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Article

Spatial Analyticity of Solutions to Korteweg–de Vries Type Equations

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Abstract: The Korteweg–de Vries equation (KdV) is a mathematical model of waves on shallow water surfaces. It is given as third-order nonlinear partial differential equation and plays a very important role in the theory of nonlinear waves. It was obtained by Boussinesq in 1877, and a detailed analysis was performed by Korteweg and de Vries in 1895. In this article, by using multi-linear estimates in Bourgain type spaces, we prove the local well-posedness of the initial value problem associated with the Korteweg–de Vries equations. The solution is established online for analytic initial data w_0 that can be extended as holomorphic functions in a strip around the x-axis. A procedure for constructing a global solution is proposed, which improves upon earlier results.

Keywords: KdV equation; radius of spatial analyticity; approximate conservation law

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1. Introduction and Main Results

The study of Korteweg–de Vries (KdV) type equations in Bourgain type spaces is an important task, both from a theoretical point of view (existence and uniqueness of solution theorems) and from the point of view of applications. In this research, we consider the nonlinear Cauchy problem:

$$\begin{cases} \partial_t w + \partial_x^3 w + \eta \mathcal{L} w + \partial_x (w)^4 = 0, \ (x,t) \in \mathbb{R}^2, \\ w(x,0) = w_0(x), \end{cases} \tag{1}$$

where the function w is real-valued, and η is a positive constant, $w(x,t) \in \mathbb{R}$.

We define the linear operator $\mathcal{L}$ via the Fourier transform by

$$\widehat{\mathcal{L} f}(\zeta) = -\phi(\zeta) \widehat{f}(\zeta). \tag{2}$$

We denote by ϕ the phase function

$$\phi(\nu) = \sum_{j=0}^{n} \sum_{i=0}^{2m} c_{i,j} \nu^i |\nu|^j, \ c_{i,j} \in \mathbb{R}, \ c_{2m,n} = -1, \tag{3}$$

where $\phi(\zeta) < c$ for some constant c.

Generally, this equation is an evolution type equation. We recall one of this type, which is the Korteweg–de Vries–Kuramoto–Sivashinsky equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \eta(\partial_x^2 u + \partial_x^4 u) + u \partial_x u = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{4}$$

which is known as a model for long waves in a viscous fluid flowing down an inclined plane; see [1–8]. It is considered as a particular case of the Benney–Lin equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \eta(\partial_x^2 u + \partial_x^4 u) + \beta \partial_x^5 u + u\partial_x u = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{5}$$

when $\beta = 0$ (please see [8,9]).

We mention a second example related to the Korteweg–de Vries–Burgers equation:

$$\begin{cases} \partial_t u + \partial_x^3 u - \eta \partial_x^2 u + u\partial_x u = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{6}$$

considered in [10]. The authors showed that problem (6) admits a local solution for some given data in $H^s, s > 1$.

For other problems, close results were obtained in the generalized Ostrovsky–Stepanyams–Tsimring equation for $x \in \mathbb{R},\ t \geq 0,\ k \in \mathbb{Z}^+$

$$\begin{cases} \partial_t u + \partial_x^3 u - \eta(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u) + u^k \partial_x u = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{7}$$

where $\mathcal{H}$ denotes the Hilbert transform; please see [11–13].

An effective method for studying lower bounds on the radius of analyticity, including this type of problem, was introduced in [14] for 1D Dirac–Klein–Gordon equations. It was applied in [15] to the modified Kawahara equation and in [16] to the non-periodic KdV equation. (For more details, please see [17–20].)

This article is a continuation of a number of previous works that were previously published in the same direction [15,16]. The main aim in the present paper is to treat the question of the well-posedness of (1), where $w_0(x)$ is analytic on the line and can be extended as holomorphic functions in a strip around the $x-$axis. The most suitable analytic function spaces in this case are the analytic Gevrey spaces $G^{\theta,s}(\mathbb{R})$ introduced in [21], defined as

$$G^{\theta,s}(\mathbb{R}) = \Big\{ f \in L^2(\mathbb{R}); \|f\|_{G^{\theta,s}}(\mathbb{R}) < \infty \Big\}, \tag{8}$$

where

$$\|f\|^2_{G^{\theta,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\theta|\xi|} \langle \zeta \rangle^{2s} |\widehat{f}(\zeta)|^2 d\zeta,$$

for $s \in \mathbb{R}$, $\theta \geq 0$ and $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$, if $\theta = 0$, the space $G^{\theta,s}$ coincides with the standard Sobolev space H^s.

For all $0 < \theta' < \theta$ and $s, s' \in \mathbb{R}$, we have

$$G^{\theta,s}(\mathbb{R}) \subset G^{\theta',s'}(\mathbb{R}),\ i.e.,\ \|f\|_{G^{\theta',s'}(\mathbb{R})} \leq c_{s,s',\theta,\theta'} \|f\|_{G^{\theta,s}(\mathbb{R})}, \tag{9}$$

which is the embedding property of the Gevrey spaces.

Proposition 1. *(Paley–Wiener Theorem, [22]) Let $\theta > 0$, $s \in \mathbb{R}$. Then, $f \in G^{\theta,s}$ if and only if it is the restriction to the real line of a function F, which is holomorphic in the strip $\{x + iy : x, y \in \mathbb{R}, |y| < \theta\}$ and satisfies*

$$\sup_{|y|<\theta} \|F(x+iy)\|_{H^s_x} < \infty.$$

In view of the Paley–Wiener theorem, it is natural to take initial data in $G^{\theta,s}$ and obtain a better understanding of the behavior of solutions as we try to extend it globally in time. This means that, given $w_0 \in G^{\theta,s}$ for some initial radius $\theta > 0$, we want to estimate the behavior of the radius of analyticity $\theta(T)$ over time.

The next theorem is related to the local well-posedness.

Theorem 1. *Let $\theta > 0$ and $s > -1/6$. Then, for any $w_0 \in G^{\theta,s}$, there exists $T = T(\| w_0 \|_{G^{\theta,s}}) > 0$ and a unique solution w of (1) on $(-T, T)$ such that*

$$w \in C\Big([-T,T], G^{\theta,s}\Big).$$

which depends on w_0, where

$$T = \frac{c_0}{(1+ \| w_0 \|^3_{G^{\theta,s}})^{\beta}}, \tag{10}$$

for some constants $c_0 > 0$ and $\beta > 1$ depending only on s. Furthermore, the solution w satisfies

$$\|w\|_{X^T_{\theta,s,b}} \leq 2C\|w_0\|_{G^{\theta,s+p(b-1/2)}} \quad 1/2 < b < 1, \tag{11}$$

$X^T_{\theta,s,b}$ will be introduced below, with constant $C > 0$ depending only on s and b.

The second result for problem (1) is given in the next theorem.

Theorem 2. *Let $s > -1/6$ and $\theta_0 > 0$. Assume that $w_0 \in G^{\theta_0,s}$; then, the solutions in Theorem 1 can be extended to be global in time, and, for any $T' > 0$, we have*

$$w \in C\Big([-T',T'], G^{\theta(T'),s}\Big) \text{ with } \theta(T') = \min\Big\{\theta_0, C_1 T'^{(6-\sigma_0)}\Big\},$$

where $\sigma_0 > 0$ can be taken as arbitrarily small and $C_1 > 0$ is a constant depending on w_0, θ_0, s and σ_0. (Here, "T'" has nothing to do with the time derivative.)

The novelty is due to the embedding property (9) of analytic Gevrey spaces and the lemmas from [23] to obtain the existence in the most suitable analytic function spaces—in this case, in the analytic Gevrey spaces. It will be very interesting to consider the generalization for $\theta > 0$, where the space will be larger than in the present paper and the one considered [23,24]. This subject is an open problem and will be the focus of our next work.

In Section 2, we define the function spaces, linear estimates and bilinear estimates. In Section 3, we prove Theorem 1, using the bilinear estimate and the linear estimate, together with the contraction mapping principle. In Section 4, we prove the existence of a fundamental approximate conservation law. In the concluding Section 5, Theorem 2 will be proven using the approximate conservation law.

2. Function Spaces and Preliminary Estimates

2.1. Function Spaces

Now, we introduce the analytic Gevrey–Bourgain spaces associated with the KdV equation. First, we consider the linear Kdv equation:

$$\begin{cases} v_t + v_{xxx} = 0, \quad x,t \in \mathbb{R} \\ v(x,0) = v_0. \end{cases} \tag{12}$$

The solutions to (12) are given by $v(x,t) = [W(t)v_0](x)$, where

$$\widehat{W(t)v_0}(\zeta) = e^{it\xi^3}\widehat{v}_0(\zeta). \tag{13}$$

The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X_{\theta,s,b}(\mathbb{R}^2) = X_{\theta,s,b}$, with respect to the

$$\begin{aligned} \|v\|_{X_{\theta,s,b}(\mathbb{R}^2)} &= \|Av\|_{X_{s,b}} \equiv \|W(-t)Av\|_{H^{s,b}} = \|\langle\rho\rangle^b\langle\zeta\rangle^s\widehat{W(-t)Av}(\zeta,\rho)\|_{L_{\zeta,\rho}} \\ &= \|e^{\theta|\zeta|}\langle\rho-\zeta^3\rangle^b\langle\zeta\rangle^s\widehat{v}(\zeta,\rho)\|_{L_{\zeta,\rho}} \\ &= \left(\int_{\mathbb{R}^2} e^{2\theta|\zeta|}\left\langle\rho-\zeta^3\right\rangle^{2b}\langle\zeta\rangle^{2s}\mid\widehat{v}(\zeta,\rho)\mid^2 d\zeta d\rho\right)^{\frac{1}{2}}, \end{aligned} \tag{14}$$

where the operator A, defined by

$$\widehat{Av}^x(\zeta,t) = e^{\theta|\zeta|}\widehat{v}^x(\zeta,t). \tag{15}$$

$\widehat{Av}^x$ is the Fourier transform in the spatial variable.

For $\theta = 0$, the spaces $X_{0,s,b}$ coincide with the Bourgain spaces $X_{s,b}$.
The spaces $X^T_{\theta,s,b}$ denote the restriction of $X_{\theta,s,b}$ onto finite time interval $[-T,T]$, $T > 0$ and equipped with the norm

$$\|v\|_{X^T_{\theta,s,b}} = \inf\Big\{\| V \|_{X_{\theta,s,b}(\mathbb{R}^2)} \colon V \in X_{\theta,s,b}, v(t) = V(t) for -T \le t \le T\Big\}.$$

Now, we consider the IVP associated with the linear parts of (1):

$$\begin{cases} v_t + v_{xxx} + \eta\mathcal{L}v = 0, & x,t \in \mathbb{R} \\ v(0) = v_0. \end{cases} \tag{16}$$

The solutions to (16) are given by $v(x,t) = [S(t)v_0](x)$, where

$$\widehat{S(t)v_0}(\zeta) = e^{it\zeta^3+\eta|t|\phi(\zeta)}\widehat{v_0}(\zeta), \tag{17}$$

the semigroup $S(t)$ can be written as $S(t) = W(t)U(t)$, where $\widehat{U(t)u_0}(\zeta) = e^{\eta|t|\phi(\zeta)}u_0(\zeta)$, and $W(t)$ is the unitary group associated with the KdV equation.

2.2. Linear Estimates

To prove our main results, we need some multi-linear estimate in the analytic Gevrey–Bourgain spaces. Note that the spaces $X_{\theta,s,b}$ are continuously embedded in $C\big(\mathbb{R}, G^{\theta,s}(\mathbb{R})\big)$, provided $b > 1/2$.

The proofs of the next lemmas, for $\theta = 0$, are developed in [16], for $\theta > 0$, using operator A in (15).

Lemma 1. *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$ and $\theta \ge 0$. Then, $X_{\theta,s,b} \subset C\big(\mathbb{R}, G^{\theta,s}(\mathbb{R})\big)$ and*

$$\sup_{t\in\mathbb{R}} \|w(t)\|_{G^{\theta,s}} \le C\|w\|_{X_{\theta,s,b}}, \tag{18}$$

where C depends only on b.

Lemma 2. *Let $s \in \mathbb{R}$, $\theta \ge 0$ and $-\frac{1}{2} < b_1 \le b_1' < 0$. Then, $\forall T > 0$, and we have*

$$\|w\|_{X^T_{\theta,s,b_1}} \le CT^{b_1'-b_1}\|w\|_{X^T_{\theta,s,b_1'}}, \tag{19}$$

in (19), and the constant C depends only on b and b'.

Lemma 3. *Let $s \in \mathbb{R}$, $\theta \geq 0$, $-\frac{1}{2} < b < \frac{1}{2}$ and $T > 0$. Then, for any time interval $I \subset [-T, T]$, we have*

$$\|\chi_I(t)w\|_{X_{\theta,s,b}} \leq C\|w\|_{X^T_{\theta,s,b}}, \tag{20}$$

where $\chi_I(t)$ is the characteristic function of I, and C depends only on b.

Next, consider the linear Cauchy problem (1), for given $G(x,t)$ and $w_0(x)$,

$$\begin{cases} \partial_t w + \partial_x^3 w + \eta \mathcal{L} w = G, \quad k = 2, 4 \\ w(x, 0) = w_0(x). \end{cases} \tag{21}$$

By Duhamel's principle, the solution can be then written as

$$w(t) = S(t)w_0 - \int_0^t S(t-\mu)\Big(\partial_x w^4(\mu)\Big)d\mu. \tag{22}$$

Lemma 4. *Let $s \in \mathbb{R}$, $b > 1/2$, $-1/2 < b' \leq 0$, $\theta \geq 0$, $p = 2m + n$ and $0 < T \leq 1$; then, there is a constant $C > 0$, such that*

$$\|S(t)w_0\|_{X_{\theta,s,b}} \leq C\|w_0\|_{G^{\theta,s+p(b-1/2)}}. \tag{23}$$

If $1/2 < b \leq b'/3 + 2/3$, then

$$\Big\| \int_0^t S(t-\mu)F(\mu)\mathrm{d}\mu \Big\|_{X_{\theta,s,b}} \leq C\|F\|_{X_{\theta,s,b'}}. \tag{24}$$

Proof. The proof is similar to that in [23] when using the operator A. □

2.3. Multi-Linear Estimate

Corollary 1 ([25])**.** *Let $\theta \geq 0$ and $s > -1/6$. $\exists \gamma \in (1/2, 1), r(s) > 0$ and if b, b' are two numbers such that $1/2 < b \leq b'_1 + 1 < \gamma$ and $b'_1 + 1/2 \leq r(s)$, then, for $w \in X_{\theta,s,b}$, we have*

$$\| \partial_x(w^4) \|_{X_{\theta,s,b'_1}} \leq C \| w \|^4_{X_{\theta,s,b}}, \tag{25}$$

3. Proof of Theorem 1

Now, we are ready to estimate all terms in (22) by using the multi-linear estimate in the above lemmas. Let $\theta > 0$, $s - p(b - 1/2) > -1/6$, and $w_0 \in G^{\theta,s}$, with $b = 1/2 + \epsilon$, $b'_1 = -1/2 + 4\epsilon$ and $0 < \epsilon \ll 1$, satisfying

$$0 < \epsilon < \min\left\{\frac{s + 1/6}{p}, \frac{1}{4}(\gamma - \frac{1}{2}), \frac{r(s)}{4}\right\}, \tag{26}$$

where $p = 2m + n'$ γ and $r(s)$ are as in Corollary 1.

3.1. Existence of Solution

We construct local solution w of (1); to this end, we proceed by an iteration argument in the space $X^T_{\theta,s,b}$.

Let $\{w^{(n)}\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} \partial_t w^{(0)} + \partial_x^3 w^{(0)} + \eta \mathcal{L} w^{(0)} = 0, \\ w^{(0)}(0) = w_0, \end{cases} \tag{27}$$

and for $n \in \{1, 2, \cdots\}$, we have

$$\begin{cases} \partial_t w^{(n)} + \partial_x^3 w^{(n)} + \eta \mathcal{L} w^{(n)} = -(\partial_x w^{(n-1)})^4, \\ w^{(n)}(0) = w_0. \end{cases} \tag{28}$$

Based on Lemma 4, we have

$$w^{(0)}(x,t) = S(t)w_0(x),$$

and

$$w^{(n)}(x,t) = S(t)w_0(x) - \int_0^t S(t-\mu)\partial_x \left(w^{(n-1)}(x,\mu)\right)^4 d\mu.$$

Then, from Lemma 4, Lemma 2 and Corollary 1, we have

$$\begin{aligned} \|w^{(0)}\|_{X^T_{\theta,s-p(b-1/2),b}} &\leq C\|w_0\|_{G^{\theta,s}} \\ \|w^{(n)}\|_{X^T_{\theta,s-p(b-1/2),b}} &\leq C\|w_0\|_{G^{\theta,s}} + CT^{b_1'-b_1}\|\partial_x \left(w^{(n-1)}\right)^4\|_{X^T_{\theta,s-p(b-1/2),b_1'}} \\ &\leq C\|w_0\|_{G^{\theta,s}} + CT^{b_1'-b_1}\|w^{(n-1)})\|^4_{X^T_{\theta,s-p(b-1/2),b_1'}}, \end{aligned} \tag{29}$$

with $1/2 < b \leq b_1' + 1 < \gamma$ and $b_1' + 1/2 \leq r(s)$. Then, by induction, we have

$$\|w^{(n)}\|_{X^T_{\theta,s-p(b-1/2),b}} \leq 2C\|w_0\|_{G^{\theta,s}}, \forall n \in \mathbb{N}. \tag{30}$$

The constant $T \in (0,1]$ will be chosen to be so small that

$$T \leq \frac{1}{(2^9 C^4 \|w_0\|^3_{G^{\theta,s}})^{\frac{1}{b_1'-b_1}}}. \tag{31}$$

Using Corollary 1 together with (30) and (29), we obtain

$$\begin{aligned} \|w^{(n)} - w^{(n-1)}\|_{X^T_{\theta,s-p(b-1/2),b}} &\leq CT^{b'-b}\left\|\partial_x\left[\left(w^{(n-1)}\right)^4 - \left(w^{(n-2)}\right)^4\right]\right\|_{X^T_{\theta,s-p(b-1/2),b'}} \\ &\leq CT^{b'-b}\|f\|_{X^T_{\theta,s-p(b-1/2),b}}\|w^{(n-1)} - w^{(n-2)})\|_{X^T_{\theta,s,b}} \\ &\leq \frac{1}{2}\|w^{(n-1)} - w^{(n-2)})\|_{X^T_{\theta,s-p(b-1/2),b}}, \end{aligned}$$

where

$$f = \left(w^{(n-1)}\right)^3 + \left(w^{(n-2)}\right)^3 + \left(w^{(n-1)}\right)\left(w^{(n-2)}\right)^2 + \left(w^{(n-2)}\right)\left(w^{(n-1)}\right)^2.$$

3.2. Continuous Dependence on the Initial Data

Suppose that w and v are two solutions to problem (1) with w_0, v_0, respectively. Then, with T and for any T' such that $0 < T' < T$, we obtain

$$\| w - v \|_{X^{T'}_{\theta,s-p(b-1/2),b}} \leq C \| w_0 - v_0 \|_{G^{\theta,s}} + \frac{1}{2} \| w - v \|_{X^{T'}_{\theta,s-p(b-1/2),b}}, \tag{32}$$

provided that $\| w_0 - v_0 \|_{G^{\theta,s}}$ is sufficiently small. This ends the proof of continuous dependence.

4. Approximate Conservation Law

We will now show an approximate conservation law for a solution to our problem (1) owing to the conservation of the $L^2(\mathbb{R})$ norm of the solution in Theorem 3.

Theorem 3. *Let $\kappa \in [0, 1/6)$ and T be as in Theorem 1; there exist $C > 0, b \in (1/2, 1)$ such that $\forall \theta > 0$ and, for any solution $w \in X^T_{\theta,0,b}$ to the problem (1) on the time interval $[0, T]$,*

$$\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\theta,0}} \leq \|w(0)\|^2_{G^{\theta,0}} + C\theta^{\kappa}\|w(0)\|^5_{G^{\theta,p(b-1/2)}}. \tag{33}$$

We need the following estimate.

Lemma 5 ([25]). *Given $\kappa \in [0, 1/6)$, there exist $C > 0, b \in (1/2, 1)$ such that $\forall T > 0$ and $u \in X_{\theta,0,b}$; we have*

$$\|G\|_{X_{0,b-1}} \leq C\theta^{\kappa}\|w\|^4_{X_{\theta,0,b}}, \tag{34}$$

where $G = \partial_x\left[(Aw)^4 - A(w)^4\right]$ and the operator A given by (15).

Proof. (Of Theorem 3) Let $V(t, x) = Aw(t, x)$, which is real-valued since the multiplier A is even and u is real-valued. Applying A to (1), we obtain

$$\partial_t V + \partial_x^3 V + \eta\mathcal{L}V + 4V^3\partial_x V = G, \tag{35}$$

where

$$G = \partial_x\left[(Aw)^4 - A(w)^4\right],$$

then

$$\int_{\mathbb{R}} V\partial_t V dx + \int_{\mathbb{R}} V\partial_x^3 V dx + \eta\int_{\mathbb{R}} V\mathcal{L}V dx + k\int_{\mathbb{R}} V^k\partial_x V dx = \int_{\mathbb{R}} VG dx.$$

Noting that $\partial_x^j V(x, t)$ goes to 0 as $|x|$ goes to ∞ (see [16]), by using the integration by parts, we obtain

$$\frac{1}{2}\partial_t\int_{\mathbb{R}} V^2 dx = \int_{\mathbb{R}} VG dx. \tag{36}$$

Integrating (36) with respect to $t \in [0, T]$, we obtain

$$\int_{\mathbb{R}} V^2(T, x)dx = \int_{\mathbb{R}} V^2(0, x)dx + 2\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG dx dt.$$

Thus,

$$\|w(T)\|^2_{G^{\theta,0}} = \|w(0)\|^2_{G^{\theta,0}} + 2\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG dx dt.$$

By using Holder's inequality, Lemma 3, Lemma 5 and since $1 - b < b, b > 1/2$, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG dx dt\right| &\leq \|\chi_{[0,T]}(t)V\|_{X_{0,1-b}}\|\chi_{[0,T]}(t)G\|_{X_{0,b-1}} \\
&\leq \|V\|_{X^T_{0,1-b}}\|G\|_{X^T_{0,b-1}} \\
&\leq CT^{\kappa}\|w\|^5_{X^T_{\theta,0,b}}.
\end{aligned}$$

□

5. Proof of Theorem 2

Let $\theta_0 > 0$, $s > a_k$, $\kappa \in (0, -a_k)$ be fixed and $w_0 \in G^{\theta_0,s}$. Since the invariance property of the KdV-type equation under the reflection $(t,x) \to (-t,-x)$, we can restrict it to $t > 0$. Then, we need to prove that the solution w of (1) satisfies

$$w \in C\Big([0,T'], G^{\theta(T'),s}\Big), \tag{37}$$

where

$$\theta(T') = \min\Big\{\theta_0, C_1 T'^{-1/\kappa}\Big\}, \quad for\ all\ T' > 0,$$

and $C_1 > 0$ is a constant depending on w_0, θ_0, s and κ. By Theorem 1, there is a maximal time $T^* = T^*(w_0, \theta_0, s) \in (0, \infty]$ such that

$$w \in C\Big([0,T^*], G^{\theta_0,s}\Big).$$

If $T^* = \infty$, it is done.

If $T^* < \infty$, as we assume henceforth, it remains to prove

$$w \in C\Big([0,T'], G^{C_1 T'^{-1/\kappa},s}\Big), \quad for\ all\ T' \geq T^*. \tag{38}$$

5.1. The Case $s = 0$

Fix $T' \geq T^*$. We will show that, if $\theta > 0$ and is sufficiently small

$$\sup_{t\in[0,T']} \|w(0)\|^2_{G^{\theta,0}} \leq 2\|w(0)\|^2_{G^{\theta_0,0}}. \tag{39}$$

In this case, by Theorem 1 and Theorem 3 with

$$T = \frac{c_0}{(1 + 2 \parallel w(0) \parallel^3_{G^{\theta_0,0}})^\beta}, \tag{40}$$

the smallness conditions on θ will be

$$\theta < \theta_0 \ \ and \ \ \frac{2T'}{T} C\theta^\kappa 2^{\frac{5}{2}} \parallel w(0) \parallel^3_{G^{\theta_0,0}} \leq 1, C > 0. \tag{41}$$

Here, C is the constant in Theorem 3. By induction, we check that

$$\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\theta,0}} \leq \|w(0)\|^2_{G^{\theta,0}} + nC\theta^\kappa 2^{\frac{5}{2}} \|w(0)\|^5_{G^{\theta_0,0}}, \tag{42}$$

and

$$\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\theta,0}} \leq 2\|w(0)\|^2_{G^{\theta_0,0}}, \tag{43}$$

for $n \in \{1, \ldots, m+1\}$, where $m \in \mathbb{N}$ is chosen, so that $T' \in [mT, (m+1)T)$, this m exists. By Theorem 1 and the definition of T^*, we have

$$T < \frac{c_0}{(1 + \parallel w(0) \parallel^3_{G^{\theta_0,0}})^\beta} < T^*, \ \ hence \ \ T < T'.$$

In the first step, we cover the interval $[0, T]$, and by Theorem 3, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\theta,0}} &\leq \|w(0)\|^2_{G^{\theta,0}} + C\theta^\kappa \|w(0)\|^5_{G^{\theta,p(b-1/2)}} \\ &\leq \|w(0)\|^2_{G^{\theta,0}} + C\theta^\kappa \|w(0)\|^5_{G^{\theta_1,0}} \\ &\leq \|w(0)\|^2_{G^{\theta,0}} + C\theta^\kappa \|w(0)\|^5_{G^{\theta_0,0}},\end{aligned}$$

since $\theta < \theta_1 \leq \theta_0$, we used

$$\|w(0)\|_{G^{\theta,0}} \leq \|w(0)\|_{G^{\theta_0,0}},$$

and

$$\|w(0)\|_{G^{\theta,p(b(1/2-\epsilon)}} \leq C\|w(0)\|_{G^{\theta_1,0}} \leq C\|w(0)\|_{G^{\theta_0,0}}.$$

This verifies (42) for $n = 1$ and, now, (43) follows using again $\|w(0)\|_{G^{\theta,0}} \leq \|w(0)\|_{G^{\theta_0,0}}$ as well as $C\theta^\kappa \|w(0)\|^3_{G^{\theta_0,0}} \leq 1$.

Suppose now that (42) and (43) hold for some $n \in \{1, \ldots, m\}$ and we show that it holds for $n + 1$.

We have to estimate

$$\begin{aligned}\sup_{t\in[nT,(n+1)T]} \|w(t)\|^2_{G^{\theta,0}} &\leq \|w(nT)\|^2_{G^{\theta,0}} + C\theta^\kappa \|w(nT)\|^5_{G^{\theta,p(b-1/2)}} \\ &\leq \|w(nT)\|^2_{G^{\theta,0}} + C\theta^\kappa 2^{\frac{5}{2}} \|w(0)\|^5_{G^{\theta_1,0}} \\ &\leq \|w(nT)\|^2_{G^{\theta,0}} + C\theta^\kappa 2^{\frac{5}{2}} \|w(0)\|^5_{G^{\theta_0,0}} \\ &\leq \|w(0)\|^2_{G^{\theta,0}} + nC\theta^\kappa 2^{\frac{5}{2}} \|w(0)\|^5_{G^{\theta_0,0}} + C\theta^\kappa 2^{\frac{5}{2}} \|w(0)\|^5_{G^{\theta_0,0}},\end{aligned}$$

verifying (42) with n replaced by $n + 1$. To obtain (43) with n replaced by $n + 1$, it is then enough to have

$$(n + 1)C\theta^\kappa 2^{\frac{5}{2}} \|w(0)\|^3_{G^{\theta_0,0}} \leq 1,$$

but this holds by (41), since $n + 1 \leq m + 1 \leq \frac{T'}{T} + 1 < \frac{2T'}{T}$.

Finally, we easily conclude that condition (41) is satisfied for $\theta \in (0, \theta_0)$ such that

$$\frac{2T'}{T} C\theta^\kappa 2^{\frac{5}{2}} \parallel w(0) \parallel^3_{G^{\theta_0,0}} = 1.$$

Thus, $\theta = C_1 T'^{-\frac{1}{\kappa}}$, where

$$C_1 = \left(\frac{c_0}{C2^{\frac{7}{2}} \parallel w(0) \parallel^3_{G^{\theta_0,0}} (1 + 2 \parallel w(0) \parallel^3_{G^{\theta_0,0}})^\beta} \right)^{\frac{1}{\kappa}}.$$

5.2. The General Case

For all s, by (9), we have $w_0 \in G^{\theta_0,s} \subset G^{\theta_0/2,0}$.

For case $s = 0$, it is proven that there is a $T_1 > 0$, such that

$$w \in C\Big([0, T_1), G^{\theta_0/2,0}\Big),$$

and

$$w \in C\Big([0, T'], G^{2\sigma T'^{-1/\kappa},0}\Big), \ \ for\ T' \geq T_1,$$

where $\sigma > 0$ depends on w_0, θ_0 and κ. Using again the embedding (9), we can conclude that

$$w \in C\Big([0, T_1), G^{\theta_0/4,s}\Big),$$

and

$$w \in C\Big([0, T'], G^{\sigma T'^{-1/\kappa},s}\Big), \;\; for\; T' \geq T_1,$$

which imply (38). The proof of Theorem 2 is now completed.

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