\geq 0 \text{ and } (-\sin(\alpha),\cos(\alpha))\bullet(Y,y)\leq 0\right\}.$$

Theorem 12. Let the position

$$(Y,y)\in\left\{(Y,y)\in\mathbb{R}^2 \mid y>0 \text{ and } (-\sin(\alpha),\cos(\alpha))\bullet(Y,y)<0\right\},$$

and let the velocity be given by

$$(W,w)=\|(W,w)\|(\cos(\phi),\sin(\phi))$$

where $\|(W,w)\|=\sqrt{W^2+w^2}$ and $\phi\in[0,2\pi)$. Hence for

(a) $\phi\in[0,\alpha]$ there will be no collision;

(b) $\phi\in(\alpha,\pi]$ there will be a ball$_M$-ball$_m$ collision;

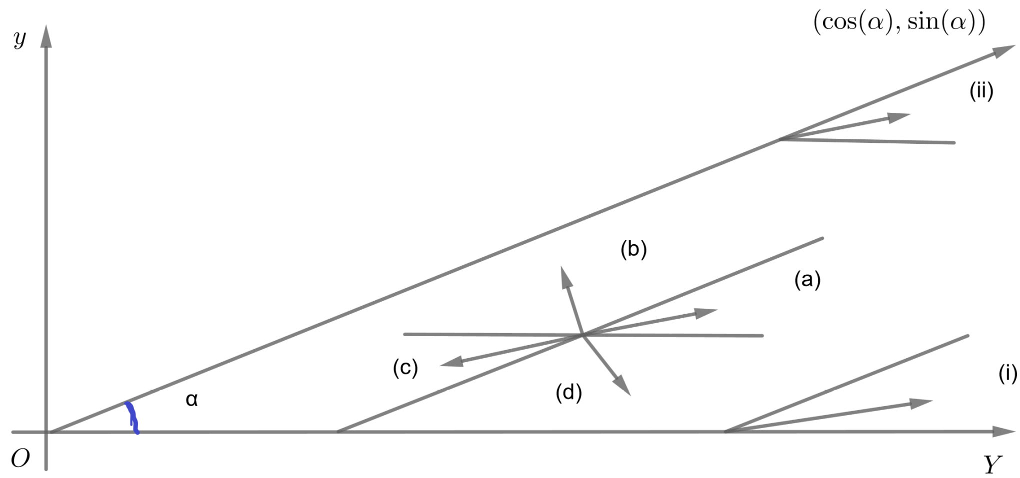

Figure 5. The coordinate system YOy.

(c) $\phi \in (-\pi, -\pi+\alpha)$ there will be either a ball$_M$-ball$_m$ or a ball$_m$-wall collision;
(d) $\phi \in [-\pi+\alpha, 0)$ there will be a ball$_m$-wall collision.

6.2. Ball-Ball System

The velocity (W, w) can also be decomposed, and the possible velocities are given by the intersection points of a circle (kinetic energy) and a line (momentum). There are no more than two intersection points.

6.2.1. Decomposition of the Velocity

Let us express the velocity in terms of the new variables and an appropriate orthonormal basis.

Theorem 13. The expression of the velocity (W, w) with respect to the orthonormal basis $\{(\cos(\alpha), \sin(\alpha)), (-\sin(\alpha), \cos(\alpha))\}$ is

$$\begin{pmatrix} W \\ w \end{pmatrix} = r\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} + s\begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix} = \mathrm{rot}(\alpha)\begin{pmatrix} r \\ s \end{pmatrix},$$

where

$$r = (\cos(\alpha), \sin(\alpha)) \bullet (W, w) = \frac{Q}{\sqrt{M+m}},$$

and

$$s = (-\sin(\alpha), \cos(\alpha)) \bullet (W, w) = \frac{-\sqrt{m}W + \sqrt{M}w}{\sqrt{M+m}} = \sqrt{\frac{Mm}{M+m}}(v - V).$$

Moreover

$$E = W^2 + w^2 = r^2 + s^2.$$

6.2.2. Compatibility Condition

The condition remains the same, but we can rewrite the expressions for the velocity.

Theorem 14. Under the condition

$$|Q| \le \sqrt{E(M+m)},$$

(i) if $|Q| < \sqrt{E(M+m)}$, there are two possible velocities

$$\begin{pmatrix} W \\ w \end{pmatrix} = \frac{Q}{\sqrt{M+m}} \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \pm \left(\frac{E(M+m) - Q^2}{M+m} \right)^{1/2} \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix};$$

(ii) if $|Q| = \sqrt{E(M+m)}$, there is only one possible velocity

$$\begin{pmatrix} W \\ w \end{pmatrix} = \frac{Q}{\sqrt{M+m}} \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}.$$

6.2.3. Elastic Collision

In elastic collisions, momentum and kinetic energy remain constant. The velocity (W, w) is therefore one of the two points on the circle as described above.

Theorem 15. Let (W_-, w_-) be the velocity before the collision. Suppose

$$(-\sin(\alpha), \cos(\alpha)) \bullet (W_-, w_-) > 0$$

to eventually have a collision between the two balls (regardless the position of the wall). Let (W_+, w_+) be the velocity after the collision. The velocities are related by the relation

$$\begin{pmatrix} W_+ \\ w_+ \end{pmatrix} = \mathrm{ref}(\alpha) \begin{pmatrix} W_- \\ w_- \end{pmatrix},$$

and the coefficients by the relation

$$\begin{pmatrix} r_+ \\ s_+ \end{pmatrix} = \mathrm{ref}(0) \begin{pmatrix} r_- \\ s_- \end{pmatrix}.$$

Moreover

$$(-\sin(\alpha), \cos(\alpha)) \bullet (W_+, w_+) < 0,$$

so the two balls move away from each other.

Proof. From Theorem 14 we have directly the relation for the coefficients r and s. From Theorem 13 we get

$$\begin{pmatrix} W_+ \\ w_+ \end{pmatrix} = \mathrm{rot}(\alpha)\,\mathrm{ref}(0)\,\mathrm{rot}(-\alpha) \begin{pmatrix} W_- \\ w_- \end{pmatrix} = \mathrm{ref}(\alpha) \begin{pmatrix} W_- \\ w_- \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} W_+ \\ w_+ \end{pmatrix} = \begin{pmatrix} W_- \\ w_- \end{pmatrix} - 2s_- \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix},$$

it follows that

$$\begin{aligned} (-\sin(\alpha), \cos(\alpha)) \bullet (W_+, w_+) &= (-\sin(\alpha), \cos(\alpha)) \bullet (W_-, w_-) - 2s_- \\ &= -(-\sin(\alpha), \cos(\alpha)) \bullet (W_-, w_-) \\ &< 0. \end{aligned}$$

We can now observe the reflection from the circle centered on the line of equation $(-\sin(\alpha), \cos(\alpha)) \bullet (Y, y) = 0$ of **Figure 6**.

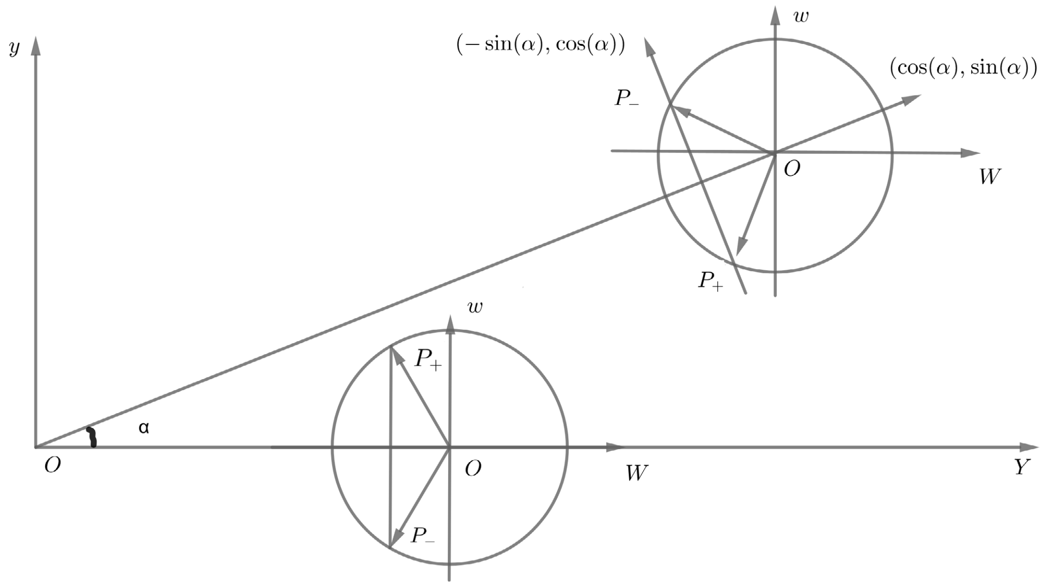

Figure 6. Effect of a collision on (W, w).

Remark. We have the following decomposition for the matrix of the linear system of Theorem 7

$$T = \begin{pmatrix} \frac{M-m}{M+m} & \frac{2m}{M+m} \\ \frac{2M}{M+m} & -\frac{M-m}{M+m} \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} \frac{M-m}{M+m} & \frac{2\sqrt{Mm}}{M+m} \\ \frac{2\sqrt{Mm}}{M+m} & -\frac{M-m}{M+m} \end{pmatrix} \Sigma = \Sigma^{-1} \operatorname{ref}(\alpha) \Sigma.$$

Since the eigenvectors of $\operatorname{ref}(\alpha)$ are $(\cos(\alpha), \sin(\alpha))$ and $(-\sin(\alpha), \cos(\alpha))$ associated to their corresponding eigenvalues 1 and −1, we obtain directly the eigenvectors and eigenvalues of T of Corollary 4.

6.3. Ball-Wall System

In the new coordinate system, for a ball$_m$-wall collision we have $w_+ = -w_-$ and $W_+ = W_-$.

Theorem 16. Let (W_-, w_-) be the velocity before the collision with the wall with $w_- < 0$. Then

$$\begin{pmatrix} W_+ \\ w_+ \end{pmatrix} = \operatorname{ref}(0) \begin{pmatrix} W_- \\ w_- \end{pmatrix},$$

and

$$\begin{pmatrix} r_+ \\ s_+ \end{pmatrix} = \operatorname{ref}(-\alpha) \begin{pmatrix} r_- \\ s_- \end{pmatrix}.$$

The momentum increases at each collision with the wall, and we have

$$Q_+ = Q_- - 2\sqrt{M+m} \sin(\alpha) w_- > Q_-.$$

Proof. The first relation is obvious. For the coefficients r and s, using Theorem 13, we get

$$\begin{pmatrix} r_+ \\ s_+ \end{pmatrix} = \operatorname{rot}(-\alpha)\operatorname{ref}(0)\operatorname{rot}(\alpha)\begin{pmatrix} r_- \\ s_- \end{pmatrix} = \operatorname{ref}(-\alpha)\begin{pmatrix} r_- \\ s_- \end{pmatrix}.$$

For the momentum

$$Q_+ = \sqrt{M}W_+ + \sqrt{m}w_+ = \sqrt{M}W_- - \sqrt{m}w_- = Q_- - 2\sqrt{m}w_-,$$

with $w_- < 0$. □

We also note that incidence and reflection angles are equal. See the circle centered on the line $y = 0$ of Figure 6.

6.4. Stopping Criterion and Trajectory on the Circle

Looking at Figure 5, we get directly result which gives conditions to have no more collision.

Theorem 17. Suppose the velocity (W, w) in polar form is

$$(W, w) = \|(W, w)\|(\cos(\phi), \sin(\phi))$$

where $\phi \in [0, 2\pi)$. There will be no new collision if (W, w)

(i) is the initial velocity and $\phi \in [0, \alpha]$;

(ii) is the velocity after a ball$_m$-wall collision (moving up vertically) and $\phi \in (0, \alpha]$;

(iii) is the velocity after a ball$_M$-ball$_m$ (moving down right) and $\phi \in [0, \alpha)$.

These conditions say that the two balls are going away from the wall with velocity $0 < v \le V$.

On Figure 7, the trajectory of the velocity (W, w) on the circle is given for the two possible ends of the process after at least one collision. We see that it moves successively to P_1, to P_2, to P_3, to P_4, and finally to (W, w) the final point.

7. Sequence of Collisions

Two sequences of collisions are analyzed here depending on the first collision. These sequences of collisions correspond to two alternating sequences of the two

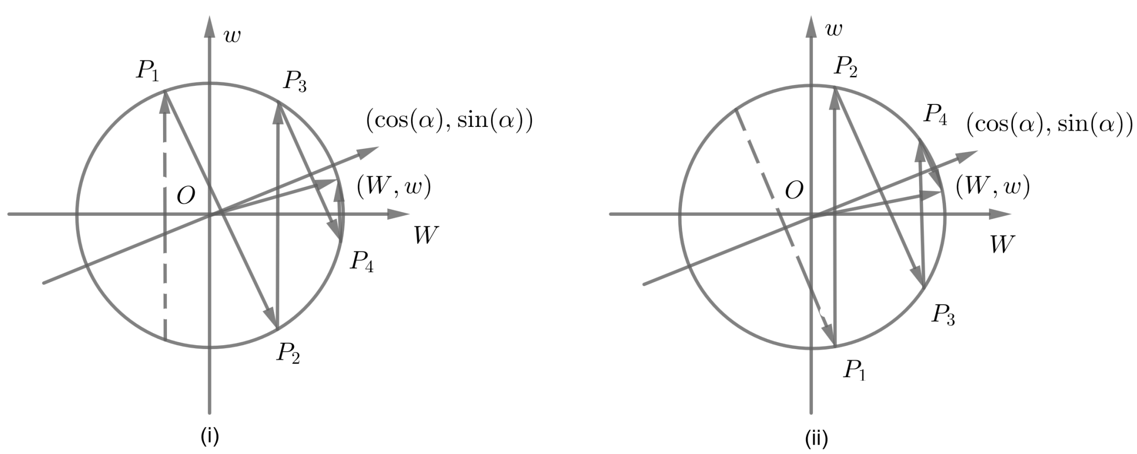

Figure 7. (W, w) trajectory on the circle and stopping region.

reflections: $\text{ref}(\alpha)$ (collision ball$_M$-ball$_m$) and $\text{ref}(0)$ (collision ball$_m$-wall). In this section we will use the notation (W_k, w_k) for the velocity after the k-th collision.

7.1. Ball-Ball Collision First

For the sequence of collisions starting with a ball$_M$-ball$_m$ collision followed by a ball$_m$-wall collision, we must have $W/\sqrt{M} = V < v = w/\sqrt{m}$, or $(-\sin(\alpha), \cos(\alpha)) \bullet (W, w) > 0$, to eventually have a ball$_M$-ball$_m$ collision, without considering the wall.

Theorem 18. A sequence of two collisions, a ball$_M$-ball$_m$ collision followed by a ball$_m$-wall collision, is a rotation of angle -2α.

Proof. For the first ball$_M$-ball$_m$ collision, we get from Lemma 10

$$\begin{pmatrix} W_1 \\ w_1 \end{pmatrix} = \text{ref}(\alpha)\begin{pmatrix} W \\ w \end{pmatrix} = \text{ref}(0)\,\text{rot}(-2\alpha)\begin{pmatrix} W \\ w \end{pmatrix},$$

and for the second ball$_m$-wall collision, using part (d) of Lemma 11, we have

$$\begin{pmatrix} W_2 \\ w_2 \end{pmatrix} = \text{ref}(0)\begin{pmatrix} W_1 \\ w_1 \end{pmatrix} = \text{ref}(0)\,\text{ref}(0)\,\text{rot}(-2\alpha)\begin{pmatrix} W \\ w \end{pmatrix} = \text{rot}(-2\alpha)\begin{pmatrix} W \\ w \end{pmatrix}.$$

So the result follows. □

Thereafter there is alternation of collisions: ball$_M$-ball$_m$, ball$_m$-wall, etc. Let us set

$$\begin{pmatrix} W \\ w \end{pmatrix} = E^{1/2}\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix},$$

where $\phi \in (\alpha, \pi + \alpha)$.

Theorem 19. The velocity after

(A) $2n$ collisions (with a last ball$_m$-wall collision) is

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = E^{1/2}\begin{pmatrix} \cos(\phi - 2n\alpha) \\ \sin(\phi - 2n\alpha) \end{pmatrix};$$

(B) $2n + 1$ collisions (with a last ball$_M$-ball$_m$ collision) is

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = E^{1/2}\begin{pmatrix} \cos(2(n+1)\alpha - \phi) \\ \sin(2(n+1)\alpha - \phi) \end{pmatrix}.$$

Proof. The process ends after $2n$ or $2n + 1$ collisions.

(A) For $2n$ collisions, we use part (a) of Lemma 11 to get

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = \underbrace{\text{rot}(-2\alpha)\cdots\text{rot}(-2\alpha)}_{n\text{-times}}\begin{pmatrix} W \\ w \end{pmatrix} = \text{rot}(-2n\alpha)\begin{pmatrix} W \\ w \end{pmatrix}.$$

(B) For $2n + 1$ collisions, one more ball$_M$-ball$_m$ collision is needed, so

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \text{ref}(\alpha)\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix}.$$

Then from part (c) of Lemma 11, we get

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \text{ref}(\alpha)\text{rot}(-2n\alpha)\begin{pmatrix} W \\ w \end{pmatrix} = \text{rot}(2(n+1)\alpha)\text{ref}(0)\begin{pmatrix} W \\ w \end{pmatrix}. \qquad \square$$

Starting with a ball$_M$-ball$_m$ collision, considering the preceding expressions for the velocity, and applying the stopping criterion, we conclude that the process will end after

(i) $2n$ collisions if the last collision is a ball$_m$-wall collision, so (W_{2n}, w_{2n}), of angle $\phi - 2n\alpha$, is on the circular arc of angle in $(0, \alpha]$;

(ii) $2n+1$ collisions if the last collision is a ball$_M$-ball$_m$ collision, so (W_{2n+1}, w_{2n+1}), of angle $2(n+1)\alpha - \phi$, is on the circular arc of angle in $[0, \alpha)$.

In both cases, if K is the number of collisions we obtain

$$\frac{\phi}{\alpha} - 1 \le K < \frac{\phi}{\alpha},$$

or

$$K = \begin{cases} \dfrac{\phi}{\alpha} - 1 & \text{if } \dfrac{\phi}{\alpha} \text{ is an integer,} \\ \left\lfloor \dfrac{\phi}{\alpha} \right\rfloor & \text{if } \dfrac{\phi}{\alpha} \text{ is not an integer.} \end{cases}$$

For example, taking $(V, v) = (-1, 0)$, so $(W, w) = (-\sqrt{M}, 0)$, $E = M$, $\phi = \pi$, and it follows that

$$\frac{\pi}{\alpha} - 1 \le K < \frac{\pi}{\alpha}.$$

7.2. Ball-Wall Collision First

For the sequence of collisions starting with a ball$_m$-wall collision followed by a ball$_M$-ball$_m$ collision, we must have (V, v) with $v < 0$, say (W, w) and $w < 0$ to eventually have a first ball$_m$-wall collision, without considering the ball$_M$.

Theorem 20. A sequence of two collisions, a ball$_m$-wall collision followed by a ball$_M$-ball$_m$ collision, is a rotation of angle 2α.

Proof. For the first collision, ball$_m$-wall, we have

$$\begin{pmatrix} W_1 \\ w_1 \end{pmatrix} = \text{ref}(0)\begin{pmatrix} W \\ w \end{pmatrix}$$

and for the second collision, a ball$_M$-ball$_m$ collision, using part (d) of Lemma 11, we get

$$\begin{pmatrix} W_2 \\ w_2 \end{pmatrix} = \text{ref}(\alpha)\begin{pmatrix} W_1 \\ w_1 \end{pmatrix} = \text{ref}(\alpha)\text{ref}(0)\begin{pmatrix} W \\ w \end{pmatrix} = \text{rot}(2\alpha)\begin{pmatrix} W \\ w \end{pmatrix}.$$

So the result follows. $\square$

Thereafter there is alternation of collisions: ball$_m$-wall, ball$_M$-ball$_m$, etc. Let us set

$$\begin{pmatrix} W \\ w \end{pmatrix} = E^{1/2}\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix},$$

where $\phi \in (-\pi, 0)$.

Theorem 21. The velocity after

(A) $2n$ collisions (with a last ball$_M$-ball$_m$ collision) is

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = E^{1/2} \begin{pmatrix} \cos(\phi + 2n\alpha) \\ \sin(\phi + 2n\alpha) \end{pmatrix}.$$

(B) $2n + 1$ collisions (with a last ball$_m$-wall collision) is

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = E^{1/2} \begin{pmatrix} \cos(-2n\alpha - \phi) \\ \sin(-2n\alpha - \phi) \end{pmatrix}.$$

Proof. The process ends after $2n$ or $2n + 1$ collisions.

(A) For $2n$ collisions, we use part (a) of Lemma 11 to get

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = \underbrace{\mathrm{rot}(2\alpha)\cdots\mathrm{rot}(2\alpha)}_{n\text{-times}} \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{rot}(2n\alpha) \begin{pmatrix} W \\ w \end{pmatrix}.$$

(B) For $2n + 1$ collisions, one more ball$_m$-wall collision is needed, so

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \mathrm{ref}(0) \begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix}.$$

Then from part (c) of Lemma 11, we get

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \mathrm{ref}(0)\,\mathrm{rot}(2n\alpha) \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{rot}(-2n\alpha)\,\mathrm{ref}(0) \begin{pmatrix} W \\ w \end{pmatrix}.$$

Starting with a ball$_m$-wall collision, considering the preceding expressions for the velocity, and applying the stopping criterion, we conclude that the process will end after

(i) $2n + 1$ collisions if the last collision is a ball$_m$-wall collision, so (W_{2n+1}, w_{2n+1}), of angle $-2n\alpha - \phi$, is on the circular arc of angle in $(0, \alpha]$;

(ii) $2n$ collisions if the last collision is a ball$_M$-ball$_m$ collision, so (W_{2n}, w_{2n}), of angle $2n\alpha + \phi$, is on the circular arc of angle in $[0, \alpha)$.

In both case, if K is the number of collisions, we have

$$-\frac{\phi}{\alpha} \le K < -\frac{\phi}{\alpha} + 1,$$

or

$$K = \begin{cases} -\dfrac{\phi}{\alpha} & \text{if } -\dfrac{\phi}{\alpha} \text{ is an integer,} \\ \left\lfloor -\dfrac{\phi}{\alpha} \right\rfloor + 1 & \text{if } -\dfrac{\phi}{\alpha} \text{ is not an integer.} \end{cases}$$

For example, taking $(V, v) = (-1, -1)$, so $(W, w) = (-\sqrt{M}, -\sqrt{m})$, $E = M + m$, $\phi = -\pi + \alpha$, and it follows that

$$\frac{\pi}{\alpha} - 1 \le K < \frac{\pi}{\alpha}.$$

7.3. The Result

From the preceding analysis we obtain the following results.

Theorem 22. The maximal number $K_{\max}$ of collisions (reflections) of the trajectory $(X(\cdot),x(\cdot))$ obtained from (X,x) in the region $D=\{(X,x)\,|\,0<x<X\}$ in the initial direction (V,v) is finite and

$$K_{\max}=\begin{cases}\dfrac{\pi}{\alpha} & \text{if } \dfrac{\pi}{\alpha} \text{ is an integer,}\\ \left\lfloor\dfrac{\pi}{\alpha}\right\rfloor+1 & \text{if } \dfrac{\pi}{\alpha} \text{ is not an integer.}\end{cases}$$

The next result describes the special situation for which the initial velocity is parallel to one of the collision lines.

Theorem 23. If the initial trajectory is parallel to one of the collision lines, that is to say

$$\begin{cases}\text{First case}:(V,v)=(-1,0)\text{ so }(W,w)=\left(-\sqrt{M},0\right),\\ \text{ or}\\ \text{Second case}:(V,v)=(-1,-1)\text{ so }(W,w)=\left(-\sqrt{M},-\sqrt{m}\right),\end{cases}$$

then the number of collisions K is

$$K=\begin{cases}\dfrac{\pi}{\alpha}-1 & \text{if } \dfrac{\pi}{\alpha} \text{ is an integer,}\\ \left\lfloor\dfrac{\pi}{\alpha}\right\rfloor & \text{if } \dfrac{\pi}{\alpha} \text{ is not an integer.}\end{cases}$$

8. Back to Galperin's Geometric Method: Position and Time of Collisions

Based on the transformation described in Section 4, and the resulting reflection properties, Galperin proposed a simple geometric method to find the number of collisions [3]. It happens that his geometric method can also be used to easily find position and time of collisions, and this observation extends the presentation done in [4] and [5].

8.1. Galperin's Method: Number of Collisions

We consider the plane YOy, and in the two situations, unfold using the symmetry with respect to the lines

$$\begin{cases}(-\sin(\alpha),\cos(\alpha))\bullet(Y,y)=0,\\ y=0,\end{cases}$$

sequentially following the collisions on the lines. Hence, since the incident and reflection angles are equal, the trajectory $(Y(\cdot),y(\cdot))$ of the balls is now a line

$$(Y(\lambda),y(\lambda))=(Y_0,y_0)+\lambda(W_0,w_0).$$

This unfolding of the collision line creates centered rays with a spacing of angle α. Then the number of collisions K is the number of times the trajectory crosses the rays, see **Figure 8**. This is the way Galperin gets the results in Theorem 22 and Theorem 23 [3]. We can now use this geometric representation to compute time and position of all collisions.

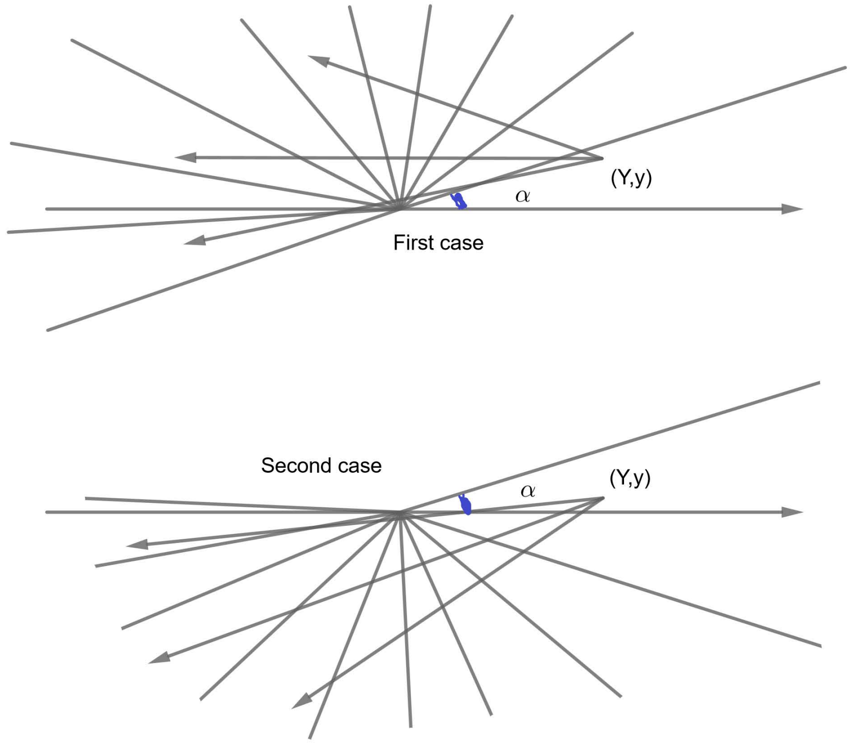

Figure 8. Geometric method.

8.2. Time and Position: Ball-Ball Collision First (Figure 9)

In systems YOy and WOw, let us consider given

(i) a position $(Y,y) = \|(Y,y)\|(\cos(\psi),\sin(\psi))$ where $\psi \in (0,\alpha)$, and

(ii) a velocity $(W,w) = \|(W,w)\|(\cos(\phi),\sin(\phi))$ with $\phi \in (\alpha,\pi+\psi)$,

such that we have a first ball$_M$-ball$_m$ collision on the line of direction $(\cos(\alpha),\sin(\alpha))$.

Let us rotate the system of an angle θ in such a way that

$$\left(\hat{W},\hat{w}\right) = \mathrm{rot}(\theta)(W,w) = \|(W,w)\|(\cos(\phi+\theta),\sin(\phi+\theta)) = \|(W,w)\|(-1,0),$$

so $\phi+\theta=\pi$, and $\theta = \pi-\phi \in (-\psi,\pi-\alpha)$. Then the position becomes

$$\left(\hat{Y},\hat{y}\right) = \mathrm{rot}(\theta)(Y,y) = \|(Y,y)\|(\cos(\psi+\theta),\sin(\psi+\theta)),$$

and $\psi+\theta \in (0,\pi-(\alpha-\psi)) \subset (0,\pi)$. The line of the ball$_M$-ball$_m$ collisions of direction $(\cos(\alpha),\sin(\alpha))$ is now of direction $(\cos(\alpha+\theta),\sin(\alpha+\theta))$. The other rays are of direction $(\cos(n\alpha+\theta),\sin(n\alpha+\theta))$ for $n=1,2,3,\cdots,K$, where the number of collisions K is such that

$$\begin{cases} K\alpha+\theta<\pi \text{ or } K\alpha<\phi, \\ (K+1)\alpha+\theta \geq \pi \text{ or } (K+1)\alpha \geq \phi, \end{cases}$$

so

$$\frac{\phi}{\alpha}-1 \leq K < \frac{\phi}{\alpha},$$

as observed in Section 7. We also observe that

$$0<\psi+\theta<\alpha+\theta\leq n\alpha+\theta\leq K\alpha+\theta<\pi,$$

for any $n=1,\cdots,K$.

The position of the n-th collision point is

$$\begin{aligned}\left(\hat{Y}_n,\hat{y}_n\right)&=\hat{y}\left(\cot\left(n\alpha+\theta\right),1\right)\\&=\left\|\left(Y,y\right)\right\|\frac{\sin\left(\psi+\theta\right)}{\sin\left(n\alpha+\theta\right)}\left(\cos\left(n\alpha+\theta\right),\sin\left(n\alpha+\theta\right)\right).\end{aligned}$$

The corresponding position in the unfolded plane is

$$\left(Y_n,y_n\right)=\operatorname{rot}\left(\theta_n\right)\operatorname{rot}\left(-\theta\right)\left(\hat{Y}_n,\hat{y}_n\right)$$

where

$$\theta_n=\begin{cases}-\left(n-1\right)\alpha & \text{if } n \text{ is odd;}\\ -n\alpha & \text{if } n \text{ is even.}\end{cases}$$

So we obtain for $n=1,\cdots,K$

$$\left(Y_n,y_n\right)=\left\|\left(Y,y\right)\right\|\frac{\sin\left(\psi+\theta\right)}{\sin\left(n\alpha+\theta\right)}\begin{cases}\left(\cos\left(\alpha\right),\sin\left(\alpha\right)\right) & \text{if } n \text{ is odd;}\\ \left(1,0\right) & \text{if } n \text{ is even.}\end{cases}$$

and

$$\left(X_n,x_n\right)=\left\|\left(Y,y\right)\right\|\frac{\sin\left(\psi+\theta\right)}{\sin\left(n\alpha+\theta\right)}\begin{cases}\dfrac{1}{\sqrt{M+m}}\left(1,1\right) & \text{if } n \text{ is odd;}\\ \dfrac{1}{\sqrt{M}}\left(1,0\right) & \text{if } n \text{ is even.}\end{cases}$$

Moreover the time of the n-th collision is

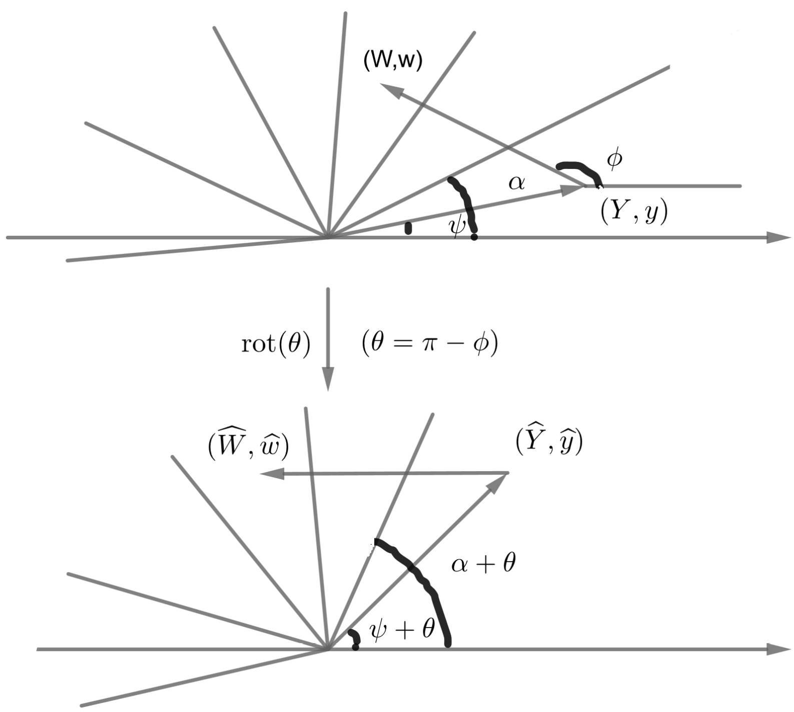

Figure 9. Rotation for the ball-ball collision first.

$$\begin{aligned} t_{0,n} &= \frac{\hat{Y}-\hat{Y}_n}{\|(W,w)\|} \\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\left[\cos(\psi+\theta)-\sin(\psi+\theta)\cot(n\alpha+\theta)\right] \\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\frac{\sin(n\alpha-\psi)}{\sin(n\alpha+\theta)}, \end{aligned}$$

where $n\alpha-\psi=(n\alpha+\theta)-(\psi+\theta)\in(0,\pi)$. The time between the n_1-*th* and n_2-*th* collisions, $1\le n_1<n_2\le K$, is

$$\begin{aligned} t_{n_1,n_2} &= t_{0,n_2}-t_{0,n_1} \\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\left[\frac{\sin(n_2\alpha-\psi)}{\sin(n_2\alpha+\theta)}-\frac{\sin(n_1\alpha-\psi)}{\sin(n_1\alpha+\theta)}\right] \\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\frac{\sin(\psi+\theta)\sin((n_2-n_1)\alpha)}{\sin(n_1\alpha+\theta)\sin(n_2\alpha+\theta)}. \end{aligned}$$

8.3. Time and Position: Ball-Wall Collision First (Figure 10)

In systems YOy and WOw, let us consider given

(i) a position $(Y,y)=\|(Y,y)\|(\cos(\psi),\sin(\psi))$ where $\psi\in(0,\alpha)$, and

(ii) a velocity $(W,w)=\|(W,w)\|(\cos(\phi),\sin(\phi))$ with $\phi\in(-\pi+\psi,0)$,

such that we have a first ball$_m$-wall collision on the OY axis. The first step is to rotate the system of an angle θ in such a way that

$$(\hat{W},\hat{w})=\mathrm{rot}(\theta)(W,w)=\|(W,w)\|(\cos(\phi+\theta),\sin(\phi+\theta))=\|(W,w)\|(-1,0)$$

so $\phi+\theta=-\pi$, and $\theta=-(\pi+\phi)\in(-\pi,-\psi)$. The position becomes

$$(\hat{Y},\hat{y})=\mathrm{rot}(\theta)(Y,y)=\|(Y,y)\|(\cos(\psi+\theta),\sin(\psi+\theta)),$$

and $\psi+\theta\in(-\pi+\psi,0)\subset(-\pi,0)$. The line of the ball$_m$-wall collisions of direction $(1,0)=(\cos(0),\sin(0))$ (in fact of the first collision) is now of direction $(\cos(\theta),\sin(\theta))$. The direction of the ray of the n-*th* collision is $(\cos(-(n-1)\alpha+\theta),\sin(-(n-1)\alpha+\theta))$ for $n=1,2,3,\cdots,K$, where K is such that

$$\begin{cases} -(K-1)\alpha+\theta>-\pi \text{ or } (K-1)\alpha<-\phi, \\ -K\alpha+\theta\le-\pi \text{ or } K\alpha\ge-\phi, \end{cases}$$

so

$$-\frac{\phi}{\alpha}\le K<-\frac{\phi}{\alpha}+1,$$

as observed in Section 7. We also observe that

$$-\pi<-(K-1)\alpha+\theta\le-(n-1)\alpha+\theta\le\theta<0,$$

for any $n=1,2,\cdots,K$.

The position of the n-*th* collision point is

$$\left(\hat{Y}_n, \hat{y}_n\right) = \hat{y}\left(\cot\left(-(n-1)\alpha + \theta\right), 1\right)$$
$$= \frac{\|(Y,y)\| \sin(\psi + \theta)}{\sin\left(-(n-1)\alpha + \theta\right)} \left(\cos\left(-(n-1)\alpha + \theta\right), \sin\left(-(n-1)\alpha + \theta\right)\right),$$

for $n = 1, \cdots, K$. The corresponding position in the unfolded plane is

$$\left(Y_n, y_n\right) = \operatorname{rot}(\theta_n)\operatorname{rot}(-\theta)\left(\hat{Y}_n, \hat{y}_n\right)$$

where

$$\theta_n = \begin{cases} (n-1)\alpha & \text{if } n \text{ is odd;} \\ n\alpha & \text{if } n \text{ is even.} \end{cases}$$

So we obtain

$$\left(Y_n, y_n\right) = \|(Y,y)\| \frac{\sin(\psi + \theta)}{\sin\left(-(n-1)\alpha + \theta\right)} \begin{cases} (1,0) & \text{if } n \text{ is odd;} \\ \left(\cos(\alpha), \sin(\alpha)\right) & \text{if } n \text{ is even.} \end{cases}$$

and

$$\left(X_n, x_n\right) = \|(Y,y)\| \frac{\sin(\psi + \theta)}{\sin\left(-(n-1)\alpha + \theta\right)} \begin{cases} \dfrac{1}{\sqrt{M}}(1,0) & \text{if } n \text{ is odd;} \\ \dfrac{1}{\sqrt{M+m}}(1,1) & \text{if } n \text{ is even.} \end{cases}$$

Moreover the time of the *n-th* collision is

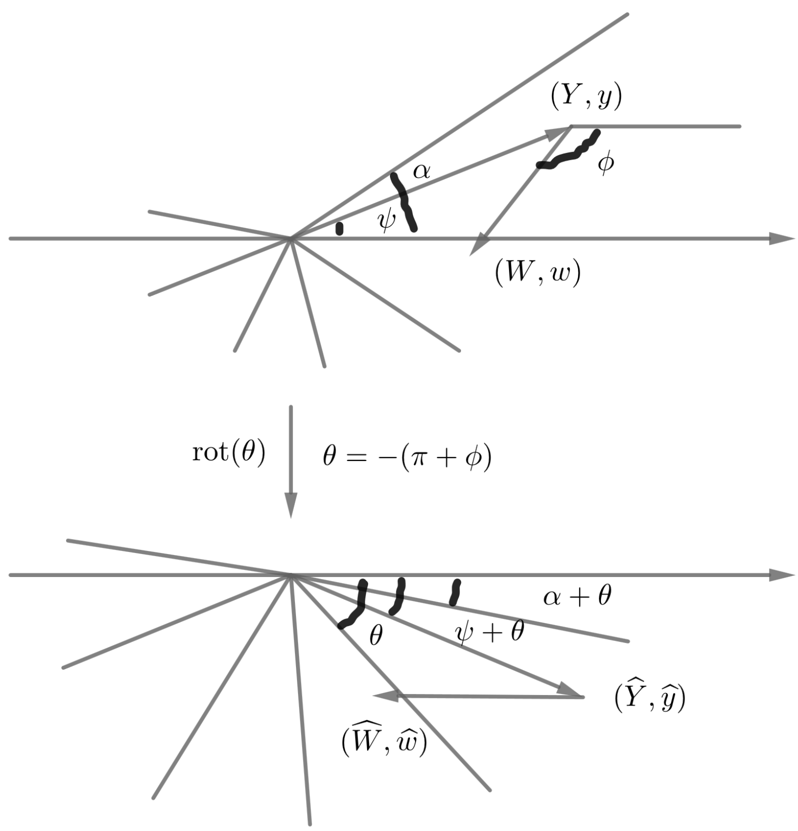

Figure 10. Rotation for the ball-wall collision first.

$$\begin{aligned}t_{0,n} &= \frac{\hat{Y}-\hat{Y}_n}{\|(W,w)\|}\\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\left[\cos(\psi+\theta)-\sin(\psi+\theta)\cot(-(n-1)\alpha+\theta)\right]\\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\frac{\sin(-(n-1)\alpha-\psi)}{\sin(-(n-1)\alpha+\theta)}\\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\frac{\sin((n-1)\alpha+\psi)}{\sin((n-1)\alpha-\theta)},\end{aligned}$$

where $-(n-1)\alpha-\psi=(-(n-1)\alpha+\theta)-(\psi+\theta)\in(-\pi,0)$ or $(n-1)\alpha+\psi=((n-1)\alpha-\theta)+(\psi+\theta)\in(0,\pi)$. The time between the n_1-*th* and n_2-*th* collisions, $1\le n_1<n_2\le K$, is

$$\begin{aligned}t_{n_1,n_2} &= t_{0,n_2}-t_{0,n_1}\\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\left[\frac{\sin((n_2-1)\alpha+\psi)}{\sin((n_2-1)\alpha-\theta)}-\frac{\sin((n_1-1)\alpha+\psi)}{\sin((n_1-1)\alpha-\theta)}\right]\\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\frac{\sin(\psi+\theta)\sin((n_1-n_2)\alpha)}{\sin((n_1-1)\alpha-\theta)\sin((n_2-1)\alpha-\theta)}\\ &= \frac{\|(Y,y)\|}{\|(W,w)\|}\frac{\sin(\pi-(\psi+\theta))\sin((n_2-n_1)\alpha)}{\sin((n_1-1)\alpha-\theta)\sin((n_2-1)\alpha-\theta)}.\end{aligned}$$

9. Digits of π

9.1. Observations

Theorem 23 suggests a way to compute the digits of π, in fact in any integer base $b\ge 2$ of a number system. For example taking $M=m$, $\alpha=\arctan(1)=\pi/4$, or $\alpha=\arcsin(1/\sqrt{2})=\pi/4$, and $K=3$ is the value of the first figures of π. Since the integer part of $\pi\cdot b^N$, noted $\lfloor \pi\cdot b^N\rfloor_b$ in base *b*, add the first *N* digits of the fractional part of π in base *b*, for an angle $\alpha\approx b^{-N}$ we will be near the goal.

To get $\alpha\approx b^{-N}$, we can consider the following two cases:

(A) $\tan(\alpha)=b^{-N}$, so $\alpha=\arctan(b^{-N})\approx b^{-N}$, which means $\sqrt{m/M}=b^{-N}$ and $M=b^{2N}m$;

(B) $\sin(\alpha)=b^{-N}$, so $\alpha=\arcsin(b^{-N})\approx b^{-N}$, which means $\sqrt{m/(M+m)}=b^{-N}$ and $M=(b^{2N}-1)m$. It remains to verify that, if K_b is the number of collisions, the following conjecture is true.

Conjecture. *For* (A) $M=b^{2N}m$ (*i.e.* $\alpha=\arctan(b^{-N})$), *or* (B) $M=(b^{2N}-1)m$ (*i.e.* $\alpha=\arcsin(b^{-N})$), *if the trajectory* (Y,y) *is parallel to the straight line of collisions, the total number* K_b *of collisions which is given by its representation in base b by*

$$K_b=\begin{cases}\left\lceil \dfrac{\pi}{\alpha}\right\rceil_b -1 & \text{if } \dfrac{\pi}{\alpha} \text{ is an integer,}\\ \left\lfloor \dfrac{\pi}{\alpha}\right\rfloor_b & \text{if } \dfrac{\pi}{\alpha} \text{ is not an integer,}\end{cases}$$

consists of the digits of the integer part of π *and the first N digits of the fractional part of* π *in base b, so* $K_b = \lfloor \pi \cdot b^N \rfloor_b$.

In the sequel we will use the following representations in base b

$$\begin{cases} \pi_b = 3_b.a_1 a_2 \cdots a_N a_{N+1} \cdots a_{2N-1} a_{2N} a_{2N+1} \cdots \\ \left[\dfrac{\pi}{3}\right]_b = 1.\cdots \end{cases}$$

and

$$\begin{cases} \left[\pi \cdot b^N\right]_b = 3_b a_1 a_2 \cdots a_N . a_{N+1} \cdots a_{2N-1} a_{2N} a_{2N+1} a_{2N+2} \cdots \\ \left[\dfrac{\pi}{3} \cdot b^{-N}\right]_b = 0.\underbrace{0 \cdots 0}_{(N-1)\text{-times}} 1 \cdots \end{cases}$$

9.2. Case (A)

The Taylor expansion of $\arctan(x)$ is

$$\arctan(x) = \sum_{i=0}^{n} (-1)^i \frac{x^{2i+1}}{2i+1} + (-1)^{n+1} B_n(x)$$

for $|x| < 1$. Also $0 < B_n(x) \leq \dfrac{x^{2n+3}}{2n+3}$ for $x > 0$. So we obtain

$$\frac{1}{x} < \frac{1}{\arctan(x)} < \frac{1}{x} + \frac{x}{3}$$

for $0 < x < 2/3$. Multiplying by π and take $x = b^{-N}$, then

$$\pi \cdot b^N < \frac{\pi}{\arctan(b^{-N})} < \pi \cdot b^N + \frac{\pi}{3} \cdot b^{-N}.$$

We first observe that $\pi \cdot b^N$ is not an integer, so

$$\lfloor \pi \cdot b^N \rfloor_b < \left[\pi \cdot b^N\right]_b < \left[\frac{\pi}{\arctan(b^{-N})}\right]_b.$$

Using the representation in base *b*, since

$$\left[\pi \cdot b^N\right]_b + \left[\frac{\pi}{3} \cdot b^{-N}\right]_b = \left[\pi \cdot b^N + \frac{\pi}{3} \cdot b^{-N}\right]_b,$$

under the condition that there exists $n \in [N+1, 2N-1]$ such that $0 \leq a_n < b-1$, we get

$$\left[\frac{\pi}{\arctan(b^{-N})}\right]_b < \left[\pi \cdot b^N + \frac{\pi}{3} \cdot b^{-N}\right]_b < \lfloor \pi \cdot b^N \rfloor_b + 1.$$

Hence

$$\lfloor \pi \cdot b^N \rfloor_b < \left[\frac{\pi}{\arctan(b^{-N})}\right]_b < \lfloor \pi \cdot b^N \rfloor_b + 1,$$

consequently, $\left[\dfrac{\pi}{\arctan(b^{-N})}\right]_b$ is not an integer and

$$\lfloor \pi \cdot b^N \rfloor_b = \left\lfloor \frac{\pi}{\arctan\left(b^{-N}\right)} \right\rfloor_b .$$

9.3. Case (B)

The Taylor expansion of $\arcsin(x)$ is

$$\arcsin(x) = \sum_{i=0}^{n} \frac{(2i)!}{4^i (i!)^2} \frac{x^{2i+1}}{2i+1} + B_n(x)$$

for $|x| < 1$. Also $0 < B_n(x) \le \frac{x^{2n+3}}{2n+3}$ for $x > 0$. We obtain

$$\frac{1}{x} - \frac{x}{3} < \frac{1}{\arcsin(x)} < \frac{1}{x},$$

for $0 < x < 1/\sqrt{2}$. Multiplying by π and take $x = b^{-N}$, we have

$$\pi \cdot b^N - \frac{\pi}{3} \cdot b^{-N} < \frac{\pi}{\arcsin\left(b^{-N}\right)} < \pi \cdot b^N .$$

Since $\pi \cdot b^N$ is not an integer

$$\left\lceil \frac{\pi}{\arcsin\left(b^{-N}\right)} \right\rceil_b < \left\lceil \pi \cdot b^N \right\rceil_b < \left\lceil \pi \cdot b^N \right\rceil_b = \left\lfloor \pi \cdot b^N \right\rfloor_b + 1.$$

Using the representation in base *b*, since

$$\left\lceil \pi \cdot b^N \right\rceil_b - \left\lceil \frac{\pi}{3} \cdot b^{-N} \right\rceil_b = \left\lceil \pi \cdot b^N - \frac{\pi}{3} \cdot b^{-N} \right\rceil_b ,$$

under the condition that there exists $n \in [N+1, 2N-1]$ such that $0 < a_n \le b-1$, we get

$$\left\lfloor \pi \cdot b^N \right\rfloor_b < \left\lceil \pi \cdot b^N - \frac{\pi}{3} \cdot b^{-N} \right\rceil_b < \left\lceil \frac{\pi}{\arcsin\left(b^{-N}\right)} \right\rceil_b .$$

So

$$\left\lfloor \pi \cdot b^N \right\rfloor_b < \left\lceil \frac{\pi}{\arcsin\left(b^{-N}\right)} \right\rceil_b < \left\lfloor \pi \cdot b^N \right\rfloor_b + 1,$$

consequently $\left\lceil \frac{\pi}{\arcsin\left(b^{-N}\right)} \right\rceil_b$ is not an integer and

$$\left\lfloor \pi \cdot b^N \right\rfloor_b = \left\lfloor \frac{\pi}{\arcsin\left(b^{-N}\right)} \right\rfloor_b .$$

9.4. Consequences

There are some consequences of the preceding results. For

Case (A): if $a_{N+1} < b-1$, then

$$\lfloor \pi \cdot b^{N-k} \rfloor_b = \left\lfloor \pi \cdot b^{N-k} + \frac{\pi}{3} \cdot b^{-(N-k)} \right\rfloor_b,$$

Case (B): if $a_{N+1} > 0$, then

$$\left\lfloor \pi \cdot b^{N-k} - \frac{\pi}{3} \cdot b^{-(N-k)} \right\rfloor_b = \lfloor \pi \cdot b^{N-k} \rfloor_b,$$

both for $k = 0, \cdots, \left\lfloor \frac{N-2}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor - 1$. So the result holds for the powers of b from $N + 1 - \left\lfloor \frac{N}{2} \right\rfloor$ up to *N*.

This last observation suggests a Cauchy induction like method [8]. With an algorithm which can find a digit of π at a precise position *N* without calculating all digits in positions less than *N*, see for example [9] [10] [11], we could deduce the result for a number of lower positions. We proceed in the following way. Suppose the property true for $n = 1, \cdots, N$. Then look for the smallest $\ell \in \{0, 1, \cdots, N\}$ such that in case (A) $a_{2N-\ell+1} < b - 1$ or in case (B) $a_{2N-\ell+1} > 0$, then the result holds for $n = 1, \cdots, 2N - \ell$.

9.5. Conjecture Almost Proved

Up to now, with modern computational facilities, and up to very large values of *N*, it has not been observed sequences such that

Case (A) $a_n = b - 1$ for $n \in [N+1, 2N-1]$,

Case (B) $a_n = 0$ for $n \in [N+1, 2N-1]$,

in the expansion of π. So the claim is verified for up to very large values of *N*.

9.6. Final Remark

There exists in fact infinitely many angles α for which we get the result $K_b = \lfloor \pi \cdot b^N \rfloor_b$. Indeed, if we use α_λ with

$$\tan^2(\alpha_\lambda) = \frac{m}{M_\lambda} = \frac{b^{-2N}}{1 - \lambda b^{-2N}}$$

for $\lambda \in [0,1]$, we also get the result. In fact we use the masses *m* and M_λ where

$$M_\lambda = (1-\lambda) M_A + \lambda M_B = \left(b^{2N} - \lambda\right) m,$$

where $M_0 = M_A = b^{2N} m$ and $M_1 = M_B = \left(b^{2N} - 1\right) m$ are the masses for case (A) and case (B). Also $\alpha_A = \alpha_0 \le \alpha_\lambda \le \alpha_1 = \alpha_B$, and

$$\frac{\pi}{\alpha_B} \le \frac{\pi}{\alpha_\lambda} \le \frac{\pi}{\alpha_A}.$$

Since the result holds for $\frac{\pi}{\alpha_A}$ and $\frac{\pi}{\alpha_B}$, it also holds for $\frac{\pi}{\alpha_\lambda}$ for any $\lambda \in [0,1]$.

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Conflicts of Interest

Author declares no competing interests.

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