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A Study on the Sum of the Squares of Generalized p-Oresme Numbers: The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^2$ for generalized p-Oresme numbers are presented. As special cases, we give sum formulas of Modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery.

Keywords: Oresme numbers; Modified p-Oresme numbers; p-Oresme-Lucas numbers; p-Oresme numbers; sum formulas.

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1 INTRODUCTION

p-Oresme numbers (see, for example, [1]) are defined by the recurence relation

$$O_{n+2} = O_{n+1} - \frac{1}{p^2}O_n, \quad O_0 = 0, O_1 = \frac{1}{p}, \;\; p \neq 0.$$

The case $p = 2$, which is called the Oresme sequence, $\{O_n\}_{n\geq 0}$, was introduced by Nicole Oresme (1320–1382) in the 14-th century. Oresme obtained the sum of the rational numbers formed by the terms $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, ..., \frac{n}{2^n}$. These numbers form a second order sequence and are defined by the recurence relation

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}.$$

In [2], Horadam presented a history and obtained an abundance of properties of these numbers. Oresme numbers have many interesting properties and applications in many fields of science (see, for example, [3,4,1,2,5,6]).

For $0 \neq p \in \mathbb{R}$, a generalized p-Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} - \frac{1}{p^2}W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = p^2 W_{-(n-1)} - p^2 W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Modified p-Oresme sequence $\{G_n\}_{n\geq 0}$, p-Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and p-Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = G_{n+1} - \frac{1}{p^2}G_n, \quad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = H_{n+1} - \frac{1}{p^2}H_n, \quad H_0 = 2, H_1 = 1, \tag{1.3}$$

$$O_{n+2} = O_{n+1} - \frac{1}{p^2}O_n, \quad O_0 = 0, O_1 = \frac{1}{p}. \tag{1.4}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= p^2 G_{-(n-1)} - p^2 G_{-(n-2)}, \\ H_{-n} &= p^2 H_{-(n-1)} - p^2 H_{-(n-2)}, \\ O_{-n} &= p^2 O_{-(n-1)} - p^2 O_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.4) hold for all integer n. For more information on generalized Oresme numbers, see Soykan [7].

In the next section, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^2$ for generalized Oresme numbers are presented. Sum of the squares of linear recurrence relations have been studied by many authors and more detail can be found in the extensive literature, see for example, [8,9,10,11,12,13,14,15].

2 THE SUM FORMULA $\sum_{k=0}^{n} x^k W_{mk+j}^2$

The following theorem presents sum formulas of generalized p-Oresme numbers (the case $r=1, s=-\frac{1}{p^2}$).

Theorem 2.1. *Suppose that $p^2 \neq 4$ so that $p^2-4 \neq 0$. Let x be a real (or complex) number. For all integers m and j, for generalized p-Oresme numbers (the case $r=1, s=-\frac{1}{p^2}$) we have the following sum formulas:*

(a) *If $(1+\frac{1}{p^{4m}}x^2 - xH_{2m})(\frac{1}{p^{2m}}x-1) = (x^2 - p^{4m}H_{2m}x + p^{4m})\left(x-p^{2m}\right)p^{-6m} \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{(p-2)(p+2)(x-p^{2m})(x^2-p^{4m}H_{2m}x+p^{4m})} \tag{2.1}$$

where
$\Omega_1 = (x-p^{4m}H_{2m})(x-p^{2m})(p-2)(p+2)x^{n+1}W_{mn+j}^2 + (x-p^{2m})(p-2)(p+2)x^{n+1}W_{mn-m+j}^2 + \left(x-p^{2m}\right)(p-2)\,p^{4m}(p+2)W_j^2 + (p^{2m}-x)(p-2)(p+2)xW_{j-m}^2 + 2p^{-2mn+4m-2j+2}(H_{2m}p^{2m} - 2)(x^n - p^{2mn})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x$

(b) *If $(x^2 - p^{4m}H_{2m}x + p^{4m})\left(x-p^{2m}\right) = (x-a)(x-b)(x-c) = 0$ for some*

$$a = \frac{p^{4m}H_{2m} + \sqrt{p^{8m}H_{2m}^2 - 4p^{4m}}}{2}, b = \frac{p^{4m}H_{2m} - \sqrt{p^{8m}H_{2m}^2 - 4p^{4m}}}{2}, c = p^{2m} \in \mathbb{C}$$

and $a \neq b \neq c$, i.e., $x=a$ or $x=b$ or $x=c$, then,

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_2}{(p-2)(p+2)\left(3x^2 - 2p^{2m}\left(p^{2m}H_{2m}+1\right)x + p^{4m} + p^{6m}H_{2m}\right)}$$

where
$\Omega_2 = (p^2-4)((x-p^{4m}H_{2m})x^{n+1} + \left(x-p^{2m}\right)((n+2)\,x - p^{4m}\,(n+1)\,H_{2m})x^n)W_{mn+j}^2 + (p^2-4)((n+2)x - p^{2m}(n+1))x^nW_{mn-m+j}^2 + p^{4m}(p^2-4)W_j^2 - (p^2-4)(2x-p^{2m})W_{j-m}^2 + 2p^{-2mn+4m-2j+2}(p^{2m}H_{2m}-2)(x^n + nx^n - p^{2mn})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)$

(c) *If $p^{8m}H_{2m}^2 - 4p^{4m} = 0$ and $(x^2 - p^{4m}xH_{2m} + p^{4m})\left(x-p^{2m}\right) = (x-a)^2(x-c) = 0$ for some $a = \dfrac{p^{4m}H_{2m}}{2}, c = p^{2m} \in \mathbb{C}$ and $a \neq c$ then if $x=c$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_3}{(p-2)(p+2)\left(3x^2 - 2p^{2m}\left(p^{2m}H_{2m}+1\right)x + p^{4m} + p^{6m}H_{2m}\right)}$$

where
$\Omega_3 = (p^2-4)((x-p^{4m}H_{2m})x^{n+1} + \left(x-p^{2m}\right)((n+2)\,x - p^{4m}\,(n+1)\,H_{2m})x^n)W_{mn+j}^2 + (p^2-4)((n+2)x - p^{2m}(n+1))x^nW_{mn-m+j}^2 + p^{4m}(p^2-4)W_j^2 - (p^2-4)(2x-p^{2m})W_{j-m}^2 + 2p^{-2mn+4m-2j+2}(p^{2m}H_{2m}-2)(x^n + nx^n - p^{2mn})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)$

and if $x=a$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_4}{2(p-2)(p+2)(3x - p^{2m} - p^{4m}H_{2m})}$$

where
$\Omega_4 = (p^2-4)((n+3)(n+2)x^2 - xp^{2m}(n+2)(n+1)(p^{2m}H_{2m}+1) + p^{6m}n(n+1)H_{2m})x^{n-1}W_{mn+j}^2 + (p^2-4)(n+1)((2+n)x^n - p^{2m}nx^{n-1})W_{mn-m+j}^2 - 2(p^2-4)W_{j-m}^2 + 2n(n+1)p^{-2mn+4m-2j+2}(p^{2m}H_{2m}-2)(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x^{n-1}$

Proof. Take $r = 1, s = -\frac{1}{p^2}$ and $H_n = H_n$ in Soykan [10, Theorem 2.1.]. Note that $(x^2 - p^{4m}xH_{2m} + p^{4m})\left(x - p^{2m}\right) = (x-a)^3 = 0$ can not be possible for some $a = p^{2m} \in \mathbb{C}$, $m \neq 0$, i.e., $x = a$, because otherwise we must have $p^{8m}H_{2m}^2 - 4p^{4m} = 0, \dfrac{p^{4m}H_{2m}}{2} = p^{2m}$ and

$$\begin{aligned}\frac{p^{4m}H_{2m}}{2} - p^{2m} &= \frac{1}{2}p^{2m}\left(p^{2m}H_{2m} - 2\right) = 0 \Rightarrow p^{2m}H_{2m} - 2 = 0\\ &\Rightarrow p^{2m} \times 2\left(\frac{1}{2}\right)^{2m} - 2 = 0\\ &\Rightarrow \left(\frac{p}{2}\right)^{2m} - 1 = 0\end{aligned}$$

in which the last equality doesn't have any integer solutions m other than $m = 0$ (so $x = p^{2m} = 1$) for the case $p \neq 2$.Here we used

$$\begin{aligned}\alpha &= \frac{r}{2},\\ H_{2m} &= 2\alpha^{2m} = 2\left(\frac{1}{2}\right)^{2m}.\end{aligned}$$

In this case (i.e. $m = 0, x = 1$), we have $\sum_{k=0}^{n} W_j^2 = (n+1)\,W_j^2$. □

Note that in the last theorem, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [16, Theorem 2.1].

Note also that (2.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_6}{(p-2)(p+2)(x-p^{2m})(x^2 - p^{4m}H_{2m}x + p^{4m})}$$

where

$\Omega_6 = (p^2-4)(x-p^{2m})(x-p^{4m}H_{2m})x^{n+1}W_{mn+j}^2 + (p^2-4)(x-p^{2m})x^{n+1}W_{mn-m+j}^2 - (p^2-4)(x-p^{2m})(x-p^{4m}H_{2m})xW_j^2 - (p^2-4)(x-p^{2m})W_{j-m}^2x + 2p^{-2mn+4m-2j+2}(x^n - p^{2mn})(p^{2m}H_{2m} - 2)(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x.$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.1. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For generalized p-Oresme numbers (the case $r = 1, s = -\frac{1}{p^2}$) we have the following sum formulas for $n \geq 0$:*

(a) $(m = 1, j = 0)$

If $\left(x - p^2\right)(x^2 - p^2(p^2-2)x + p^4) \neq 0$, *i.e.,*

$$x \neq \frac{p^2(p^2-2) + \sqrt{p^6(p^2-4)}}{2}, x \neq \frac{p^2(p^2-2) - \sqrt{p^6(p^2-4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Phi}{(p^2-4)(x-p^2)(x^2 - p^2(p^2-2)x + p^4)}$$

where

$\Phi = (x - p^2(p^2-2))(x-p^2)(p^2-4)x^{n+1}W_n^2 + (x-p^2)(p^2-4)x^{n+1}W_{n-1}^2 + \left(x-p^2\right)(p^2-4)p^4W_0^2 + p^4(p^2-x)(p^2-4)(W_0-W_1)^2x + 2p^{6-2n}((p^2-2)-2)(x^n - p^{2n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x,$

and

if $\left(x-p^{2}\right)\left(x^{2}-p^{2}(p^{2}-2)x+p^{4}\right)=0$, *i.e.,*

$$x=\frac{p^{2}(p^{2}-2)+\sqrt{p^{6}(p^{2}-4)}}{2} \text{ or } x=\frac{p^{2}(p^{2}-2)-\sqrt{p^{6}(p^{2}-4)}}{2} \text{ or } x=p^{2},$$

then

$$\sum_{k=0}^{n} x^{k}W_{k}^{2}=\frac{\Lambda}{(p^{2}-4)(3x^{2}-2p^{2}(p^{2}-1)x+p^{4}(p^{2}-1))}$$

where
$\Lambda=(p^{2}-4)((x-p^{2}(p^{2}-2))x^{n+1}+\left(x-p^{2}\right)((n+2)x-(n+1)p^{2}\left(p^{2}-2\right))x^{n})W_{n}^{2}+(p^{2}-4)((n+2)x-p^{2}(n+1))x^{n}W_{n-1}^{2}+p^{4}(p^{2}-4)W_{0}^{2}-p^{4}(2x-p^{2})(p^{2}-4)(W_{0}-W_{1})^{2}+2p^{6-2n}((p^{2}-2)-2)(x^{n}+nx^{n}-p^{2n})(W_{1}^{2}+\frac{1}{p^{2}}W_{0}^{2}-W_{1}W_{0}).$

(b) $(m=2,\ j=0)$

If $\left(x-p^{4}\right)\left(x^{2}-p^{4}\left(p^{4}-4p^{2}+2\right)x+p^{8}\right)\neq 0$, *i.e.,*

$$x\neq\frac{p^{4}(p^{4}-4p^{2}+2)+\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2}, x\neq\frac{p^{4}(p^{4}-4p^{2}+2)-\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2}, x\neq p^{4},$$

then

$$\sum_{k=0}^{n} x^{k}W_{2k}^{2}=\frac{\Phi}{(p^{2}-4)(x-p^{4})(x^{2}-p^{4}(p^{4}-4p^{2}+2)x+p^{8})}$$

where
$\Phi=(x-p^{4}(p^{4}-4p^{2}+2))(x-p^{4})(p^{2}-4)x^{n+1}W_{2n}^{2}+(x-p^{4})(p^{2}-4)x^{n+1}W_{2n-2}^{2}+\left(x-p^{4}\right)(p^{2}-4)p^{8}W_{0}^{2}+(p^{4}-x)(p^{2}-4)p^{4}(p^{2}W_{0}-W_{0}-p^{2}W_{1})^{2}x+2p^{12-4n}(p^{2}-4)(x^{n}-p^{4n})(W_{1}^{2}+\frac{1}{p^{2}}W_{0}^{2}-W_{1}W_{0})x,$

and if $\left(x-p^{4}\right)\left(x^{2}-p^{4}\left(p^{4}-4p^{2}+2\right)x+p^{8}\right)=0$ *and* $p^{2}\neq 2$ *i.e.,*

$$x=\frac{p^{4}(p^{4}-4p^{2}+2)+\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2} \text{ or } x=\frac{p^{4}(p^{4}-4p^{2}+2)-\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2} \text{ or } x=p^{4},$$

(and $p^{2}\neq 2$*) then*

$$\sum_{k=0}^{n} x^{k}W_{2k}^{2}=\frac{\Lambda}{(p^{2}-4)(3x^{2}-2p^{4}(p^{2}-1)(p^{2}-3)x+p^{8}(p^{2}-1)(p^{2}-3))}$$

where
$\Lambda=(p^{2}-4)((x-p^{4}(p^{4}-4p^{2}+2))x^{n+1}+\left(x-p^{4}\right)((n+2)x-p^{4}(p^{4}-4p^{2}+2)(n+1))x^{n})W_{2n}^{2}+(p^{2}-4)((n+2)x-p^{4}(n+1))x^{n}W_{2n-2}^{2}+p^{8}(p^{2}-4)W_{0}^{2}-(p^{2}-4)(2x-p^{4})p^{4}(p^{2}W_{0}-W_{0}-p^{2}W_{1})^{2}+2p^{12-4n}(p^{2}-4)(x^{n}+nx^{n}-p^{4n})(W_{1}^{2}+\frac{1}{p^{2}}W_{0}^{2}-W_{1}W_{0}),$

and if $\left(x-p^{4}\right)\left(x^{2}-p^{4}\left(p^{4}-4p^{2}+2\right)x+p^{8}\right)=(x-4)(x+4)^{2}=0,\ x\neq p^{4}=4$ *and* $p^{2}=2$ *i.e.,*

$$x=\frac{p^{4}(p^{4}-4p^{2}+2)+\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2}=\frac{p^{4}(p^{4}-4p^{2}+2)-\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2}=-4,$$

and $x\neq p^{4}=4$ *(and* $p^{2}=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}(-4)^{k}W_{2k}^{2}=\frac{\Theta}{32}$$

where
$\Theta=64(-4)^{n-1}\left(n^{2}-2\right)W_{2n}^{2}+(-4)^{n+1}(n+1)^{2}W_{2n-2}^{2}+16(W_{0}-2W_{1})^{2}-n(n+1)(-4)^{n-1}2^{8-2n}(W_{1}^{2}+\frac{1}{2}W_{0}^{2}-W_{1}W_{0}).$

(c) $(m=2,\ j=1)$

If $\left(x-p^{4}\right)\left(x^{2}-p^{4}\left(p^{4}-4p^{2}+2\right)x+p^{8}\right)\neq 0$, *i.e.,*

$$x\neq\frac{p^{4}(p^{4}-4p^{2}+2)+\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2}, x\neq\frac{p^{4}(p^{4}-4p^{2}+2)-\sqrt{p^{10}(p^{2}-4)(p^{2}-2)^{2}}}{2}, x\neq p^{4},$$

then

$$\sum_{k=0}^{n} x^{k}W_{2k+1}^{2}=\frac{\Phi}{(p^{2}-4)(x-p^{4})(x^{2}-p^{4}(p^{4}-4p^{2}+2)x+p^{8})},$$

where
$\Phi=(x-p^4(p^4-4p^2+2))(x-p^4)(p^2-4)x^{n+1}W_{2n+1}^2+(x-p^4)(p^2-4)x^{n+1}W_{2n-1}^2+\left(x-p^4\right)(p^2-4)p^8W_1^2+(p^4-x)(p^2-4)p^4(W_0-W_1)^2x+2p^{10-4n}(p^2-4)(x^n-p^{4n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0)x,$

and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=0$ *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=p^4,$$

(and $p^2\neq 2$*) then*

$$\sum_{k=0}^{n}x^kW_{2k+1}^2=\frac{\Lambda}{(p^2-4)(3x^2-2p^4(p^2-1)(p^2-3)x+p^8(p^2-1)(p^2-3))},$$

where
$\Lambda=(p^2-4)((x-p^4(p^4-4p^2+2))x^{n+1}+\left(x-p^4\right)((n+2)x-p^4(p^4-4p^2+2)(n+1))x^n)W_{2n+1}^2+(p^2-4)((n+2)x-p^4(n+1))x^nW_{2n-1}^2+p^8(p^2-4)W_1^2-(p^2-4)(2x-p^4)p^4(W_0-W_1)^2+2p^{10-4n}(p^2-4)(x^n+nx^n-p^{4n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0),$

and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=(x-4)(x+4)^2=0,\ x\neq p^4=4$ *and* $p^2=2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=-4,$$

and $x\neq p^4=4$ *(and* $p^2=2$*, i.e.,*$p=-\sqrt{2},\ p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}(-4)^kW_{2k+1}^2=\frac{\Theta}{32},$$

where
$\Theta=64\,(-4)^{n-1}\left(n^2-2\right)W_{2n+1}^2+(-4)^{n+1}\,(n+1)^2\,W_{2n-1}^2+16(W_0-W_1)^2-n(n+1)\,(-4)^{n-1}\,2^{7-2n}(W_1^2+\frac{1}{2}W_0^2-W_1W_0).$

(d) $(m=-1,\ j=0)$

If $(p^2x-1)p^{-2}(x^2-(p^2-2)p^{-2}x+p^{-4})\neq 0,$ *i.e.,*

$$x\neq\frac{(p^2-2)p^{-2}+\sqrt{(p^2-4)p^{-2}}}{2},\ x\neq\frac{(p^2-2)p^{-2}-\sqrt{(p^2-4)p^{-2}}}{2},\ x\neq p^{-2},$$

then

$$\sum_{k=0}^{n}x^kW_{-k}^2=\frac{\Phi}{(p^2-4)(x-p^{-2})(x^2-(p^2-2)p^{-2}x+p^{-4})},$$

where
$\Phi=(x-p^{2m})(p^2-4)x^{n+1}W_{-n+1}^2+(x-(p^2-2)p^{-2})(x-p^{-2})(p^2-4)x^{n+1}W_{-n}^2+\left(x-p^{-2}\right)(p^2-4)p^{-4}W_0^2+(p^{-2}-x)(p^2-4)xW_1^2+2p^{2n-2}(p^2-4)(x^n-p^{-2n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0)x$

and if $(p^2x-1)p^{-2}(x^2-(p^2-2)p^{-2}x+p^{-4})=0,$ *i.e.,*

$$x=\frac{(p^2-2)p^{-2}+\sqrt{(p^2-4)p^{-2}}}{2} \text{ or } x=\frac{(p^2-2)p^{-2}-\sqrt{(p^2-4)p^{-2}}}{2} \text{ or } x=p^{-2},$$

then

$$\sum_{k=0}^{n}x^kW_{-k}^2=\frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)p^{-2}x+(p^2-1)p^{-4})},$$

where
$\Lambda=(p^2-4)((n+2)x-p^{-2}(n+1))x^nW_{-n+1}^2+(p^2-4)(p^4(n+3)x^2-p^2x(p^2-1)(n+2)+(p^2-2)(n+1))p^{-4}x^nW_{-n}^2+p^{-4}(p^2-4)W_0^2-(p^2-4)(2x-p^{-2})W_1^2+2p^{2n-2}(p^2-4)(x^n+nx^n-p^{-2n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0).$

(e) $(m=-2,\ j=0)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})\neq 0,$ *i.e.,*

$$x\neq\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2},\ x\neq\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2},\ x\neq p^{-4},$$

then

$$\sum_{k=0}^{n}x^kW_{-2k}^2=\frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where
$\Phi = (x-p^{-4})(p^2-4)x^{n+1}W^2_{-2n+2} + (x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}W^2_{-2n} + \left(x-p^{-4}\right)(p^2-4)p^{-8}W_0^2 + (p^{-4}-x)(p^2-4)x(-W_0+p^2W_1)^2p^{-4} + 2p^{4n-4}(p^2-4)(x^n-p^{-4n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0)x$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, and $p^2\neq 2$ i.e.,

$$x = \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x = \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2\neq 2$, i.e., $p\neq -\sqrt{2}, p\neq\sqrt{2}$) then

$$\sum_{k=0}^{n} x^k W^2_{-2k} = \frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where
$\Lambda = (p^2-4)((n+2)x-p^{-4}(n+1))x^nW^2_{-2n+2} + (p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1} + \left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)W^2_{-2n} + p^{-8}(p^2-4)W_0^2 - (p^2-4)(2x-p^{-4})(-W_0+p^2W_1)^2p^{-4} + 2p^{4n-4}(p^2-4)(x^n+nx^n-p^{-4n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0)$,

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8}) = \frac{1}{64}(4x-1)(4x+1)^2=0$, $x\neq p^{-4}=\frac{1}{4}$ and $p^2=2$ i.e.,

$$x = \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} = \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} = -\frac{1}{4},$$

and $x\neq p^{-4}=\frac{1}{4}$ (and $p^2=2$, i.e., $p=-\sqrt{2}, p=\sqrt{2}$) then

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k W^2_{-2k} = \frac{\Theta}{2},$$

where
$\Theta = -4\left(-\frac{1}{4}\right)^n(n+1)^2W^2_{-2n+2} + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)W^2_{-2n} + (W_0-2W_1)^2 - n(n+1)2^{2n-1}\left(-\frac{1}{4}\right)^{n-1}(2W_1^2 + W_0^2-2W_1W_0)$.

(f) $(m=-2, j=1)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})\neq 0$, i.e.,

$$x \neq \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq p^{-4},$$

then

$$\sum_{k=0}^{n} x^k W^2_{-2k+1} = \frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where
$\Phi = (x-p^{-4})(p^2-4)x^{n+1}W^2_{-2n+3} + (x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}W^2_{-2n+1} + \left(x-p^{-4}\right)(p^2-4)p^{-8}W_1^2 + (p^{-4}-x)(p^2-4)p^{-4}(-W_0-W_1+p^2W_1)^2x + 2p^{4n-6}(p^2-4)(x^n-p^{-4n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0)x$,

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, and $p^2\neq 2$ i.e.,

$$x = \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x = \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2\neq 2$, i.e., $p\neq -\sqrt{2}, p\neq\sqrt{2}$) then

$$\sum_{k=0}^{n} x^k W^2_{-2k+1} = \frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where
$\Lambda = (p^2-4)((n+2)x-p^{-4}(n+1))x^nW^2_{-2n+3} + (p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1} + \left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)W^2_{-2n+1} + p^{-8}(p^2-4)W_1^2 - (p^2-4)(2x-p^{-4})p^{-4}(-W_0-W_1+p^2W_1)^2 + 2p^{4n-6}(p^2-4)(x^n+nx^n-p^{-4n})(W_1^2+\frac{1}{p^2}W_0^2-W_1W_0)$,

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8}) = \frac{1}{64}(4x-1)(4x+1)^2=0$, $x\neq p^{-4}=\frac{1}{4}$ and $p^2=2$ i.e.,

$$x = \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} = \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k W_{-2k+1}^2 = \frac{\Theta}{2},$$

where
$\Theta = -4\left(-\frac{1}{4}\right)^n (n+1)^2 W_{-2n+3}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right) W_{-2n+1}^2 + (W_0 - W_1)^2 - n(n+1)2^{2n-2}(2W_1^2 + W_0^2 - 2W_1W_0)\left(-\frac{1}{4}\right)^{n-1}.$

Note that in the last proposition, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [16, Proposition 2.1].

From the above proposition, we have the following corollary which gives sum formulas of modified p-Oresme numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1$).

Corollary 2.2. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For $n \geq 0$, modified p-Oresme numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $\left(x - p^2\right)\left(x^2 - p^2(p^2-2)x + p^4\right) \neq 0$, *i.e.,*

$$x \neq \frac{p^2(p^2-2) + \sqrt{p^6(p^2-4)}}{2}, x \neq \frac{p^2(p^2-2) - \sqrt{p^6(p^2-4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^{n} x^k G_k^2 = \frac{\Phi}{(p^2-4)(x-p^2)(x^2-p^2(p^2-2)x+p^4)}$$

where
$\Phi = (x - p^2(p^2-2))(x-p^2)(p^2-4)x^{n+1}G_n^2 + (x-p^2)(p^2-4)x^{n+1}G_{n-1}^2 - p^{-2n+4}(p^2-4)(p^{2n}(x+p^2) - 2p^2x^n)x,$

and

if $\left(x - p^2\right)\left(x^2 - p^2(p^2-2)x + p^4\right) = 0$, *i.e.,*

$$x = \frac{p^2(p^2-2) + \sqrt{p^6(p^2-4)}}{2} \text{ or } x = \frac{p^2(p^2-2) - \sqrt{p^6(p^2-4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^{n} x^k G_k^2 = \frac{\Lambda}{(p^2-4)(3x^2 - 2p^2(p^2-1)x + p^4(p^2-1))}$$

where
$\Lambda = (p^2-4)((x - p^2(p^2-2))x^{n+1} + \left(x-p^2\right)((n+2)x - (n+1)p^2\left(p^2-2\right))x^n)G_n^2 + (p^2-4)((n+2)x - p^2(n+1))x^nG_{n-1}^2 + p^{-2n+4}(p^2-4)(-p^{2n}(2x+p^2) + 2p^2(n+1)x^n).$

(b) $(m = 2, j = 0)$

If $\left(x - p^4\right)\left(x^2 - p^4\left(p^4 - 4p^2 + 2\right)x + p^8\right) \neq 0$, *i.e.,*

$$x \neq \frac{p^4(p^4-4p^2+2) + \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq \frac{p^4(p^4-4p^2+2) - \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^{n} x^k G_{2k}^2 = \frac{\Phi}{(p^2-4)(x-p^4)(x^2-p^4(p^4-4p^2+2)x+p^8)}$$

where
$\Phi = (x - p^4(p^4-4p^2+2))(x-p^4)(p^2-4)x^{n+1}G_{2n}^2 + (x-p^4)(p^2-4)x^{n+1}G_{2n-2}^2 - p^{-4n+8}(p^2-4)(p^{4n}(x+p^4) - 2p^4x^n)x,$

and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=0$ *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=p^4,$$

(and $p^2\neq 2$*) then*

$$\sum_{k=0}^{n}x^kG_{2k}^2=\frac{\Lambda}{(p^2-4)(3x^2-2p^4(p^2-1)(p^2-3)x+p^8(p^2-1)(p^2-3))}$$

where
$\Lambda=(p^2-4)((x-p^4(p^4-4p^2+2))x^{n+1}+\left(x-p^4\right)((n+2)x-p^4(p^4-4p^2+2)(n+1))x^n)G_{2n}^2+(p^2-4)((n+2)x-p^4(n+1))x^nG_{2n-2}^2+p^{-4n+8}(p^2-4)(-p^{4n}(2x+p^4)+2p^4(n+1)x^n)),$
and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=(x-4)(x+4)^2=0$, $x\neq p^4=4$ *and* $p^2=2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=-4,$$

and $x\neq p^4=4$ *(and* $p^2=2$*, i.e.,* $p=-\sqrt{2},p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}(-4)^kG_{2k}^2=\frac{\Theta}{32}$$

where
$\Theta=64\,(-4)^{n-1}\left(n^2-2\right)G_{2n}^2+(-4)^{n+1}(n+1)^2G_{2n-2}^2+16(G_0-2G_1)^2-n(n+1)(-4)^{n-1}2^{8-2n}.$

(c) $(m=2,j=1)$

If $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)\neq 0$, *i.e.,*

$$x\neq\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x\neq\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x\neq p^4,$$

then

$$\sum_{k=0}^{n}x^kG_{2k+1}^2=\frac{\Phi}{(p^2-4)(x-p^4)(x^2-p^4(p^4-4p^2+2)x+p^8)},$$

where
$\Phi=(x-p^4(p^4-4p^2+2))(x-p^4)(p^2-4)x^{n+1}G_{2n+1}^2+(x-p^4)(p^2-4)x^{n+1}G_{2n-1}^2-p^{-4n+4}(p^2-4)(p^{4n}(x^2-2p^4x+2p^6x+p^8)-2p^6x^{n+1}),$
and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=0$ *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=p^4,$$

(and $p^2\neq 2$*) then*

$$\sum_{k=0}^{n}x^kG_{2k+1}^2=\frac{\Lambda}{(p^2-4)(3x^2-2p^4(p^2-1)(p^2-3)x+p^8(p^2-1)(p^2-3))},$$

where
$\Lambda=(p^2-4)((x-p^4(p^4-4p^2+2))x^{n+1}+\left(x-p^4\right)((n+2)x-p^4(p^4-4p^2+2)(n+1))x^n)G_{2n+1}^2+(p^2-4)((n+2)x-p^4(n+1))x^nG_{2n-1}^2+2p^{-4n+4}(p^2-4)(p^{4n}(-x-p^6+p^4)+p^6(n+1)x^n),$
and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=(x-4)(x+4)^2=0$, $x\neq p^4=4$ *and* $p^2=2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=-4,$$

and $x\neq p^4=4$ *(and* $p^2=2$*, i.e.,*$p=-\sqrt{2},p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}(-4)^kG_{2k+1}^2=\frac{\Theta}{32},$$

where
$\Theta=64\,(-4)^{n-1}\left(n^2-2\right)G_{2n+1}^2+(-4)^{n+1}(n+1)^2G_{2n-1}^2+16(G_0-G_1)^2-n(n+1)(-4)^{n-1}2^{7-2n}.$

(d) $(m=-1, j=0)$

If $(p^2x-1)p^{-2}(x^2-(p^2-2)p^{-2}x+p^{-4})\neq 0$, *i.e.,*

$$x\neq \frac{(p^2-2)p^{-2}+\sqrt{(p^2-4)p^{-2}}}{2}, x\neq \frac{(p^2-2)p^{-2}-\sqrt{(p^2-4)p^{-2}}}{2}, x\neq p^{-2},$$

then

$$\sum_{k=0}^{n} x^k G_{-k}^2 = \frac{\Phi}{(p^2-4)(x-p^{-2})(x^2-(p^2-2)p^{-2}x+p^{-4})},$$

where

$\Phi=(x-p^{-2})(p^2-4)x^{n+1}G_{-n+1}^2+(x-(p^2-2)p^{-2})(x-p^{-2})(p^2-4)x^{n+1}G_{-n}^2+(p^2-4)p^{-2}(2p^{2n}x^n-p^2x-1)x$

and if $(p^2x-1)p^{-2}(x^2-(p^2-2)p^{-2}x+p^{-4})=0$, *i.e.,*

$$x=\frac{(p^2-2)p^{-2}+\sqrt{(p^2-4)p^{-2}}}{2} \text{ or } x=\frac{(p^2-2)p^{-2}-\sqrt{(p^2-4)p^{-2}}}{2} \text{ or } x=p^{-2},$$

then

$$\sum_{k=0}^{n} x^k G_{-k}^2 = \frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)p^{-2}x+(p^2-1)p^{-4})},$$

where

$\Lambda=(p^2-4)((n+2)x-p^{-2}(n+1))x^nG_{-n+1}^2+(p^2-4)(p^4(n+3)x^2-p^2x(p^2-1)(n+2)+(p^2-2)(n+1))p^{-4}x^nG_{-n}^2+(p^2-4)p^{-2}(2p^{2n}(n+1)x^n-2p^2x-1)$

(e) $(m=-2, j=0)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})\neq 0$, *i.e.,*

$$x\neq \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq p^{-4},$$

then

$$\sum_{k=0}^{n} x^k G_{-2k}^2 = \frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where

$\Phi=(x-p^{-4})(p^2-4)x^{n+1}G_{-2n+2}^2+(x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}G_{-2n}^2+(p^2-4)p^{-4}(2p^{4n}x^n-p^4x-1)x$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=p^{-4},$$

(and $p^2\neq 2$, *i.e.,* $p\neq -\sqrt{2}, p\neq\sqrt{2}$*) then*

$$\sum_{k=0}^{n} x^k G_{-2k}^2 = \frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where

$\Lambda=(p^2-4)((n+2)x-p^{-4}(n+1))x^nG_{-2n+2}^2+(p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1}+\left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)G_{-2n}^2+(p^2-4)p^{-4}(2p^{4n}(n+1)x^n-2p^4x-1)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=\frac{1}{64}(4x-1)(4x+1)^2=0, x\neq p^{-4}=\frac{1}{4}$ *and* $p^2=2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=-\frac{1}{4},$$

and $x\neq p^{-4}=\frac{1}{4}$ *(and* $p^2=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k G_{-2k}^2=\frac{\Theta}{2},$$

where

$\Theta=-4\left(-\frac{1}{4}\right)^n(n+1)^2G_{-2n+2}^2+\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)G_{-2n}^2+4(n(n+1)2^{2n}\left(-\frac{1}{4}\right)^n+1)$

(f) $(m=-2, j=1)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8}) \neq 0$, *i.e.,*

$$x \neq \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x \neq \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^{n} x^k G_{-2k+1}^2 = \frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where

$\Phi = (x-p^{-4})(p^2-4)x^{n+1}G_{-2n+3}^2+(x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}G_{-2n+1}^2+(p^2-4)p^{-12}(-p^8(p-1)^2(p+1)^2x^2+p^4(p^4-4p^2+2)x+2p^{4n+6}x^{n+1}-1)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, *and* $p^2 \neq 2$ *i.e.,*

$$x = \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x = \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=p^{-4},$$

(and $p^2 \neq 2$, *i.e.,* $p \neq -\sqrt{2}, p \neq \sqrt{2}$*) then*

$$\sum_{k=0}^{n} x^k G_{-2k+1}^2 = \frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where

$\Lambda = (p^2-4)((n+2)x-p^{-4}(n+1))x^nG_{-2n+3}^2+(p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1}+\left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)G_{-2n+1}^2+(p^2-4)p^{-8}(-2p^4x(p-1)^2(p+1)^2+2p^{4n+2}(n+1)x^n+p^4-4p^2+2)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8}) = \frac{1}{64}(4x-1)(4x+1)^2=0, x\neq p^{-4}=\frac{1}{4}$ *and* $p^2=2$ *i.e.,*

$$x = \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} = \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4}=\frac{1}{4}$ *(and* $p^2=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k G_{-2k+1}^2 = \frac{\Theta}{2},$$

where

$\Theta = -4\left(-\frac{1}{4}\right)^n(n+1)^2G_{-2n+3}^2+\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)G_{-2n+1}^2+2n(n+1)2^{2n}\left(-\frac{1}{4}\right)^n+1.$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2-4 \neq 0$ is considered. The sum formulas for the case $p^2=4$ so that $p^2-4=0$ is given in Soykan [16, Corollary 2.2].

Taking $W_n = H_n$ with $H_0=2, H_1=1$ in the last proposition, we have the following corollary which presents sum formulas of p-Oresme-Lucas numbers.

Corollary 2.3. *Suppose that* $p^2 \neq 4$ *so that* $p^2-4 \neq 0$. *For* $n \geq 0$, *p-Oresme-Lucas numbers have the following properties:*

(a) $(m=1, j=0)$

If $\left(x-p^2\right)(x^2-p^2(p^2-2)x+p^4) \neq 0$, *i.e.,*

$$x \neq \frac{p^2(p^2-2)+\sqrt{p^6(p^2-4)}}{2}, x \neq \frac{p^2(p^2-2)-\sqrt{p^6(p^2-4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{\Phi}{(p^2-4)(x-p^2)(x^2-p^2(p^2-2)x+p^4)}$$

where

$\Phi = (x - p^2(p^2-2))(x-p^2)(p^2-4)x^{n+1}H_n^2 + (x-p^2)(p^2-4)x^{n+1}H_{n-1}^2 + p^{-2n+4}(p^2-4)(p^{2n}(p^2(3x-4) - x(x+4)) - 2(p^2-4)x^{n+1})$,

and

if $\left(x-p^2\right)\left(x^2 - p^2(p^2-2)x + p^4\right) = 0$, *i.e.,*

$$x = \frac{p^2(p^2-2) + \sqrt{p^6(p^2-4)}}{2} \text{ or } x = \frac{p^2(p^2-2) - \sqrt{p^6(p^2-4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{\Lambda}{(p^2-4)(3x^2 - 2p^2(p^2-1)x + p^4(p^2-1))}$$

where

$\Lambda = (p^2-4)((x-p^2(p^2-2))x^{n+1} + \left(x-p^2\right)((n+2)x - (n+1)p^2\left(p^2-2\right))x^n)H_n^2 + (p^2-4)((n+2)x - p^2(n+1))x^nH_{n-1}^2 + p^{-2n+4}(p^2-4)(p^{2n}(-2x+3p^2-4) - 2(n+1)(p^2-4)x^n)$.

(b) $(m = 2, j = 0)$

If $\left(x-p^4\right)(x^2 - p^4\left(p^4 - 4p^2 + 2\right)x + p^8) \neq 0$, *i.e.,*

$$x \neq \frac{p^4(p^4-4p^2+2) + \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq \frac{p^4(p^4-4p^2+2) - \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{\Phi}{(p^2-4)(x-p^4)(x^2 - p^4(p^4-4p^2+2)x + p^8)}$$

where

$\Phi = (x - p^4(p^4-4p^2+2))(x-p^4)(p^2-4)x^{n+1}H_{2n}^2 + (x-p^4)(p^2-4)x^{n+1}H_{2n-2}^2 + p^{-4n+4}(p^2-4)(p^{4n}(4p^2x^2 - p^4x^2 - 4x^2 + 8p^4x - 12p^6x + 3p^8x - 4p^8) - 2p^6(p^2-4)x^{n+1})$

and if $\left(x-p^4\right)(x^2 - p^4\left(p^4-4p^2+2\right)x + p^8) = 0$ *and* $p^2 \neq 2$ *i.e.,*

$$x = \frac{p^4(p^4-4p^2+2) + \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x = \frac{p^4(p^4-4p^2+2) - \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$*) then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{\Lambda}{(p^2-4)(3x^2 - 2p^4(p^2-1)(p^2-3)x + p^8(p^2-1)(p^2-3))}$$

where

$\Lambda = (p^2-4)((x-p^4(p^4-4p^2+2))x^{n+1} + \left(x-p^4\right)((n+2)x - p^4(p^4-4p^2+2)(n+1))x^n)H_{2n}^2 + (p^2-4)((n+2)x - p^4(n+1))x^nH_{2n-2}^2 + p^{-4n+4}(p^2-4)(p^{4n}(-8x+8p^2x - 2p^4x + 8p^4 - 12p^6 + 3p^8) - 2p^6(p^2-4)(n+1)x^n)$

and if $\left(x-p^4\right)(x^2 - p^4\left(p^4-4p^2+2\right)x + p^8) = (x-4)(x+4)^2 = 0$, $x \neq p^4 = 4$ *and* $p^2 = 2$ *i.e.,*

$$x = \frac{p^4(p^4-4p^2+2) + \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} = \frac{p^4(p^4-4p^2+2) - \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ *(and* $p^2 = 2$, *i.e.,* $p = -\sqrt{2}, p = \sqrt{2}$*) then*

$$\sum_{k=0}^{n} (-4)^k H_{2k}^2 = \frac{\Theta}{32}$$

where

$\Theta = 64(-4)^{n-1}\left(n^2-2\right)H_{2n}^2 + (-4)^{n+1}(n+1)^2H_{2n-2}^2 - n(n+1)2^{-2n+8}(-4)^{n-1}$

(c) $(m = 2, j = 1)$

If $\left(x-p^4\right)(x^2 - p^4\left(p^4-4p^2+2\right)x + p^8) \neq 0$, *i.e.,*

$$x \neq \frac{p^4(p^4-4p^2+2) + \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq \frac{p^4(p^4-4p^2+2) - \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{\Phi}{(p^2-4)(x-p^4)(x^2 - p^4(p^4-4p^2+2)x + p^8)},$$

where
$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}H_{2n+1}^2 + (x - p^4)(p^2 - 4)x^{n+1}H_{2n-1}^2 - p^{-4n+4}(p^2 - 4)(p^{4n}(x^2 + 6p^4x - 2p^6x + p^8) + 2p^4(p^2 - 4)x^{n+1})$
and if $\left(x - p^4\right)(x^2 - p^4\left(p^4 - 4p^2 + 2\right)x + p^8) = 0$ *and* $p^2 \neq 2$ *i.e.,*

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$*) then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))},$$

where
$\Lambda = (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + \left(x - p^4\right)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)H_{2n+1}^2 + (p^2 - 4)((n + 2)x - p^4(n + 1))x^nH_{2n-1}^2 - 2p^{-4n+4}(p^2 - 4)(p^{4n}(x + 3p^4 - p^6) + p^4(p^2 - 4)(n + 1)x^n)$
and if $\left(x - p^4\right)(x^2 - p^4\left(p^4 - 4p^2 + 2\right)x + p^8) = (x - 4)(x + 4)^2 = 0,\ x \neq p^4 = 4$ *and* $p^2 = 2$ *i.e.,*

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ *(and* $p^2 = 2$, *i.e.,* $p = -\sqrt{2}, p = \sqrt{2}$*) then*

$$\sum_{k=0}^{n} (-4)^k H_{2k+1}^2 = \frac{\Theta}{32},$$

where
$\Theta = 64(-4)^{n-1}\left(n^2 - 2\right)H_{2n+1}^2 + (-4)^{n+1}(n + 1)^2 H_{2n-1}^2 + 2^{-2n+4}(2^{2n} + 2n(n + 1)(-4)^n).$

(d) $(m = -1, j = 0)$
If $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) \neq 0$, *i.e.,*

$$x \neq \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq p^{-2},$$

then

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-2})(x^2 - (p^2 - 2)p^{-2}x + p^{-4})},$$

where
$\Phi = (x - p^{-2})(p^2 - 4)x^{n+1}H_{-n+1}^2 + (x - (p^2 - 2)p^{-2})(x - p^{-2})(p^2 - 4)x^{n+1}H_{-n}^2 - (p^2 - 4)p^{-6}(p^6x^2 + 4p^2x - 3p^4x + 2p^{2n+2}(p^2 - 4)x^{n+1} + 4)$
and if $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) = 0$, *i.e.,*

$$x = \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = p^{-2},$$

then

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)p^{-2}x + (p^2 - 1)p^{-4})},$$

where
$\Lambda = (p^2 - 4)((n + 2)x - p^{-2}(n + 1))x^nH_{-n+1}^2 + (p^2 - 4)(p^4(n + 3)x^2 - p^2x(p^2 - 1)(n + 2) + (p^2 - 2)(n + 1))p^{-4}x^nH_{-n}^2 + (p^2 - 4)p^{-4}(-2p^{2n}(p^2 - 4)(n + 1)x^n - 2p^4x + 3p^2 - 4)$

(e) $(m = -2, j = 0)$
If $\left(x - p^{-4}\right)(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, *i.e.,*

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$\Phi = (x-p^{-4})(p^2-4)x^{n+1}H^2_{-2n+2}+(x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}H^2_{-2n}+(p^2-4)p^{-12}(-p^8(p^2-2)^2x^2+p^4(3p^4-12p^2+8)x-2p^{4n+6}(p^2-4)x^{n+1}-4)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=p^{-4},$$

(and $p^2\neq 2$, *i.e.,* $p\neq-\sqrt{2}, p\neq\sqrt{2}$*) then*

$$\sum_{k=0}^{n}x^kH^2_{-2k}=\frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where

$\Lambda=(p^2-4)((n+2)x-p^{-4}(n+1))x^nH^2_{-2n+2}+(p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1}+\left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)H^2_{-2n}+(p^2-4)p^{-8}(-2p^2x^np^{4n}(p^2-4)(n+1)-2p^4(p^2-2)^2x+3p^4-12p^2+8)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=\frac{1}{64}(4x-1)(4x+1)^2=0$, $x\neq p^{-4}=\frac{1}{4}$ *and* $p^2=2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=-\frac{1}{4},$$

and $x\neq p^{-4}=\frac{1}{4}$ *(and* $p^2=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^kH^2_{-2k}=\frac{\Theta}{2},$$

where

$\Theta=-4\left(-\frac{1}{4}\right)^n(n+1)^2H^2_{-2n+2}+\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)H^2_{-2n}-n(n+1)2^{2n}\left(-\frac{1}{4}\right)^{n-1}$

(f) $(m=-2, j=1)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})\neq 0$, *i.e.,*

$$x\neq\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq p^{-4},$$

then

$$\sum_{k=0}^{n}x^kH^2_{-2k+1}=\frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where

$\Phi=(x-p^{-4})(p^2-4)x^{n+1}H^2_{-2n+3}+(x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}H^2_{-2n+1}+(p^2-4)p^{-12}(-p^8(p^2-3)^2x^2+p^4(p^4-4p^2+2)x-2p^{4n+4}(p^2-4)x^{n+1}-1)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=p^{-4},$$

(and $p^2\neq 2$, *i.e.,* $p\neq-\sqrt{2}, p\neq\sqrt{2}$*) then*

$$\sum_{k=0}^{n}x^kH^2_{-2k+1}=\frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where

$\Lambda=(p^2-4)((n+2)x-p^{-4}(n+1))x^nH^2_{-2n+3}+(p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1}+\left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)H^2_{-2n+1}+(p^2-4)p^{-8}(-2p^{4n}(p^2-4)(n+1)x^n-2p^4\left(p^2-3\right)^2x+p^4-4p^2+2)$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=\frac{1}{64}(4x-1)(4x+1)^2=0$, $x\neq p^{-4}=\frac{1}{4}$ *and* $p^2=2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=-\frac{1}{4},$$

and $x\neq p^{-4}=\frac{1}{4}$ *(and* $p^2=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^kH^2_{-2k+1}=\frac{\Theta}{2},$$

where

$\Theta=-4\left(-\frac{1}{4}\right)^n(n+1)^2H^2_{-2n+3}+\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)H^2_{-2n+1}+n(n+1)2^{2n+1}\left(-\frac{1}{4}\right)^n+1.$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [16, Corollary 2.3].

From the above proposition, we have the following corollary which gives sum formulas of p-Oresme numbers (take $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{p}$).

Corollary 2.4. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For $n \geq 0$, p-Oresme numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $\left(x - p^2\right)\left(x^2 - p^2(p^2 - 2)x + p^4\right) \neq 0$, *i.e.,*

$$x \neq \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2}, x \neq \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^{n} x^k O_k^2 = \frac{\Phi}{(p^2 - 4)(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4)}$$

where

$\Phi = (x - p^2(p^2 - 2))(x - p^2)(p^2 - 4)x^{n+1}O_n^2 + (x - p^2)(p^2 - 4)x^{n+1}O_{n-1}^2 - p^{-2n+2}(p^2 - 4)$ $(p^{2n}(x + p^2) - 2p^2x^n)x$

and

if $\left(x - p^2\right)\left(x^2 - p^2(p^2 - 2)x + p^4\right) = 0$, *i.e.,*

$$x = \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^{n} x^k O_k^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^2(p^2 - 1)x + p^4(p^2 - 1))}$$

where

$\Lambda = (p^2 - 4)((x - p^2(p^2 - 2))x^{n+1} + \left(x - p^2\right)\left((n+2)\,x - (n+1)p^2\left(p^2 - 2\right)\right)x^n)O_n^2 + (p^2 - 4)((n+2)x - p^2(n+1))x^nO_{n-1}^2 + p^{-2n+2}(p^2 - 4)(-p^{2n}(2x + p^2) + 2p^2(n+1)x^n)$

(b) $(m = 2, j = 0)$

If $\left(x - p^4\right)\left(x^2 - p^4\left(p^4 - 4p^2 + 2\right)x + p^8\right) \neq 0$, *i.e.,*

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^{n} x^k O_{2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)}$$

where

$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}O_{2n}^2 + (x - p^4)(p^2 - 4)x^{n+1}O_{2n-2}^2 - p^{-4n+6}(p^2 - 4)(p^{4n}(x + p^4) - 2p^4x^n)x$

and if $\left(x - p^4\right)\left(x^2 - p^4\left(p^4 - 4p^2 + 2\right)x + p^8\right) = 0$ *and* $p^2 \neq 2$ *i.e.,*

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^{n} x^k O_{2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))}$$

where

$\Lambda = (p^2-4)((x-p^4(p^4-4p^2+2))x^{n+1}+\left(x-p^4\right)((n+2)x-p^4(p^4-4p^2+2)(n+1))x^n)O_{2n}^2+(p^2-4)((n+2)x-p^4(n+1))x^nO_{2n-2}^2+p^{-4n+6}(p^2-4)(-p^{4n}(2x+p^4)+2p^4(n+1)x^n)$

and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=(x-4)(x+4)^2=0$, $x\neq p^4=4$ *and* $p^2=2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=-4,$$

and $x\neq p^4=4$ *(and* $p^2=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}(-4)^kO_{2k}^2=\frac{\Theta}{32}$$

where

$\Theta=64(-4)^{n-1}\left(n^2-2\right)O_{2n}^2+(-4)^{n+1}(n+1)^2O_{2n-2}^2+p^{-2}2^{-2n+6}(2^{2n}+n(n+1)(-4)^n)$

(c) $(m=2, j=1)$

If $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)\neq 0$, *i.e.,*

$$x\neq\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x\neq\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x\neq p^4,$$

then

$$\sum_{k=0}^{n}x^kO_{2k+1}^2=\frac{\Phi}{(p^2-4)(x-p^4)(x^2-p^4(p^4-4p^2+2)x+p^8)},$$

where

$\Phi=(x-p^4(p^4-4p^2+2))(x-p^4)(p^2-4)x^{n+1}O_{2n+1}^2+(x-p^4)(p^2-4)x^{n+1}O_{2n-1}^2-p^{-4n+2}(p^2-4)(p^{4n}(p^8+2p^6x-2p^4x+x^2)-2p^6x^{n+1})$,

and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=0$ *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2} \text{ or } x=p^4,$$

(and $p^2\neq 2$*) then*

$$\sum_{k=0}^{n}x^kO_{2k+1}^2=\frac{\Lambda}{(p^2-4)(3x^2-2p^4(p^2-1)(p^2-3)x+p^8(p^2-1)(p^2-3))},$$

where

$\Lambda=(p^2-4)((x-p^4(p^4-4p^2+2))x^{n+1}+\left(x-p^4\right)((n+2)x-p^4(p^4-4p^2+2)(n+1))x^n)O_{2n+1}^2+(p^2-4)((n+2)x-p^4(n+1))x^nO_{2n-1}^2+2p^{-4n+2}(p^2-4)(p^{4n}(-x-p^6+p^4)+p^6(n+1)x^n)$,

and if $\left(x-p^4\right)(x^2-p^4\left(p^4-4p^2+2\right)x+p^8)=(x-4)(x+4)^2=0$, $x\neq p^4=4$ *and* $p^2=2$ *i.e.,*

$$x=\frac{p^4(p^4-4p^2+2)+\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=\frac{p^4(p^4-4p^2+2)-\sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}=-4,$$

and $x\neq p^4=4$ *(and* $p^2=2$, *i.e.,*$p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}(-4)^kO_{2k+1}^2=\frac{\Theta}{32},$$

where

$\Theta=64(-4)^{n-1}\left(n^2-2\right)O_{2n+1}^2+(-4)^{n+1}(n+1)^2O_{2n-1}^2+2^{-2n+4}p^{-2}(2^{2n}+2n(n+1)(-4)^n))$.

(d) $(m=-1, j=0)$

If $(p^2x-1)p^{-2}(x^2-(p^2-2)p^{-2}x+p^{-4})\neq 0$, *i.e.,*

$$x\neq\frac{(p^2-2)p^{-2}+\sqrt{(p^2-4)p^{-2}}}{2}, x\neq\frac{(p^2-2)p^{-2}-\sqrt{(p^2-4)p^{-2}}}{2}, x\neq p^{-2},$$

then

$$\sum_{k=0}^{n}x^kO_{-k}^2=\frac{\Phi}{(p^2-4)(x-p^{-2})(x^2-(p^2-2)p^{-2}x+p^{-4})},$$

where

$\Phi=(x-p^{-2})(p^2-4)x^{n+1}O_{-n+1}^2+(x-(p^2-2)p^{-2})(x-p^{-2})(p^2-4)x^{n+1}O_{-n}^2+(p^2-4)p^{-4}(2p^{2n}x^n-p^2x-1)x$

and if $(p^2x-1)p^{-2}(x^2-(p^2-2)p^{-2}x+p^{-4})=0$, *i.e.,*

$$x=\frac{(p^2-2)p^{-2}+\sqrt{(p^2-4)p^{-2}}}{2} \text{ or } x=\frac{(p^2-2)p^{-2}-\sqrt{(p^2-4)p^{-2}}}{2} \text{ or } x=p^{-2},$$

then

$$\sum_{k=0}^{n} x^k O_{-k}^2=\frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)p^{-2}x+(p^2-1)p^{-4})},$$

where

$\Lambda=(p^2-4)((n+2)x-p^{-2}(n+1))x^nO_{-n+1}^2+(p^2-4)(p^4(n+3)x^2-p^2x(p^2-1)(n+2)+(p^2-2)(n+1))p^{-4}x^nO_{-n}^2+(p^2-4)p^{-4}(2p^{2n}(n+1)x^n-2p^2x-1)$

(e) $(m=-2, j=0)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})\neq 0$, *i.e.,*

$$x\neq \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq p^{-4},$$

then

$$\sum_{k=0}^{n} x^k O_{-2k}^2=\frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where

$\Phi=(x-p^{-4})(p^2-4)x^{n+1}O_{-2n+2}^2+(x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}O_{-2n}^2+(p^2-4)p^{-6}(2p^{4n}x^n-p^4x-1)x$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0$, *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=p^{-4},$$

(and $p^2\neq 2$, *i.e.,* $p\neq -\sqrt{2}, p\neq\sqrt{2}$*) then*

$$\sum_{k=0}^{n} x^k O_{-2k}^2=\frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where

$\Lambda=(p^2-4)((n+2)x-p^{-4}(n+1))x^nO_{-2n+2}^2+(p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1}+\left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)O_{-2n}^2+(p^2-4)p^{-6}(2p^{4n}(n+1)x^n-2p^4x-1),$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=\frac{1}{64}(4x-1)(4x+1)^2=0, x\neq p^{-4}=\frac{1}{4}$ *and* $p^2=2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=-\frac{1}{4},$$

and $x\neq p^{-4}=\frac{1}{4}$ *(and* $p^2=2$, *i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k O_{-2k}^2=\frac{\Theta}{2},$$

where

$\Theta=-4\left(-\frac{1}{4}\right)^n(n+1)^2O_{-2n+2}^2+\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)O_{-2n}^2+4p^{-2}(2^{2n}n(n+1)\left(-\frac{1}{4}\right)^n+1).$

(f) $(m=-2, j=1)$

If $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})\neq 0$, *i.e.,*

$$x\neq \frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq \frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}, x\neq p^{-4},$$

then

$$\sum_{k=0}^{n} x^k O_{-2k+1}^2=\frac{\Phi}{(p^2-4)(x-p^{-4})(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})},$$

where

$\Phi=(x-p^{-4})(p^2-4)x^{n+1}O_{-2n+3}^2+(x-(p^4-4p^2+2)p^{-4})(x-p^{-4})(p^2-4)x^{n+1}O_{-2n+1}^2+(p^2-4)p^{-14}(-p^8(p-1)^2(p+1)^2x^2+p^4(p^4-4p^2+2)x+2p^{4n+6}x^{n+1}-1),$

and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=0,$ *and* $p^2\neq 2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2} \text{ or } x=p^{-4},$$

(and $p^2\neq 2$*, i.e.,* $p\neq-\sqrt{2}, p\neq\sqrt{2}$*) then*

$$\sum_{k=0}^{n}x^kO_{-2k+1}^2=\frac{\Lambda}{(p^2-4)(3x^2-2(p^2-1)(p^2-3)p^{-4}x+(p^2-1)(p^2-3)p^{-8})},$$

where
$\Lambda=(p^2-4)((n+2)x-p^{-4}(n+1))x^nO_{-2n+3}^2+(p^2-4)((x-(p^4-4p^2+2)p^{-4})x^{n+1}+\left(x-p^{-4}\right)((n+2)x-(p^4-4p^2+2)p^{-4}(n+1))x^n)O_{-2n+1}^2+(p^2-4)p^{-10}(2p^{4n+2}(n+1)x^n-2p^4(p-1)^2(p+1)^2x+p^4-4p^2+2),$
and if $\left(x-p^{-4}\right)(x^2-(p^4-4p^2+2)p^{-4}x+p^{-8})=\frac{1}{64}(4x-1)(4x+1)^2=0, x\neq p^{-4}=\frac{1}{4}$ *and* $p^2=2$ *i.e.,*

$$x=\frac{(p^4-4p^2+2)p^{-4}+\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=\frac{(p^4-4p^2+2)p^{-4}-\sqrt{(p^2-4)((p^2-2)^2p^{-6}}}{2}=-\frac{1}{4},$$

and $x\neq p^{-4}=\frac{1}{4}$ *(and* $p^2=2$*, i.e.,* $p=-\sqrt{2}, p=\sqrt{2}$*) then*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^kO_{-2k+1}^2=\frac{\Theta}{2},$$

where
$\Theta=-4\left(-\frac{1}{4}\right)^n(n+1)^2O_{-2n+3}^2+\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}\left(n^2-2\right)O_{-2n+1}^2+p^{-2}(n(n+1)2^{2n+1}\left(-\frac{1}{4}\right)^n+1).$

Note that in the last corollary, the case $p^2\neq 4$ so that $p^2-4\neq 0$ is considered. The sum formulas for the case $p^2=4$ so that $p^2-4=0$ is given in Soykan [16, Corollary 2.4].

Taking $x=1$ in the last three corollaries we get the following corollary.

Corollary 2.5. *Suppose that* $p^2\neq 4$ *so that* $p^2-4\neq 0$*. For* $n\geq 0$*, modified p-Oresme numbers and p-Oresme-Lucas numbers and p-Oresme numbers have the following properties:*

1. *modified p-Oresme numbers:*

   (a) *If* $p^2\neq 1$ *then*

$$\sum_{k=0}^{n}G_k^2=\frac{\Phi}{(p^2-4)(1-p^2)(2p^2+1)}$$

   where
   $\Phi=(-p^4+2p^2+1)(1-p^2)(p^2-4)G_n^2+(1-p^2)(p^2-4)G_{n-1}^2-p^{-2n+4}(p^2-4)(p^{2n}(1+p^2)-2p^2)$
   and if $p^2=1$ *then*

$$\sum_{k=0}^{n}G_k^2=\frac{1}{3}(2G_n^2+G_{n-1}^2+2n-1).$$

   (b) *If* $p^2\neq 1$ *then*

$$\sum_{k=0}^{n}G_{2k}^2=\frac{\Phi}{(p^2-4)(1-p^4)(2p^2+1)(2p^4-2p^2+1)}$$

   where
   $\Phi=(-p^8+4p^6-2p^4+1)(1-p^4)(p^2-4)G_{2n}^2+(1-p^4)(p^2-4)G_{2n-2}^2-p^{-4n+8}(p^2-4)(p^{4n}(1+p^4)-2p^4)$
   and if $p^2=1$ *then*

$$\sum_{k=0}^{n}G_{2k}^2=\frac{1}{3}(2G_{2n}^2+G_{2n-2}^2+2n-1).$$

(c) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} G_{2k+1}^2 = \frac{\Phi}{(p^2-4)(1-p^4)\,(2p^2+1)\,(2p^4-2p^2+1)}$$

where
$\Phi = (-p^8+4p^6-2p^4+1)(1-p^4)(p^2-4)G_{2n+1}^2 + (1-p^4)(p^2-4)G_{2n-1}^2 - p^{-4n+4}(p^2-4)(p^{4n}(1-2p^4+2p^6+p^8)-2p^6)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} G_{2k+1}^2 = \frac{1}{3}(2G_{2n+1}^2 + G_{2n-1}^2 + 2n).$$

(d) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} G_{-k}^2 = \frac{\Phi}{(p^2-4)(1-p^{-2})(2p^{-2}+p^{-4})}$$

where
$\Phi = (1-p^{-2})(p^2-4)G_{-n+1}^2 + 2p^{-2}(1-p^{-2})(p^2-4)G_{-n}^2 + (p^2-4)p^{-2}(2p^{2n}-p^2-1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} G_{-k}^2 = \frac{1}{3}(G_{-n+1}^2 + 2G_{-n}^2 + 2n - 1).$$

(e) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} G_{-2k}^2 = \frac{\Phi}{(p^2-4)(1-p^{-4})(1-(p^4-4p^2+2)p^{-4}+p^{-8})}$$

where
$\Phi = (1-p^{-4})(p^2-4)G_{-2n+2}^2 + (1-(p^4-4p^2+2)p^{-4})(1-p^{-4})(p^2-4)G_{-2n}^2 + (p^2-4)p^{-4}(2p^{4n}-p^4-1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} G_{-2k}^2 = \frac{1}{3}(G_{-2n+2}^2 + 2G_{-2n}^2 + 2n - 1).$$

(f) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} G_{-2k+1}^2 = \frac{\Phi}{(p^2-4)(1-p^{-4})(1-(p^4-4p^2+2)p^{-4}+p^{-8})}$$

where
$\Phi = (1-p^{-4})(p^2-4)G_{-2n+3}^2 + (1-(p^4-4p^2+2)p^{-4})(1-p^{-4})(p^2-4)G_{-2n+1}^2 + (p^2-4)p^{-12}(-p^8(p-1)^2(p+1)^2 + p^4(p^4-4p^2+2) + 2p^{4n+6} - 1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} G_{-2k+1}^2 = \frac{1}{3}(G_{-2n+3}^2 + 2G_{-2n+1}^2 + 2n + 1).$$

2. *p-Oresme-Lucas numbers:*

(a) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} H_k^2 = \frac{\Phi}{(p^2-4)(1-p^2)(2p^2+1)}$$

where
$\Phi = (p^2-1)(p^2-4)(p^4-2p^2-1)H_n^2+(1-p^2)(p^2-4)H_{n-1}^2+p^{-2n+4}(p^2-4)(8-2p^2-p^{2n}(p^2+5))$
and if $p^2=1$ *then*

$$\sum_{k=0}^{n} H_k^2 = \frac{1}{3}(2H_n^2+H_{n-1}^2+6n+3).$$

(b) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} H_{2k}^2 = \frac{\Phi}{(p^2-4)(1-p^4)(1-p^4(p^4-4p^2+2)+p^8)}$$

where
$\Phi = (1-p^4(p^4-4p^2+2))(1-p^4)(p^2-4)H_{2n}^2+(1-p^4)(p^2-4)H_{2n-2}^2+p^{-4n+4}(p^2-4)(p^{4n}(-p^8-12p^6+7p^4+4p^2-4)-2p^6(p^2-4))$
and if $p^2=1$ *then*

$$\sum_{k=0}^{n} H_{2k}^2 = \frac{1}{3}(2H_{2n}^2+H_{2n-2}^2+6n+3).$$

(c) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} H_{2k+1}^2 = \frac{\Phi}{(p^2-4)(1-p^4)(1-p^4(p^4-4p^2+2)+p^8)}$$

where
$\Phi = (1-p^4(p^4-4p^2+2))(1-p^4)(p^2-4)H_{2n+1}^2+(1-p^4)(p^2-4)H_{2n-1}^2-p^{-4n+4}(p^2-4)(p^{4n}(1+6p^4-2p^6+p^8)+2p^4(p^2-4))$
and if $p^2=1$ *then*

$$\sum_{k=0}^{n} H_{2k+1}^2 = \frac{1}{3}(2H_{2n+1}^2+H_{2n-1}^2+6n).$$

(d) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} H_{-k}^2 = \frac{\Phi}{(p^2-4)(1-p^{-2})(2p^{-2}+p^{-4})}$$

where
$\Phi = (1-p^{-2})(p^2-4)H_{-n+1}^2+2p^{-2}(1-p^{-2})(p^2-4)H_{-n}^2-(p^2-4)p^{-6}(p^6+4p^2-3p^4+2p^{2n+2}(p^2-4)+4)$
and if $p^2=1$ *then*

$$\sum_{k=0}^{n} H_{-k}^2 = \frac{1}{3}(H_{-n+1}^2+2H_{-n}^2+6n+3).$$

(e) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} H_{-2k}^2 = \frac{\Phi}{(p^2-4)(1-p^{-4})(1-(p^4-4p^2+2)p^{-4}+p^{-8})}$$

where
$\Phi = (1-p^{-4})(p^2-4)H_{-2n+2}^2+(1-(p^4-4p^2+2)p^{-4})(1-p^{-4})(p^2-4)H_{-2n}^2+(p^2-4)p^{-12}(-p^8(p^2-2)^2+p^4(3p^4-12p^2+8)-2p^{4n+6}(p^2-4)-4)$
and if $p^2=1$ *then*

$$\sum_{k=0}^{n} H_{-2k}^2 = \frac{1}{3}(H_{-2n+2}^2+2H_{-2n}^2+6n+3).$$

(f) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} H_{-2k+1}^2 = \frac{\Phi}{(p^2-4)(1-p^{-4})(1-(p^4-4p^2+2)p^{-4}+p^{-8})}$$

where
$\Phi = (1-p^{-4})(p^2-4)H_{-2n+3}^2 + (1-(p^4-4p^2+2)p^{-4})(1-p^{-4})(p^2-4)H_{-2n+1}^2 + (p^2-4)p^{-12}(-p^8(p^2-3)^2 + p^4(p^4-4p^2+2) - 2p^{4n+4}(p^2-4) - 1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} H_{-2k+1}^2 = \frac{1}{3}(H_{-2n+3}^2 + 2H_{-2n+1}^2 + 6n - 3).$$

3. *p-Oresme numbers:*

(a) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} O_k^2 = \frac{\Phi}{(p^2-4)(1-p^2)(2p^2+1)}$$

where
$\Phi = (-p^4+2p^2+1)(1-p^2)(p^2-4)O_n^2 + (1-p^2)(p^2-4)O_{n-1}^2 - p^{-2n+2}(p^2-4)(p^{2n}(1+p^2) - 2p^2)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} O_k^2 = \frac{1}{3}(2O_n^2 + O_{n-1}^2 + 2n - 1).$$

(b) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} O_{2k}^2 = \frac{\Phi}{(p^2-4)(1-p^4)(1-p^4(p^4-4p^2+2)+p^8)}$$

where
$\Phi = (1-p^4(p^4-4p^2+2))(1-p^4)(p^2-4)O_{2n}^2 + (1-p^4)(p^2-4)O_{2n-2}^2 - p^{-4n+6}(p^2-4)(p^{4n}(1+p^4) - 2p^4)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} O_{2k}^2 = \frac{1}{3}(2O_{2n}^2 + O_{2n-2}^2 + 2n - 1).$$

(c) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} O_{2k+1}^2 = \frac{\Phi}{(p^2-4)(1-p^4)(1-p^4(p^4-4p^2+2)+p^8)}$$

where
$\Phi = (1-p^4(p^4-4p^2+2))(1-p^4)(p^2-4)O_{2n+1}^2 + (1-p^4)(p^2-4)O_{2n-1}^2 + p^2(p^2-4)(2p^4p^{4n} - 2p^6p^{4n} - p^8p^{4n} - p^{4n} + 2p^6)p^{-4n}$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} O_{2k+1}^2 = \frac{1}{3}(2O_{2n+1}^2 + O_{2n-1}^2 + 2n).$$

(d) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} O_{-k}^2 = \frac{\Phi}{(p^2-4)(1-p^{-2})(2p^{-2}+p^{-4})}$$

where
$\Phi = (1-p^{-2})(p^2-4)O_{-n+1}^2 + 2p^{-2}(1-p^{-2})(p^2-4)O_{-n}^2 + (p^2-4)p^{-4}(2p^{2n}-p^2-1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} O_{-k}^2 = \frac{1}{3}(O_{-n+1}^2 + 2O_{-n}^2 + 2n - 1).$$

(e) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} O_{-2k}^2 = \frac{\Phi}{(p^2-4)(1-p^{-4})(1-(p^4-4p^2+2)p^{-4}+p^{-8})}$$

where
$\Phi = (1-p^{-4})(p^2-4)O_{-2n+2}^2 + (1-(p^4-4p^2+2)p^{-4})(1-p^{-4})(p^2-4)O_{-2n}^2 + (p^2-4)p^{-6}(2p^{4n}-p^4-1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} O_{-2k}^2 = \frac{1}{3}(O_{-2n+2}^2 + 2O_{-2n}^2 + 2n - 1).$$

(f) *If* $p^2 \neq 1$ *then*

$$\sum_{k=0}^{n} O_{-2k+1}^2 = \frac{\Phi}{(p^2-4)(1-p^{-4})(1-(p^4-4p^2+2)p^{-4}+p^{-8})}$$

where
$\Phi = (1-p^{-4})(p^2-4)O_{-2n+3}^2 + (1-(p^4-4p^2+2)p^{-4})(1-p^{-4})(p^2-4)O_{-2n+1}^2 + (p^2-4)p^{-14}(-p^8(p-1)^2(p+1)^2 + p^4(p^4-4p^2+2) + 2p^{4n+6} - 1)$
and if $p^2 = 1$ *then*

$$\sum_{k=0}^{n} O_{-2k+1}^2 = \frac{1}{3}(O_{-2n+3}^2 + 2O_{-2n+1}^2 + 2n + 1).$$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [16, Corollary 2.5].

3 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized p-Oresme sequence, and then we have presented the formulas as special cases the corresponding identity for the modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

We can summarize the sections as follows:

- In section 1, we present some background about generalized p-Oresme numbers.

- In section 2, summation formulas have been presented for the generelized p-Oresme numbers. As special cases, summation formulas of modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers have been given.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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