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Research Article

Boundedness and Asymptotic Behavior to a Chemotaxis System with Indirect Signal Generation and Singular Sensitivity

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In this paper, we consider the following indirect signal generation and singular sensitivity $\begin{cases} n_t = \Delta n + \chi\nabla\cdot(n/\varphi(c)\nabla c), & x\in\Omega, t>0, \\ c_t = \Delta c - c + w, & x\in\Omega, t>0, \\ w_t = \Delta w - w + n, & x\in\Omega, t>0, \end{cases}$ in a bounded domain $\Omega\subset R^N (N=2,3)$ with smooth boundary $\partial\Omega$. Under the nonflux boundary conditions for n, c, and w, we first eliminate the singularity of $\varphi(c)$ by using the Neumann heat semigroup and then establish the global boundedness and rates of convergence for solution.

1. Introduction

One of the first mathematical models of chemotaxis was introduced by Keller and Segel [1] to describe the aggregation of certain types of bacteria. In mathematics, it is described as a fully parabolic system

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\chi(n,c)\nabla c), & x\in\Omega, t>0, \\ c_t = \Delta c - c + n, & x\in\Omega, t>0. \end{cases} \tag{1}$$

Here, the unknowns $n = n(t,x)$ and $c(t,x)$ denote the cell density and chemical concentration, respectively. The given function $\chi(n,c)$ is the chemotactic sensitivity. The physical domain $\Omega\subset\mathbb{R}^N (N=2,3)$ is a bounded domain with smooth boundary. This model describes a biological process in which cells move towards their preferred environment and a signal being produced by the cells themselves. When the diffusion of chemical signals is much faster than that of cells, the system can be simplified as

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\chi(n,c)\nabla c), & x\in\Omega, t>0, \\ 0 = \Delta c - c + n, & x\in\Omega, t>0. \end{cases} \tag{2}$$

Another important chemotaxis model is formed with singular sensitivity function, such as $\chi(n,c) = \chi/c$. This model is proposed by the Weber-Fechner law of stimulus perception [2] and supported by experimental [3] and theoretical evidence [4]. The articles about singular sensitive function can be referred to reference [5–9].

Considering the proliferation and death of cells, many scholars have done corresponding research on the above model to add the logistic source. We refer the reader to the survey [10–15] and the references therein. There are also some models involving nonlinear diffusion and rotation terms, which can be referred to [16–19].

It is also important to consider the indirect signal model because the attractive signal and repulsive signal exist simultaneously in some Keller-Segel models. Lin-Mu-Wang established the global existence and large-time behavior in [20].

The blow-up solution was studied by Fujie and Senba in [21]. Tao and Wang [22] considered the global solvability, boundedness, blow-up, existence of nontrivial stationary solutions, and asymptotic behavior. Stinner et al. [23] have given the global existence and some basic boundedness of weak solutions for a PDE-ODE system

Considering the singular sensitivity function, we study the following singular chemotaxis model of indirect signal generation

$$\begin{cases} n_t = \Delta n + \chi\nabla\cdot\left(\dfrac{n}{\varphi(c)}\nabla c\right), & x\in\Omega, t>0,\\ c_t = \Delta c - c + w, & x\in\Omega, t>0,\\ w_t = \Delta w - w + n, & x\in\Omega, t>0,\end{cases} \tag{3}$$

where the parameter χ is a positive constant and φ is a known function. On the other hand, the case of$\Omega\subset\mathbb{R}^N (N=2,3)$is a bounded domain, under the assumption of the no-flux Neumann boundary condition for n,c and w, i.e.,

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x\in\partial\Omega, t>0, \tag{4}$$

where ν is the unit outward normal vector on $\partial\Omega$ and of the initial conditions

$$n(x,0)=n_0(x), c(x,0)=c_0(x), w(x,0)=w_0(x), \quad x\in\Omega \tag{5}$$

satisfy

$$\begin{cases} 0\le n_0(x)\in C^0(\bar\Omega) \text{ and } n_0(x)\not\equiv 0, x\in\bar\Omega,\\ c_0(x)\in W^{1,\infty}(\Omega) \text{ is nonnegative and } \inf_{x\in\Omega} c_0(x)>0,\\ w_0\in W^{1,\infty}(\Omega) \text{ is nonnegative},\\ \varphi(x)\in C^1(0,+\infty), \varphi'(x)>0, x\in(0,+\infty) \text{ and } \lim_{x\longrightarrow 0^+}\varphi(x)=0.\end{cases} \tag{6}$$

There are some sensitivity functions φ satisfying the fourth conditions of (6). For example, $\varphi(x)=x^\alpha, \alpha>0$ or-
$\varphi(x)=\log(1+x), \varphi(x)=\arctan x, \varphi(x)=x^\alpha \log(1+x), \varphi(x)=\int_0^x \tau^\alpha \log(1+\tau)d\tau$, and so on are all satisfied with conditions of (6).

Under these assumptions, we give the well-posedness and asymptotic behavior results as follows.

Theorem 1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that n_0, c_0, w_0, φ satisfy (6). Then, for any $q>1$, systems (3)–(4) possess a global classical solution (n,c,w) which enjoys the regularity properties:*

$$\begin{cases} n\,\epsilon C^0(\bar\Omega\times\lfloor 0,\infty))\cap c^{2,1}(\bar\Omega\times(0,\infty)),\\ c\,\epsilon C^0(\bar\Omega\times\lfloor 0,\infty))\cap c^{2,1}(\bar\Omega\times(0,\infty))\cap L^\infty((0,\infty);W^{1,q}(\Omega)),\\ w\,\epsilon C^0(\bar\Omega\times\lfloor 0,\infty))\cap c^{2,1}(\bar\Omega\times(0,\infty))\cap L^\infty((0,\infty);W^{1,q}(\Omega)).\end{cases} \tag{7}$$

Moreover, this solution is uniformly bounded in the sense that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,q}(\Omega)} + \|w(\cdot,t)\|_{W^{1,q}(\Omega)} \le C, \quad \text{for all } t\in(0,\infty), \tag{8}$$

with some positive constant C.

Theorem 2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that (6) holds. Then, there exists $\epsilon_0>0$ such that if m satisfies*

$$m<\epsilon \tag{9}$$

for some $0<\epsilon<\epsilon_0$, the solution of (3) has the following decay estimates:

$$\begin{aligned} &\left\|n(\cdot,t)-\frac{m}{|\Omega|}\right\|_{L^\infty(\Omega)} \longrightarrow 0,\\ &\left\|c(\cdot,t)-\frac{m}{|\Omega|}\right\|_{L^\infty(\Omega)} \longrightarrow 0,\\ &\left\|w(\cdot,t)-\frac{m}{|\Omega|}\right\|_{L^\infty(\Omega)} \longrightarrow 0,\end{aligned} \tag{10}$$

where $m:=\|n_0(\cdot)\|_{L^1(\Omega)}$ and $|\Omega|$ is Lebesgue measure.

2. Preliminaries and Bounded Estimates

We first establish the local existence result; then the global existence of the solutions is obtained by using a priori estimate.

Lemma 1. *For $N \in \{2, 3\}$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that n_0, c_0, w_0, φ satisfy (6). Then, there exist $T_{\max} \in (0,\infty]$ and a classical solution (n, c, w) of (3)–(4) in $\Omega \times (0, T_{\max})$ such that*

$$\begin{cases} n \in C^0(\bar{\Omega} \times \lfloor 0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ c \in C^0(\bar{\Omega} \times \lfloor 0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty((0,\infty); \mathrm{W}^{1,q}(\Omega)), \\ w \in C^0(\bar{\Omega} \times \lfloor 0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty((0,\infty); \mathrm{W}^{1,q}(\Omega)), \end{cases}$$
$$T_{\max} = \infty \text{ or } \lim_{t \longrightarrow T_{\max}} \left(\|n(.,t)\| L^\infty(\Omega) + \|c(\cdot, t)\|_{W^{1,q}(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} \right) = \infty. \tag{11}$$

Proof. Let $c_* = (1/e) \inf_{x \in \Omega} c_0(x) > 0$. With adaptations of the methods akin to those used in [24] and ([25], Thm. 2.3 i) to deal with the singular sensitivity, $R > 0$ and $T \in (0, 1)$ to be specified below, in Banach's space

$$X := \mathrm{L}^\infty((\mathrm{O}, T); C^0(\Omega) \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega)), \quad \text{for all } q > 0, \tag{12}$$

we consider the closed set

$$\mathrm{S} := \Big\{ (n, c, w) \in X \mid \|n\|_{L^\infty(\Omega)} + \|c\|_{W^{1,q}(\Omega)} + \|w\|_{w^{1,q}(\Omega)} \le R, \text{for } a.e. t \in (0, T)^\circ \Big\} \tag{13}$$

and introduce a mapping $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ on S by defining

$$\begin{aligned} \Phi_1(n, c, w) &:= e^{t\Delta} n_0 - \chi \int_{t_0}^t e^{(t-s)\Delta} \nabla \bullet \left(\frac{n}{\varphi(c)} \nabla c \right) ds, \\ \Phi_2(n, c, w) &:= e^{t(\Delta-1)} c_0 + \int_{t_0}^t e^{(t-s)(\Delta-1)} w(\cdot, s) ds, \\ \Phi_3(n, c, w) := \Phi_2(n, c, w) &:= e^{t(\Delta-1)} w_0 + \int_{t_0}^t e^{(t-s)(\Delta-1)} n(\cdot, s) ds, \end{aligned} \tag{14}$$

for $(n, c, w) \in S$ and $t \in (0, T)$. Using the reasoning (see [26], Lemma 1) based on Banach's fixed point theorem applied in a closed bounded set in $L^\infty((0, T); C^0(\bar{\Omega}) \times W^{1,q}(\Omega) \times W^{1,q}(\Omega))$ for suitably small $T > 0$, the following regularity arguments, proving this local existence and uniqueness result. □

In order to get time-independent pointwise lower bounds of w and c, we need to use the L^1-conservation of n. The purpose of this method is to eliminate the singularity of the function $1/\varphi(C)$ at zero.

Lemma 2. *For any $t \in (0, T_{\max})$, there exist $C > 0$, $\eta > 0$, and $m > 0$ such that*

$$\|n(\cdot, t)\|_{L^1(\Omega)} = \|n_0(\cdot)\|_{L^1(\Omega)}, \tag{15}$$

$$\min \quad \{w(.,t), c.,t)\} \ge \eta. \tag{16}$$

Moreover, we have

$$\|w(\cdot, t)\|_{L^1(\Omega)} \le m + \|w_0(x)\|_{L^1(\Omega)} \cdot e^{-t}, \tag{17}$$

$$\|c(\cdot, t)\|_{L^1(\Omega)} \le m + \|c_0(x)\|_{L^1(\Omega)} \cdot e^{-t}. \tag{18}$$

Proof. Integrate the first equation of (3) to obtain (15).

Using the representation formula of Neumann heat semigroup and point lower bound estimation in [27], we have

$$\begin{aligned} w(\cdot, t) &= e^{t(\Delta-1)} u_0 + \int_0^t e^{(t-s)(\Delta-1)} n(\cdot, s) ds \\ &\ge \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} e^{-\left((t-s) + \left((\mathrm{diam}\Omega)^2/(4(t-s))\right)\right)} \\ &\quad \cdot \|n(\cdot, s)\|_{L^1(\Omega)} ds = m \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \\ &\quad \cdot e^{-\left((t-s) + \left((\mathrm{diam}\Omega)^2/(4(t-s))\right)\right)} ds \\ &= m \int_0^{t_0} \frac{1}{(4\pi\tau)^{n/2}} e^{-\left(\tau + \left((\mathrm{diam}\Omega)^2/4\tau\right)\right)} := \eta_1 > 0, \end{aligned} \tag{19}$$

where η_1 is a positive constant and $\mathrm{diam}\Omega := \max_{x,y \in \Omega} |x - y|$. In the same way, we see that

$$
\begin{aligned}
c(\cdot,t) &= e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}w(\cdot,s)ds \\
&\geq \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} e^{-\left((t-s)+\left((\operatorname{diam}\Omega)^2/(4(t-s))\right)\right)} \\
&\quad \cdot \|w(\cdot,s)\|_{L^1(\Omega)} ds \geq \eta_1|\Omega| \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \\
&\quad \cdot e^{-\left((t-s)+\left((\operatorname{diam}\Omega)^2/(4(t-s))\right)\right)} ds \\
&= \eta_1|\Omega| \int_0^{t_0} \frac{1}{(4\pi\tau)^{n/2}} e^{-\left(\tau+\left((\operatorname{diam}\Omega)^2/4\tau\right)\right)} d\tau := \eta_2 > 0,
\end{aligned} \tag{20}
$$

where η_2 is a positive constant. Taking $\eta = \min\{\eta_1, \eta_2\} > 0$, we get (16).

We integrate the third equation of (3) to obtain

$$
\frac{d}{dt}\int_\Omega w(x,t)dx + \int_\Omega w(x,t)dx = \int_\Omega n(x,t)dx = m. \tag{21}
$$

Applying Lemma 3.4 in [23], we obtain (17). In a similar way, we can get (18). □

Lemma 3. *Let*

$$
\bar{p} = \begin{cases} +\infty, & N = 2, \\ 3, & N = 3. \end{cases} \tag{22}
$$

For any $p \in (0, \bar{p})$, there exists constant C such that

$$
\|w(\cdot,t)\|_{L^p(\Omega)} \leq C, \quad \text{for all } t \in (0, T_{\max}). \tag{23}
$$

Moreover, if $T_{\max} = \infty$, then,

$$
\|w(\cdot,t)\|_{L^p(\Omega)} \leq Cm, \quad \text{as } t \longrightarrow \infty. \tag{24}
$$

Proof. We represent w according to

$$
w(\cdot,t) = e^{t(\Delta-1)}u_0 + \int_0^t e^{(t-s)(\Delta-1)}n(\cdot,s)ds, \quad \text{for all } 0 < t < T_{\max}. \tag{25}
$$

Using the properties of fractional powers $(-\Delta+1)^\theta$ with a dense domain $D((-\Delta+1)\theta)$, $\theta \in (0,1)$ in [28], we see from $N/2(1-(1/p)) < 1$ that

$$
\begin{aligned}
\|w(\cdot,t)\|_{L^p(\Omega)} &\leq C_1 \left\|(-\Delta+1)^\theta w(\cdot,t)\right\|_{L^p(\Omega)} \\
&\leq C_1 \left\|(-\Delta+1)^\theta e^{t(\Delta-1)}w_0\right\|_{L^p(\Omega)} \\
&\quad + C_1 \int_0^t \left\|(-\Delta+1)^\theta e^{(t-s)(\Delta-1)}n(\cdot,s)\right\|_{L^p(\Omega)} ds \\
&\leq C_2 t^{-\theta} e^{-\lambda_1 t}\|w_0\|_{L^p(\Omega)} + C_2 \\
&\quad \cdot \int_0^{+\infty} (t-s)^{-\theta-(N/2)(1-1/p)} e^{-\lambda_1(t-s)}\|n\|_{L^1(\Omega)} ds \\
&\leq C_3 \left(t^{-\theta}e^{-\lambda_1 t} + \|n_0\|_{L^1(\Omega)}\right),
\end{aligned} \tag{26}
$$

where $\lambda_1 \in (0,1)$ and C_1, C_2, $C_3 > 0$ are constants. If $T_{\max} = \infty$, we can take the time t large enough such that $\|w(\cdot,t)\|_{L^p(\Omega)} \leq Cm$. □

Lemma 4. *For any $q \in (0, +\infty)$, there exists constant C such that*

$$
\|c\|_{W^{1,q}(\Omega)} \leq C, \quad \text{for all } t \in (0, T_{\max}). \tag{27}
$$

Moreover, if $T_{\max} = \infty$, then

$$
\|c\|_{W^{1,q}(\Omega)} \leq Cm, \quad \text{as } t \longrightarrow \infty. \tag{28}
$$

Proof. By applying the representation formula, we have

$$
c(\cdot,t) = e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}w(\cdot,s)ds, \quad t > 0. \tag{29}
$$

We apply $(-\Delta+1)^\theta$ to both sides of equation (29) to obtain

$$
\begin{aligned}
\|c(\cdot,t)\|_{L^q(\Omega)} &\leq C_1 \left\|(-\Delta+1)^\theta c(\cdot,t)\right\|_{L^q(\Omega)} \\
&\leq C_1 \left\|(-\Delta+1)^\theta e^{t(\Delta-1)}c_0\right\|_{L^q(\Omega)} \\
&\leq C_1 \int_0^t \left\|(-\Delta+1)^\theta e^{(t-s)(\Delta-1)}w(\cdot,s)\right\|_{L^q(\Omega)} ds \\
&\leq C_2 t^{-\theta} e^{-\lambda_1 t}\|(c_0)\|_{L^q(\Omega)} \\
&\leq C_2 \int_0^{+\infty} (t-s)^{-\theta-(N/2)(1/p-1/q)} e^{-\lambda_1(t-s)}\|w\|_{L^p(\Omega)} ds \\
&\leq C_3 \left(t^{-\theta}e^{-\lambda_1 t} + \|w\|_{L^p(\Omega)}\right).
\end{aligned} \tag{30}
$$

Then by

$$\begin{aligned}\|\nabla c(\cdot,t)\|_{L^q(\Omega)} &\le C_1\left\|\nabla e^{t(\Delta+1)}c_0\right\|_{L^q(\Omega)} \\ &\quad + C_1\int_0^t\left\|\nabla e^{(t-s)(\Delta+1)}w(\cdot,s)\right\|_{L^q(\Omega)}ds \\ &\le C_1\left(1+t^{-1/2}e^{-\lambda_1 t}\|c_0\|_{L^q(\Omega)}\right) \\ &\quad + C_2\int_0^{+\infty}\left(1+(t-s)^{-1/2-(N/2)(1/p-1/q)}\right) \\ &\quad \cdot e^{-\lambda_1(t-s)}\|w\|_{L^p(\Omega)}ds \\ &\le C_3\left(\left(1+t^{-1/2}\right)e^{-\lambda_1 t}+\|w\|_{L^q(\Omega)}\right).\end{aligned} \tag{31}$$

If $T_{\max}=\infty$, taking the time t large enough and by virtue of Lemma 3, we can complete the proof. □ □

Lemma 5. *For any $r>1$, there exists constant C such that*

$$\begin{aligned}\|n\|_{L^r(\Omega)} &\le C\Big(m^{2(q-N)/Nq(r-1)+4q-2N}\Big(1 \\ &\quad + m^{N[q(r-1)+2]/2[Nq(r-1)+4q-2N]} \\ &\quad + m^{2Nq(r-1)+4q/2[Nq(r-1)+4q-2N]}\Big) \\ &\quad + \|n(\cdot,t_0)\|_{L^r(\Omega)}e^{-1/r(t-t_0)}\Big), \quad \textit{for all } t\ge t_0,\end{aligned} \tag{32}$$

with some fixed $t_0>0$.

Proof. Multiplying n^{r-1} by the first equation of (3) and integration by parts, using Hölder's inequality and Young inequality, we have that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n^r dx+\frac{4(r-1)}{r}\int_\Omega\left|\nabla n^{r/2}\right|^2dx \\ &\quad = \chi r(r-1)\int_\Omega\frac{n^{r-1}}{\varphi(c)}\nabla n\cdot\nabla c dx \\ &\quad \le 2\chi(r-1)\frac{1}{\varphi(\eta)}\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}\left\|n^{r/2}\nabla c\right\|_{L^2(\Omega)} \\ &\quad \le \frac{2(r-1)}{r}\int_\Omega\left|\nabla n^{r/2}\right|^2dx+\frac{\chi^2 r(r-1)}{2\varphi^2(\eta)}\int_\Omega\left|n^{r/2}\nabla c\right|^2dx.\end{aligned} \tag{33}$$

That is,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n^r dx+\frac{2(r-1)}{r}\int_\Omega\left|\nabla n^{r/2}\right|^2dx \\ &\quad \le \frac{\chi^2 r(r-1)}{2\varphi^2(\eta)}\int_\Omega\left|n^{r/2}\nabla c\right|^2dx.\end{aligned} \tag{34}$$

To handle the right-hand side of (34), we use Hölder's inequality and Gagliardo-Nirenberg inequality to get

$$\begin{aligned}\left\|n^{r/2}\nabla c\right\|_{L^2(\Omega)} &\le \left\|n^{r/2}\right\|_{L^{2q/q-2}(\Omega)}\|\nabla c\|_{L^q(\Omega)} \\ &\le \left(C_{\mathrm{GN}}\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^{2N+Nq(r-1)/2q+Nq(r-1)}\right. \\ &\quad \cdot\left.\left\|n^{r/2}\right\|_{L^2(\Omega)}^{2(r-1)/2q+Nq(r-1)}+C_{\mathrm{GN}}\left\|n^{r/2}\right\|_{L^{2/r}(\Omega)}\right) \\ &\quad \cdot\|\nabla c\|_{L^q(\Omega)}\le C_{\mathrm{GN}}\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^{2N+Nq(r-1)/2q+Nq(r-1)} \\ &\quad \cdot\|n\|_{L^1(\Omega)}^{r(q-N)/2q+Nq(r-1)}\|\nabla c\|_{L^q(\Omega)} \\ &\quad + C_{\mathrm{GN}}\|n\|_{L^1(\Omega)}^{r/2}\|\nabla c\|_{L^q(\Omega)} \\ &= C_{\mathrm{GN}}m^{r(q-N)/2q+Nq(r-1)}\|\nabla c\|_{L^q(\Omega)} \\ &\quad \cdot\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^{2N+Nq(r-1)/2q+Nq(r-1)} \\ &\quad + C_{\mathrm{GN}}m^{r/2}\|\nabla c\|_{L^q(\Omega)},\end{aligned} \tag{35}$$

where $C_{\mathrm{GN}}>0$ is constant and $q>n$.

Similarly, using the Gagliardo-Nirenberg inequality, there is $C_{\mathrm{GN}}>0$ such that

$$\begin{aligned}\|n\|_{L^2(\Omega)}^r &= \left\|n^{r/2}\right\|_{L^2(\Omega)}^2\le C_{\mathrm{GN}}\left\|n^{r/2}\right\|_{L^1(\Omega)}^{2r/N(r-1)+2} \\ &\quad \cdot\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^{2N(r-1)/N(r-1)+2}+C_{\mathrm{GN}}\|n\|_{L^1(\Omega)}^r \\ &= C_{\mathrm{GN}}\|n\|_{L^1(\Omega)}^{2r/N(r-1)+2}\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^{2N(r-1)/N(r-1)+2} \\ &\quad + C_{\mathrm{GN}}\|n\|_{L^1(\Omega)}^r = C_{\mathrm{GN}}\Big(m^{2r/N(r-1)+2} \\ &\quad \cdot\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^{2N(r-1)/N(r-1)+2}+m^r\Big).\end{aligned} \tag{36}$$

From (35) and (36), we obtain $C_4>0$ such that

$$\begin{aligned}\left\|n^{r/2}\nabla c\right\|_{L^2(\Omega)}^2 &\le \frac{2\varphi^2(\eta)}{\chi^2 r(r-1)}\left(\frac{r-1}{2r}\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^2\right. \\ &\quad + C_4 m^{2r(q-N)/Nq(r-1)+4q-2N} \\ &\quad \cdot\left.\left(1+m^{rN[q(r-1)+2]/2[Nq(r-1)+4q-2N]}\right)\right),\end{aligned} \tag{37}$$

$$\|n\|_{L^r(\Omega)}^r\le\frac{r-1}{2r}\left\|\nabla n^{r/2}\right\|_{L^2(\Omega)}^2\le C_4 m^r. \tag{38}$$

We now substitute (37)–(38) into (34) to obtain that

$$\begin{aligned}&\frac{d}{dt}\|n\|_{L^r(\Omega)}^r+\|n\|_{L^r(\Omega)}^r+\frac{r-1}{r}\left\|\nabla n^{r/2}\right\|_{L^r(\Omega)}^2 \\ &\quad \le C_4 m^{2r(q-N)/Nq(r-1)+4q-2N} \\ &\quad \cdot\Big(1+m^{rN[q(r-1)+2]/2[Nq(r-1)+4q-2N]} \\ &\quad + m^{r[2Nq(r-1)+4q]/2[Nq(r-1)+4q-2N]}\Big).\end{aligned} \tag{39}$$

Applying Gronwall's inequality, we see that

$$\begin{aligned}\|n\|^r_{L^r(\Omega)} &\le C_4 m^{2r(q-N)/Nq(r-1)+4q-2N} \\ &\cdot \left(1 + m^{rN[q(r-1)+2]rN[q(r-1)+2]/2[Nq(r-1)+4q-2N]} \right.\\ &\left. + m^{r[2Nq(r-1)+4q]/2[Nq(r-1)+4q-2N]}\right) \\ &+ \|n(\cdot, t_0)\|^r_{L^r(\Omega)} e^{-(t-t_0)}, \quad \text{for all } t \ge t_0,\end{aligned} \tag{40}$$

with some fixed $t_0 > 0$. Due to $\|n\|_{L^r(\Omega)}$ being uniformly bounded, we can obtain (32) immediately. □

Lemma 6. *For any $p \in (0, \infty)$, there exists constant C such that*

$$\|w(\cdot, t)\|_{W^{1,p}(\Omega)} \le C, \quad \text{for all } t \in (0, T_{\max}). \tag{41}$$

Proof. Using the variation-of-constant formula for w again, we obtain

$$w(\cdot, t) = e^{t(\Delta-1)} u_0 + \int_0^t e^{(t-s)(\Delta-1)} n(\cdot, s) ds, \quad \text{for all } 0 < t < T_{\max}. \tag{42}$$

Therefore, the estimate of $\|n\|_{L^r(\Omega)}$ provides us with $C_5 > 0$ and $C_6 > 0$, for any $t \in (0, T_{\max})$ satisfying

$$\begin{aligned}\|w(\cdot, t)\|_{L^p(\Omega)} &\le e^{-t} \left\|e^{t\Delta} w_0\right\|_{L^p(\Omega)} + \int_0^t e^{-(t-s)} \left\|e^{(t-s)\Delta} n(\cdot, s)\right\|_{L^{r_1}(\Omega)} ds \\ &\le C_5 \|w_0\|_{L^\infty(\Omega)} + C_5 \int_0^t e^{-(t-s)} \\ &\quad \cdot \left(1 + (t-s)^{-1/2-N/s(1/r_2-1/r_1)}\right) \|n(\cdot, s)\|_{L^r(\Omega)} ds \\ &\le C_6 + C_6 \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-1/2-N/s(1/r-1/r_1)}\right) ds,\end{aligned} \tag{43}$$

wherein the last integral is finite since $1/2 + N/2((1/r) - (1/r_1)) < (1/2)$. Similarly, we can deduce that

$$\begin{aligned}\|\nabla w(\cdot, t)\|_{L^p(\Omega)} &\le C_5 \left\|\nabla e^{t(\Delta-1)} w_0\right\|_{L^p(\Omega)} + C_5 \int_0^t \left\|\nabla e^{(t-s)(\Delta-1)} n(\cdot, s)\right\|_{L^p(\Omega)} ds \\ &\le C_5 \left(1 + t^{-1/2}\right) e^{-\lambda_1 t} \|w_0\|_{L^p(\Omega)} \\ &\quad + C_6 \int_0^{+\infty} \left(1 + (t-s)^{-1/2-N/2(1/r-1/p)}\right) e^{-\lambda_1 (t-s)} \|n\|_{L^p(\Omega)} ds \\ &\le C_7, \quad \text{for all } t \in (0, T_{\max}),\end{aligned} \tag{44}$$

with some $C_7 > 0$, where we can select some $p > r > 1$ such that $N/2((1/r) - (1/p)) < (1/2)$ Thus, by virtue of (43) and (44), we finish the proof of Lemma 6. □ □

Proof of Theorem 1. In light of the prior estimates obtained in Lemma 2–Lemma 6 and the local existence results obtained in Lemma 1, we can complete the proof of Theorem 1. □

3. Asymptotic Behavior

To simplify notation, we shall abbreviate the deviations from the nonzero homogeneous steady state by the following transformation:

$$\begin{cases} U(x, t) = n(x, t) - \dfrac{m}{|\Omega|}, \\ V(x, t) = c(x, t) - \dfrac{m}{|\Omega|}, \\ W(x, t) = w(x, t) - \dfrac{m}{|\Omega|}, \end{cases} \tag{45}$$

for all $x \in \Omega$ and $t > 0$. Through simple calculation, we see that (U, V, W) satisfies the following initial boundary value problem:

$$\begin{cases} U_t = \Delta U + \nabla \cdot \left(\dfrac{n}{\varphi(c)} \nabla V\right), & x \in \Omega, t > 0, \\ V_t = \Delta V - V + W, & x \in \Omega, t > 0, \\ W_t = \Delta W - W + U, & x \in \Omega, t > 0, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = \dfrac{\partial W}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ U(x, 0) = n_0(x) - \dfrac{m}{|\Omega|}, V(x, 0) = c_0(x) - \dfrac{m}{|\Omega|}, W(x, 0) = u_0(x) - \dfrac{m}{|\Omega|}, & x \in \Omega. \end{cases} \tag{46}$$

In order to prove Theorem 2, we need several lemmas.

Lemma 7. *For any $r > 1, q > N$, there exists constant C such that*

$$\lim_{t\longrightarrow\infty} \|U(\cdot,t)\|_{L^\infty(\Omega)} \le Cm^{1+2(q-N)/Nq(r-1)+4q-2N}. \quad (47)$$

Proof. By using the variation-of-constant representation,

$$U(\cdot,t) = e^{(t-t_2)\Delta}U(\cdot,t_2) - \int_{t_2}^{t} e^{(t-s)\Delta}\nabla\cdot\left(\frac{n(\cdot,s)}{\varphi(c(\cdot,s))}\nabla V(\cdot,s)\right)ds, \quad (48)$$

for all $t > t_2$, we obtain

$$\begin{aligned}\|U(\cdot,t)\|_{L^\infty(\Omega)} &= \left\|e^{(t-t_2)\Delta}U(\cdot,t_2)\right\|_{L^\infty(\Omega)} \\ &\quad + \int_{t_2}^{t}\left\|e^{(t-s)\Delta}\nabla\cdot\left(\frac{n(\cdot,s)}{\varphi(c(\cdot,s))}\nabla V(\cdot,s)\right)\right\|_{L^\infty(\Omega)}ds \\ &:= I_1 + I_2, \quad \text{for all } t > t_2.\end{aligned} \quad (49)$$

For I_1, there is a constant $c_1 > 0$ such that

$$\begin{aligned}\|U(\cdot,t_2)\|_{L^r(\Omega)} &= \left\|n(x,t_2) - \frac{m}{|\Omega|}\right\|_{L^r(\Omega)} \\ &\le \|n(x,t_2)\|_{L^r(\Omega)} + \left\|\frac{m}{|\Omega|}\right\|_{L^r(\Omega)} \le c_1.\end{aligned} \quad (50)$$

Noticing that $\int_\Omega U(\cdot,t)dx = 0$, we have

$$\begin{aligned}I_1 &= \left\|e^{(t-t_2)\Delta}U(\cdot,t_2)\right\|_{L^\infty(\Omega)} \\ &\le c_1\left(1 + (t-t_2)^{-N/2r}\right)e^{-\lambda_1(t-t_2)} \\ &= \|U(\cdot,t_2)\|_{L^r(\Omega)} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.\end{aligned} \quad (51)$$

For I_2, taking $r > r_1 > N, q > N$, using the estimate of Neumann heat semigroup and Hölder's inequality, we obtain

$$\begin{aligned}I_2 &= \int_{t_2}^{t}\left\|e^{(t-s)\Delta}\nabla\cdot\left(\frac{n(\cdot,s)}{\varphi(c)}\nabla V(\cdot,s)\right)\right\|_{L^\infty(\Omega)}ds \\ &\le c_2\int_{t_2}^{t}\left(1 + (t-s)^{-1/2-N/2r_1}\right)e^{-\lambda_1(t-s)}\left\|\frac{n(\cdot,s)}{\varphi(c)}\nabla V(\cdot,s)\right\|_{L^{r_1}(\Omega)}ds \\ &\le c_2\eta\int_{t_2}^{t}\left(1 + (t-s)^{-1/2-N/2r_1}\right)e^{-\lambda_1(t-s)} \\ &\quad \cdot \|n(\cdot,s)\|_{L^r(\Omega)}\|\nabla V(\cdot,s)\|_{rr_1/L^{r-r_1}(\Omega)}ds \\ &\le c_2\eta\int_{t_2}^{t}\left(1 + (t-s)^{-1/2-N/2r_1}\right)e^{-\lambda_1(t-s)} \\ &\quad \cdot \|n(\cdot,s)\|_{L^r(\Omega)}\|c(\cdot,s)\|_{W^1 rr_1/r-r_1(\Omega)}ds \\ &\le c_3 m^{1+2(q-N)/Nq(r-1)+4q-2N},\end{aligned} \quad (52)$$

where $c_2, c_3 > 0$ are constants. We now substitute (51)–(52) into (49) to complete the proof. □ □

Next, we want to extend $\tilde{T}_0$ to infinity. Applying the Lemma 7, we can select $t_3 = t_3(n,c,u) > 0$ to obtain

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le 2c_3 m^{1+2(q-N)/Nq(r-1)+4q-2N}, \quad (53)$$

for some $r > 1, q > N$.

For any $p \in (1,\bar{p})$, one has

$$\begin{aligned}&\|W(\cdot,t)\|_{L^p(\Omega)} \\ &\le \left\|e^{(t-t_2)(\Delta-1)}W(\cdot,t)\right\|_{L^p(\Omega)} + \int_{t_2}^{t}\left\|e^{(t-t_2)(\Delta-1)}U(\cdot,t)\right\|_{L^p(\Omega)}ds \\ &\le c_3(t-t_2)^{-\theta}e^{\lambda_1 t}\|W(\cdot,t_2)\|_{L^p(\Omega)} \\ &\quad + c_3\int_{t_2}^{t}(t-s)^{-\theta-n/2(1-1/p)}e^{-\lambda_1(t-s)}\|U(\cdot,s)\|_{L^1(\Omega)}ds \\ &= c_3(t-t_2)^{-\theta}e^{-\lambda_1 t}\|W(\cdot,t_2)\|_{L^p(\Omega)} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.\end{aligned} \quad (54)$$

By combining Lemma 3 and (45), we see that

$$\|W(\cdot,t)\|_{L^p(\Omega)} \le 2c_3 m, \quad \text{for all } t > t_3. \quad (55)$$

Applying the Lemma 4, we can get

$$\|\nabla V(\cdot,t)\|_{L^p(\Omega)} = \|\nabla c(\cdot,t)\|_{L^p(\Omega)} \le c_3 m, \quad \text{for all } t > t_3. \quad (56)$$

We now choose m small enough such that

$$c_3 m^{2(q-N)/Nq(r-1)+4q-2N} \le \frac{1}{2}. \quad (57)$$

It is easy to see that

$$\|U(\cdot,t_3)\|_{L^\infty(\Omega)} \le \frac{1}{2}\epsilon, \quad \text{for all } t \ge t_3. \quad (58)$$

Let

$$\tilde{T}_0 := \left\{T \ge t_3 \mid \|U(\cdot,t)\|_{L^\infty(\Omega)} \le \epsilon e^{-\lambda 1(t-t_3)}, \text{for all } t \in [t_3, T_0]\right\}, \quad (59)$$

where T_0 is a given positive constant. Then, $\tilde{T}_0$ is well-defined since (49), (51), and (58). In order to extend $\tilde{T}_0$ to infinity, we give the following lemmas.

Lemma 8. *For any $p \in (1, \bar{p})$, there exists a constant $c_4 > 0$ satisfying*

$$\|W(\cdot, t)\|_{L^p(\Omega)} \le 2c_4 \epsilon e^{-\lambda_1 (t-t_3)}, \quad \text{for all } t \in (t_3, T). \tag{60}$$

Proof. We first use (46) to represent W according to

$$W(\cdot, t) = e^{(t-t_3)(\Delta-1)} W(\cdot, t_1) + \int_{t_3}^{t} e^{(t-s)(\Delta-1)} U(\cdot, s) ds \tag{61}$$

and the fact that $\lambda_1 < 1$ and (55) to estimate

$$\begin{aligned}\left\| e^{(t-t_3)(\Delta-1)} W(\cdot, t_3) \right\|_{L^p(\Omega)} &\le e^{-(t-t_3)} \left\| e^{(t-t_3)\Delta} W(\cdot, t_3) \right\|_{L^p(\Omega)} \\ &\le c_4 \epsilon e^{-\lambda_1 (t-t_3)}, \quad \text{for all } t > t_3. \end{aligned} \tag{62}$$

Furthermore, using Hölder's inequality and the definitions of $\dot{T}$ and c_5 entails that

$$\begin{aligned}&\left\| \int_{t_3}^{t} e^{(t-s)(\Delta-1)} U(\cdot, s) ds \right\|_{L^p(\Omega)} \\ &\le c_3 \int_{t_3}^{t} e^{-(t-s)} \left\| e^{(t-s)\Delta} U(\cdot, s) \right\|_{L^p(\Omega)} ds \\ &\le c_3 \int_{t_3}^{t} \left(1 + (t-s)^{-N/2(1-1/p)}\right) e^{-(\lambda_1+1)(t-s)} \|U(\cdot, s)\|_{L^1(\Omega)} ds \\ &\le c_3 |\Omega| \int_{t_3}^{t} \left(1 + (t-s)^{-N/2(1-1/p)}\right) e^{-(\lambda_1+1)(t-s)} \|U(\cdot, s)\|_{L^\infty(\Omega)} ds \\ &\le c_3 \epsilon |\Omega| \int_{t_3}^{t} \left(1 + (t-s)^{-N/2(1-1/p)}\right) e^{-(\lambda_1+1)(t-s)} e^{-\lambda_1 (s-t_3)} ds \\ &\le c_3 \epsilon |\Omega| \int_{0}^{t-t_3} \frac{\left(1 + (t-\sigma-t_3)^{-N/2(1-1/p)}\right)}{e^{-(\lambda_1+1)(t-\sigma-t_3)} e^{-\lambda_1 \sigma}} d\sigma \\ &\le c_3 \epsilon |\Omega| e^{-(\lambda_1+1)(t-t_3)} \int_{0}^{t-t_3} \left(1 + (t-\sigma-t_3)^{-N/2(1-1/p)}\right) e^{\sigma} d\sigma \\ &\le c_4 \epsilon e^{-\lambda_1 (t-t_3)} \quad \text{for all } t \in (t_3, T). \end{aligned} \tag{63}$$

Thus, substituting (62) and (63) into (61), we obtain the Lemma 8. □ □

Lemma 9. *For any $q \in (1, +\infty)$, there exists constant c_5 such that*

$$\|\nabla V(\cdot, t)\|_{L^q(\Omega)} \le c_5 \epsilon e^{-\lambda_1 1 (t-t_3)}, \quad \text{for all } t \in (t_3, T). \tag{64}$$

Proof. By means of the variation-of-constant representation for V, combined with (56) and Lemma 8, we show that

$$\begin{aligned}\|\nabla V(\cdot, t)\|_{L^q(\Omega)} &\le \left\| \nabla e^{(t-t_3)(\Delta-1)} V(\cdot, t_3) \right\|_{L^q(\Omega)} \\ &\quad + \int_{t_3}^{t} \left\| \nabla e^{(t-s)(\Delta-1)} W(\cdot, s) \right\|_{L^q(\Omega)} ds \\ &= e^{-(t-t_3)} \left\| \nabla e^{(t-t_3)\Delta} V(\cdot, t_3) \right\|_{L^q(\Omega)} \\ &\quad + \int_{t_3}^{t} \left\| e^{-(t-s)} \nabla e^{(t-s)\Delta} W(\cdot, s) \right\|_{L^q(\Omega)} ds \\ &\le c_1 e^{-(\lambda_1+1)(t-t_3)} \|\nabla V(\cdot, t_3)\|_{L^q(\Omega)} + c_2 \\ &\quad \cdot \int_{t_3}^{t} \left(1 + (t-s)^{-1/2-N/2(1/p-1/q)}\right) \\ &\quad \cdot e^{-(\lambda_1+1)(t-s)} \|W(\cdot, s)\|_{L^p(\Omega)} ds \\ &\le c_1 c_3 \int e^{-(\lambda_1+1)(t-t_3)} \\ &\quad + c_2 \int_{t_3}^{t} \left(1 + (t-s)^{-1/2-N/2(1/p-1/q)}\right) \\ &\quad \cdot e^{-(\lambda_1+1)(t-s)} 2c_4 \int e^{-\lambda_1 (s-t_3)} ds \\ &\le c_1 c_3 \int e^{-(\lambda_1+1)(t-t_3)} + 2c_2 c_4 \int e^{-\lambda_1 (s-t_3)} c_2 \\ &\quad \cdot \int_{0}^{t-t_3} \left(1 + \sigma^{-1/2-N/2(1/p-1/q)}\right) e^{-\sigma} ds \\ &\le c_5 \int e^{-\lambda_1 (t-t_3)}, \quad \text{for all } t \in (t_3, T), \end{aligned} \tag{65}$$

with some $c_5 > 0$. □ □

Lemma 10. *Let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then, there exists constant c_6 such that*

$$\|U(\cdot, t)\|_{L^\infty(\Omega)} \le c_6 e^{-\lambda_1 (t-t_3)}, \quad \text{for all } t > t_3. \tag{66}$$

Proof. Notice that the fact of U has the following estimate:

$$\|U(\cdot, t)\|_{L^\infty(\Omega)} \le c_6 e^{-\lambda_1 (t-t_3)}, \quad \text{for all } t \in (t_3, T). \tag{67}$$

Furthermore, we can use (45) to obtain

$$\begin{aligned}\|n(\bullet, t)\|_{L^\infty(\Omega)} &= \left\| U(\bullet, t) + \frac{m}{(\Omega)} \right\|_{L^\infty(\Omega)} \le \|U(\bullet, t)\|_{L^\infty(\Omega)} + \frac{m}{(\Omega)} \\ &\le \epsilon \left(e^{-\lambda_1 (t-t_3)} + \frac{m}{(\Omega)} \right). \end{aligned} \tag{68}$$

We next write

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \left\|e^{(t-t_3)\Delta}U(\cdot,t_3)\right\|_{L^\infty(\Omega)} \cdot \int_{t_3}^{t}\left\|e^{(t-S)\Delta}\nabla\left(\frac{n(\cdot,s)}{\varphi(c)}\nabla \mathrm{V}(\cdot,s)\right)\right\|_{L^\infty(\Omega)} ds \tag{69}$$

and employ the estimate (53) to obtain

$$\begin{aligned}\left\|e^{(t-t_3)\Delta}U(\cdot,t_3)\right\|_{L^\infty(\Omega)} &\le c_5 e^{-\lambda_1(t-t_3)}\|U(\cdot,t_3)\|_{L^\infty(\Omega)}\\ &\le 2c_3c_5 m^{1+2(q-N)/Nq(r-1)+4q-2N}e^{-\lambda_1(t-t_3)}\\ &\le 2c_3c_5\int^{1+2(q-N)/Nq(r-1)+4q-2N}e^{-\lambda_1(t-t_3)}.\end{aligned} \tag{70}$$

We next recall (18) and (45) and employ the estimates (64) and (68) to see that

$$\begin{aligned}&\int_{t_3}^{t}\left\|e^{(t-s)\Delta}\nabla\cdot\left(\frac{n(\cdot,s)}{\varphi(c)}\nabla V(\cdot,s)\right)\right\|_{L^\infty(\Omega)} ds\\ &\le c_5\int_{t_3}^{t}\left(1+(t-s)^{-1/2-N/2r}\right)e^{-\lambda_1(t-s)}\left\|\frac{n(\cdot,s)}{\varphi(c)}\nabla V(\cdot,s)\right\|_{L^r(\Omega)} ds\\ &\le \frac{c_5}{\varphi\eta}\int_{t_3}^{t}\left(1+(t-s)^{-1/2-N/2r}\right)e^{-\lambda_1(t-s)}\\ &\quad\cdot\|n(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla V(\cdot,s)\|_{L^r(\Omega)}ds\\ &\le \frac{c_5}{\varphi\eta}\int_{t_3}^{t}\left(1+(t-s)^{-1/2-N/2r}\right)e^{-\lambda_1(t-s)}\\ &\quad\cdot\int\left(e^{-\lambda_1(s-t_3)}+\frac{1}{|\Omega|}\right)c_5\int e^{-\lambda_1(s-t_3)}ds\\ &\le \frac{c_5^2\int^2}{\varphi(\eta)}e^{-\lambda_1(t-t_3)}\int_{t_3}^{t}\left(1+(t-s)^{-1/2-N/2r}\right)\left(e^{-\lambda_1(s-t_3)}+\frac{1}{|\Omega|}\right)ds\\ &\le \frac{c_5^2\int^2}{\varphi(\eta)}e^{-2\lambda_1(t-t_3)}\int_{0}^{t-t_3}\left(1+\sigma^{-1/2-N/2r}\right)\left(e^{\lambda_1\sigma}+\frac{1}{|\Omega|}\right)d\sigma\\ &\le \frac{c_5^2c_7\int^2}{\varphi(\eta)}e^{-\lambda_1(t-t_3)},\end{aligned} \tag{71}$$

for all $r > N$ and $c_7 >$ is a constant.

Thus, substituting (70) and (71) into (69), we have

$$\|U(\bullet,t)\|_{L^\infty(\Omega)} \le \frac{1}{2}c_8\epsilon^{1+2(q-N)/Nq(r-1)+(4q-2N)}e^{-\lambda_1(t-t_3)}, \quad \text{for all } t\in(t_3,T), \tag{72}$$

where c_8 is a positive constant. Then, we select $\epsilon_0 > 0$ as sufficiently small to fulfilling

$$c_8\epsilon^{1+2(q-N)/Nq(r-1)+(4q-2N)} \le \frac{1}{2}. \tag{73}$$

In conjunction with (57) and (73), this yields

$$\|U(\bullet,t)\|_{L^\infty(\Omega)} \le \frac{1}{2}\epsilon e^{-\lambda_1(t-t_3)}, \quad \text{for all } t\in(t_3,T). \tag{74}$$

By the continuity of U, we can extend $\tilde{T}_0=\infty$. So, we complete the proof. □ □

Lemma 11. *Let $\lambda_1\in(0,1)$. Then, there is constant $c_9>0$ satisfying*

$$\begin{aligned}\left\|c(\cdot,t)-\frac{m}{|\Omega|}\right\|_{L^\infty(\Omega)} &\le c_9e^{-\lambda_1/2t},\\ \left\|w(\cdot,t)-\frac{m}{|\Omega|}\right\|_{L^\infty(\Omega)} &\le c_9e^{-\lambda_1/2t}.\end{aligned} \tag{75}$$

for all $t>0$.

Proof. Let $(x,t) := c(x,t)-(m/|\Omega|)$. From the second equation of (3), we can get the following system:

$$\begin{cases}\psi_t-\Delta\psi+\psi=u-\dfrac{m}{|\Omega|}, & x\in\Omega, t>0,\\ \dfrac{\partial\psi}{\partial\nu}=0, & x\in\partial\Omega, t>0,\\ \psi(x,0)=c_0(x)-\dfrac{m}{|\Omega|}:=\psi_0(x), & x\in\Omega.\end{cases} \tag{76}$$

Let ψ^* be the solution of the following initial value problem:

$$\begin{cases}\psi_t^*+\psi^*=c_{10}e^{-\lambda_1 t}, & t>0,\\ \psi^*(0)=\|\psi^*\|_{L^\infty(\Omega)}.\end{cases} \tag{77}$$

Using the comparison principle in [29], we see that $\psi^*(t)$ is a supersolution of the system (76), and thus,

$$\psi(x,t)\le\psi^*(t), \quad \text{for all } x\in\Omega, t>0. \tag{78}$$

Similarly, we have $\psi(x,t)\ge-\psi^*(t)$ for all $x\in\Omega, t>0$. Hence, we furthermore obtain that

$$|\psi(x,t)|\le\psi^*(t), \quad \text{for all } x\in\Omega, t>0. \tag{79}$$

On the other side, direct computation shows that there are some constants c_{11} and c_{12} such that

$$0\le\psi^*(t)\le c_{11}\left(1+\|\psi^*\|_{L^\infty(\Omega)}\right)e^{-\lambda_1 t}c_{12}e^{-\lambda_1/2t}, \quad \text{for all } t>0. \tag{80}$$

Thus, we can deduce that

$$\left\|c(\cdot,t)-\frac{m}{(\Omega)}\right\|_{L^\infty(\Omega)}=\|\psi(\cdot,t)\|_{L^\infty(\Omega)} \le \psi^*(t)c_{12}e^{-\lambda_1/2t},\quad \text{for all } t>0. \tag{81}$$

In a similar way, we can get the convergence of w. Thus, we complete the proof. □ □

Proof of Theorem 1. Using the estimates of Lemma 10 and Lemma 11, we obtain the decay estimates of n, c, and w. Hence, the proof is completed.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no competing interests.

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