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Research Article

Wave Breaking and Global Existence for the Generalized Periodic Camassa-Holm Equation with the Weak Dissipation

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In this paper, a family of the weakly dissipative periodic Camassa-Holm type equation cubic and quartic nonlinearities is considered. The precise blow-up scenarios of strong solutions and several conditions on the initial data to guarantee blow-up of the induced solutions are described in detail. Finally, we establish a sufficient condition for global solutions.

1. Introduction

In this paper, we are concerned with the periodic Cauchy problem of the generalized Camassa-Holm (CH) type equation the weak dissipation as follows:

$$\begin{aligned} u_t - u_{txx} + 3uu_x + \lambda(u - u_{xx}) &= 2u_x u_{xx} + uu_{xxx} + \alpha u_x \\ &\quad + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}, \end{aligned}$$

$$u(t, x+1) = u(t, x), t \geq 0, x \in \mathbb{R},$$

$$u(0, x) = u_0(x), x \in \mathbb{R}, \tag{1}$$

where $\alpha, \beta, \gamma, \lambda$, and Γ are arbitrary constants.

When $\lambda = \alpha = \beta = \gamma = \Gamma = 0$, the generalized CH type equation in (1) recovers the well-known integrable Camassa-Holm equation [1]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xx}, \tag{2}$$

which was derived by Camassa and Holm for shallow water waves [2] and is named after them. In fact, the CH equation was originally derived by Fokas and Fuchssiteiner [1] as a bi-Hamiltonian generalization of KdV. It is completely integrable and has infinitely many conservation laws [2–4]. In the recent years, the CH equation has caught a lot of attention from different perspectives, such as well-posedness, blow-up phenomena of solutions, and their stability. For example, the local well-posedness and the global strong solutions for certain class of initial data were studied [4, 5]. Existence and uniqueness results for classical solution of the periodic CH equation were established in [6]. The blow-up phenomena of the periodic CH equation were also investigated in a number of papers (see [2, 4–8] and references therein). Orbital stability of the peakons for the CH equation was studied by Lenells in [9, 10].

When $\lambda = \beta = \gamma = 0$ and $\alpha\Gamma \neq 0$, we have the Dullin-Gottwald-Holm (DGH) equation [11, 12]. And if $\alpha = \beta = \gamma = \Gamma = 0$ and $\lambda > 0$, it reduces to the weakly dissipative CH equation, whereas if $\beta = \gamma = 0$ and $\lambda > 0, \alpha\Gamma \neq 0$, it reduces to the weakly dissipative DGH equation. The well-posedness and blow-up phenomena of solutions of the Cauchy problem for the weakly dissipative DGH equation were studied, see, for example, [13, 14]. The similar research of the dissipative DGH equation higher-order nonlinearities and arbitrary coefficients and the other related models was also discussed in [15–19]. For some new and important developments for searching for solving numerical solutions for some PDE, the reader is referred to [20, 21] and references therein.

If $\lambda = 0$, the first equation of (1) becomes the equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} + \alpha u_x + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}, \tag{3}$$

which is related to the following physically relevant model:

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} - cu_x + \frac{\beta_0}{\beta} u_{xxx} - \frac{\omega_1}{\alpha^2} u^2 u_x - \frac{\omega_2}{\alpha^3} u^3 u_x, \tag{4}$$

with

$$\begin{aligned} c &= \sqrt{1+\Omega^2} - \Omega, \\ \alpha &= \frac{c^2}{1+c^2}, \\ \beta_0 &= \frac{c\left(c^4 + 6c^2 - 1\right)}{6\left(c^2+1\right)^2}, \\ \beta &= \frac{3c^4 + 8c^2 - 1}{6\left(c^2+1\right)^2}, \\ \omega_1 &= -\frac{3c\left(c^2-1\right)\left(c^2-2\right)}{2\left(c^2+1\right)^3}, \\ \omega_2 &= \frac{\left(c^2-1\right)^2\left(c^2-2\right)\left(8c^2-1\right)}{2\left(c^2+1\right)^5}, \end{aligned} \tag{5}$$

satisfying $c \longrightarrow 1, \beta \longrightarrow 5/12, \beta_0 \longrightarrow 1/4, \omega_1, \omega_2 \longrightarrow 0$, and $\alpha \longrightarrow 1/2$ when $\Omega \longrightarrow 0$. Here, Ω is a parameter related to the rotational frequency due to Coriolis effect which is typically a manifestation of rotation when Newton's laws are applied to model physical phenomena on Earth's surface. The Cauchy problem of Equations (3) and (4) has been studied in ref. [16, 22–26].

When $\lambda > 0$, the first equation of (1) can be seen as a weakly dissipative Camassa-Holm (CH) type equation. For the weakly dissipative CH type equations: the local well-posedness, global existence, and blow-up phenomena of the Cauchy problem of the weakly dissipative CH equation

$$u_t - u_{txx} + 3uu_x + \lambda\left(u - u_{xx}\right) = 2u_x u_{xx} + uu_{xxx}, \tag{6}$$

on the line [27] and on the circle [27] were studied. They found that the behaviors of Equation (44) are similar to the CH equation in a finite interval of time, such as the local well-posedness and the blow-up phenomena, and that there are considerable differences between them in their long time behaviors. Thereafter, a new global existence result and a new blow-up result for strong solutions to the equation certain profiles were presented in [28]. The obtained results improve considerably the previous results. Later on, a new blow-up result for positive strong solutions of (6) was presented by Novruzov [29]. In particular, they used a condition where the initial data $u_0(x)$ and its derivative are not simultaneously involved and the parameter λ is not bounded from above. The well-posedness and wave-breaking phenomena to the weakly dissipative CH equation quadratic and cubic nonlinearities

$$u_t - u_{txx} + 3u^2 u_x + \lambda\left(u - u_{xx}\right) = 2u_x u_{xx} + uu_{xxx}, \tag{7}$$

were considered by Freire et al. [30]. The novelty of their work is the method of group analysis was applied in order to construct conserved currents, and therefore, the conserved quantities were established as an extremely natural consequence of them. Subsequently, the periodic Cauchy problem of Equation (7) was considered by Ji and Zhou [31] and their local well-posedness was established via Kato's theory [32]; then, a sufficient condition on the initial data to guarantee the wave breaking was given and the global existence of solutions was given finally. Recently, Freire [33] considered the Cauchy problem of the weakly dissipative CH Equation (1). In their paper, some time-dependent energy functionals of solutions were proved, then the existence of wave-breaking phenomena was investigated, and necessary conditions for its existence were also obtained.

In general, it is difficulty to avoid energy dissipation mechanics in a real world. So it is reasonable to investigate the model with energy dissipation in propagation of nonlinear waves, see [34, 35] and references therein. Inspired by the previous work, the aim of the paper is to investigate whether the periodic Cauchy problem of the equation in (1) has the similar remarkable properties as that on the entire line. The outline of the paper is as follows. In Section 2, we obtain the local well-posedness and wave-breaking criterion. In Section 3, a blow-up scenario for strong solutions is described and some sufficient conditions for wave breaking of strong solutions in finite time are established. Furthermore, the blow-up rate of blow-up solutions of (1) is also derived. In Section 4, a sufficient condition for global solutions is provided.

1.1. Notations. Throughout this paper, we identity all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e., $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. The norm of the Sobolev space $H^s(\mathbb{S})$, $s \in \mathbb{R}$, by $\|\cdot\|_{H^s}$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity.

2. Preliminaries and Local Well-Posedness

Let us introduce the subject of investigation of this paper. In order to establish the local well-posedness result by Kato' theorem. Rewrite problem (1) as follows:

$$\begin{gathered} m_t + (u+\Gamma) m_x + 2u_x m + \lambda m - \partial_x h(u) = 0, t > 0, x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), t \geq 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), x \in \mathbb{R}, \end{gathered} \tag{8}$$

where

$$m = u - u_{xx}, \quad h(u) := (\alpha + \Gamma)u + \frac{\beta}{3}u^3 + \frac{\gamma}{4}u^4. \tag{9}$$

Denote $G(x) := \cosh (x - [x] - 1/2)/2 \sinh (1/2), x \in \mathbb{R}$. Then $(1 - \partial_x^2)^{-1} f = \Lambda^{-2} f = G * f$ for all $f \in L^2(\mathbb{S})$. Our system (8) can be written in the following transport type:

$$\begin{gathered} u_t + (u + \Gamma)u_x = \partial_x G^* h(u) - \partial_x G^* \left(u^2 + \frac{u_x^2}{2}\right) \\ - \lambda u, t > 0, x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), t \geq 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), x \in \mathbb{R}, \end{gathered} \tag{10}$$

or equivalently,

$$\begin{gathered} u_t + (u + \Gamma)u_x = \partial_x \Lambda^{-2} h(u) - \partial_x \Lambda^{-2} \left(u^2 + \frac{u_x^2}{2}\right) - \lambda u, t > 0, x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), t \geq 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), x \in \mathbb{R}. \end{gathered} \tag{11}$$

We begin by presenting the local well-posedness result for the periodic Cauchy problem (11). Concerning the generalized CH equation in (1) is suitable for applying Kato's theory [32]; one may follow the similar argument as in [30, 33] to obtain the following theorem.

Theorem 1. *Given $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$. Then, there exists a maximal time $T = T(u_0) > 0$ and unique solution u to (1) satisfying the initial condition $u(0, x) = u_0(x)$ such that*

$$u(\cdot, u_0) \in C([0, T) ; H^s(\mathbb{S})) \cap C^1 \left([0, T) ; H^{s-1}(\mathbb{S})\right). \tag{12}$$

Moreover, the solution depends continuously on the initial data, in the sense that the mapping $u_0 \longrightarrow u(\cdot, u_0)$: $H^s(\mathbb{S}) \longrightarrow C([0, T) ; H^s(\mathbb{S})) \cap C^1([0, T) ; H^{s-1}(\mathbb{S}))$ is continuous and T does not depend on $\mathbb{S}$.

Remark 2. The equation in (1), for $\lambda > 0$, can be seen as the weakly dissipative periodic Camassa-Holm type equation. However, no matter the value of parameter λ, the problem (1) is locally well-posed as shown in Theorem 1.

Lemma 3. (see [8]).

(i) For every $f \in H^1(\mathbb{S})$, we have

$$\max_{x \in [0,1]} f^2(x) = \frac{e + 1}{2(e - 1)} \|f\|_{H^1(\mathbb{S})}^2, \tag{13}$$

where the constant $(e + 1)/2(e - 1)$ is sharp.

(ii) For every $f \in H^3(\mathbb{S})$, we have

$$\max_{x \in [0,1]} f^2(x) \leq c \|f\|_{H^1(\mathbb{S})}^2, \tag{14}$$

with the best possible constant c lying within the range $(1, 13/12]$. Moreover, the best constant c is $(e + 1)/2(e - 1)$.

The next theorem will establish the time-dependent conserved quantities of solutions to problem (1), which is crucial in the investigation of wave-breaking phenomena and global existence of solutions.

Theorem 4. *Assume that $u(t, x)$ be the solution to problem (1) with the initial data $u_0(x) \in H^s(\mathbb{S}), s > 3/2$ such that $u(t, x)$ and its derivatives up to second order go to 0 as $x \longrightarrow 0$ and $x \longrightarrow 1$. Let*

$$H_0(t) := \int_{\mathbb{S}} u(t, x) dx, \quad H_1(t) := \frac{1}{2} \int_{\mathbb{S}} \left(u^2(t, x) + u_x^2(t, x)\right) dx. \tag{15}$$

Then, for any $t \in [0, T)$, we have

$$H_0(t) = e^{-\lambda t} H_0(0), \quad H_1(t) = e^{-2\lambda t} H_1(0). \tag{16}$$

Proof. Integrating the first equation of system (1) by parts, in view of the periodicity of u, we have

$$\frac{d}{dt} \int_{\mathbb{S}} u(t, x) \, dx + \lambda \int_{\mathbb{S}} u(t, x) \, dx = 0. \tag{17}$$

Then integrating (17) with respect to t over $(0, t)$ implies

$$H_0(t) = \int_{\mathbb{S}} u(t, x) dx = e^{-\lambda t} \int_{\mathbb{S}} u_0(x) dx = e^{-\lambda t} H_0(0). \tag{18}$$

For the proof of the other conserved quantity, multiplying both sides of the first equation in (1) by $2u$, we have

$$\begin{aligned} 2uu_t - 2uu_{txx} + 6u^2 u_x + 2\lambda \left(u^2 - uu_{xx}\right) \\ = 4uu_x u_{xx} + 2u^2 u_{xxx} + 2\alpha uu_x + 2\beta u^3 u_x \\ + 2\gamma u^4 u_x + 2\Gamma uu_{xxx}. \end{aligned} \tag{19}$$

Integrating (19) with respect to t over $(0, t)$ yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{S}} \left(u^2 + u_x^2\right) dx + 2\lambda \int_{\mathbb{S}} \left(u^2 + u_x^2\right) dx \\ & = -2 \int_{\mathbb{S}} \frac{\partial}{\partial_x} \left(u^3 - u^2 u_{xx} - uu_{tx} + \frac{\Gamma}{2} u_x^2 - \Gamma uu_{xx} \right. \\ & \left. - \frac{\alpha}{2} u^2 - \frac{\beta}{4} u^4 - \frac{\gamma}{5} u^5 - \lambda uu_x \right) dx = 0, \end{aligned} \tag{20}$$

i.e., $H_1(t) = e^{-2\lambda t} H_1(0)$, which completes the proof of the theorem. □

It is worth mentioning that if $\lambda > 0$, the results of Theorem 1 implies

$$H_0(t) \le H_0(0), \quad H_1(t) \le H_1(0), \tag{21}$$

which will be of great relevance in our investigation of wave breaking. Combining this observation with the Sobolev Embedding Theorem, we give the following remark.

Remark 5. If $u_0(x) \in H^s(\mathbb{S})$, $s > 3/2$, and $\lambda > 0$, then the H^1-norm of the corresponding solution of system (1) is bounded above by $\|u_0\|_{H^1(\mathbb{S})}$.

Once we have presented conditions for having locally well-posed solutions, a natural question is whether (1) admits wave breaking, which can be assured by the following blow-up criterion. In the following proof, we only prove the case when $s = 3$, since the same conclusion for general case $s > 3/2$ can be obtained by using denseness.

Theorem 6. *Let $u_0(x) \in H^s$, $s > 3/2$ be given and assume that $T > 0$ is the maximal existence time of the corresponding solution $u(t, x)$ to problem (1). Then, T is finite if and only if $\liminf_{t\longrightarrow T}\{\inf_{x\in\mathbb{S}} u_x(t, x)\} = -\infty$.*

Proof. Note that $m = u - u_{xx}$, a simple computation yields

$$\begin{aligned} \|m\|_{L^2}^2 &= \int_{\mathbb{S}} (u - u_{xx})^2 dx = \int_{\mathbb{S}} \left(u^2 + 2u_x^2 + u_{xx}^2\right) dx, \\ \|m_x\|_{L^2}^2 &= \int_{\mathbb{S}} (u_x - u_{xxx})^2 dx = \int_{\mathbb{S}} \left(u_x^2 + 2u_{xx}^2 + u_{xxx}^2\right) dx. \end{aligned} \tag{22}$$

Therefore, we can conclude that

$$\|u\|_{H^3}^2 \le \|m\|_{H^1}^2 \le 3\|u\|_{H^3}^2. \tag{23}$$

Multiplying $2m$ to both sides of the first equation in (8), we get

$$2mm_t + 2(u + \Gamma)mm_x + 4u_x m^2 + 2\lambda m^2 - 2m\partial_x h(u) = 0. \tag{24}$$

Integrating (24) with respect to x over $\mathbb{S}$ yields

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} m^2 dx &= -\int_{\mathbb{S}} \left(2(u+\Gamma)mm_x + 4u_x m^2 + 2\lambda m^2 - 2m\partial_x h(u)\right) dx \\ &= -\int_{\mathbb{S}} \left(3u_x m^2 + 2\lambda m^2 - 2m\partial_x h(u)\right) dx, \end{aligned} \tag{25}$$

where we used the relations

$$\int_{\mathbb{S}} 2mm_x dx = \int_{\mathbb{S}} dm^2 = 0, \quad \int_{\mathbb{S}} 2umm_x dx = -\int_{\mathbb{S}} u_x m^2 dx = 0. \tag{26}$$

If $u_0 \in H^4$, differentiating the first equation in (8) with respect to x, we obtain

$$m_{tx} + 3u_x m_x + (u + \Gamma)m_{xx} + 2u_{xx}m + \lambda m_x - \partial_x^2 h(u) = 0. \tag{27}$$

Multiplying $2m_x$ to both sides of Equation (27), then integrating the result with respect to x over $\mathbb{S}$, it follows that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx = &-6\int_{\mathbb{S}} u_x m_x^2 dx - 2\int_{\mathbb{S}} (u+\Gamma)m_x m_{xx} dx \\ &- 4\int_{\mathbb{S}} u_{xx} m m_x dx - 2\lambda\int_{\mathbb{S}} m_x^2 dx \\ &+ 2\int_{\mathbb{S}} m_x \partial_x^2 h(u) dx. \end{aligned} \tag{28}$$

Since that

$$\begin{aligned} 2\int_{\mathbb{S}} (u+\Gamma)m_x m_{xx} dx = -\int_{\mathbb{S}} u_x m_x^2 dx, \quad 4\int_{\mathbb{S}} u_{xx} m m_x \\ = -2\int_{\mathbb{S}} u_x m^2 dx, \end{aligned} \tag{29}$$

we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx = &-5\int_{\mathbb{S}} u_x m_x^2 dx + 2\int_{\mathbb{S}} u_x m^2 dx \\ &- 2\lambda\int_{\mathbb{S}} m_x^2 dx + 2\int_{\mathbb{S}} m_x \partial_x^2 h(u) dx, \end{aligned} \tag{30}$$

with

$$\partial_x^2 h(u) = (\alpha + \Gamma)u_{xx} + 2\beta u u_x^2 + \beta u^2 u_{xx} + 3\gamma u^3 u_{xx}, \tag{31}$$

which yields that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx = &-5\int_{\mathbb{S}} u_x m_x^2 dx + 2\int_{\mathbb{S}} u_x m^2 dx \\ &- 2\lambda\int_{\mathbb{S}} m_x^2 dx + 2\int_{\mathbb{S}} m_x \partial_x^2 h(u) dx. \end{aligned} \tag{32}$$

Then, by approximating u_0 in H^3 by function $u_0^n \in H^4$ $(n \ge 1)$, (32) also holds for $u_0 \in H^3$. In fact, let u^n be the solution to problem (1) with the initial data u_0^n. By local well-posedness theorem, we know that $u^n \in C([0, T_n]; H^4) \cap C^1([0, T_n]; H^3)$ for $n \ge 1, u^n(t, x) \longrightarrow u(t, x)$ in H^3 and $T_n \longrightarrow T$ as $n \longrightarrow \infty$. Since $u_0^n(x) \in H^4$, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} (m_x^n)^2 dx = &-5\int_{\mathbb{S}} u_x^n (m_x^n)^2 dx + 2\int_{\mathbb{S}} u_x^n (m^n)^2 dx \\ &- 2\lambda\int_{\mathbb{S}} (m_x^n)^2 dx + 2\int_{\mathbb{S}} m_x^n \partial_x^2 h(u^n) dx. \end{aligned} \tag{33}$$

Since $u^n \longrightarrow u$ in H^3 as $n \longrightarrow \infty$, it follows that

$u_x^n \longrightarrow u_x, u_{xx}^n \longrightarrow u_{xx}$ in L^∞ as $n \longrightarrow \infty$. Meanwhile, $m^n \longrightarrow m$ in H^1 and $m_x^n \longrightarrow m_x$ in L^2 as $n \longrightarrow \infty$. Letting $n \longrightarrow \infty$ in (33), it follows that (32) holds for $u_0(x) \in H^3$.

Thus, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx = -5\int_{\mathbb{S}} u_x m_x^2 dx - \int_{\mathbb{S}} u_x m^2 dx \\ - 2\lambda\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx \\ + 2\int_{\mathbb{S}}\left(m\partial_x h(u) + m_x\partial_x^2 h(u)\right)dx.\end{aligned} \tag{34}$$

Note that

$$\begin{aligned}2\int_{\mathbb{S}}\left(m\partial_x h(u) + m_x\partial_x^2 h(u)\right)dx = 2\int_{\mathbb{S}} m\Lambda^2\partial_x h(u)dx \\ \le \|m\|_{L^2}^2 + \left\|\Lambda^2\partial_x h(u)\right\|_{L^2}^2 \le c\|m\|_{H^1}^2.\end{aligned} \tag{35}$$

Here, we have used the facts that

$$\begin{aligned}\left\|\Lambda^2\partial_x h(u)\right\|_{L^2} = \|\partial_x h(u)\|_{H^2} \le \|h(u)\|_{H^3}, \\ \|h(u)\|_{H^3} \le c\|u\|_{H^3} = c\left\|\Lambda^{-2}u\right\|_{H^1} = c\|m\|_{H^1}.\end{aligned} \tag{36}$$

Therefore, (34) is reduced to

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx = -5\int_{\mathbb{S}} u_x m_x^2 dx - \int_{\mathbb{S}} u_x m^2 dx \\ - 2\lambda\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx + c\|m\|_{H^1}^2.\end{aligned} \tag{37}$$

If there exists some positive constant $M > 0$ such that $u_x > -M$, then we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx \le 5M\int_{\mathbb{S}} m_x^2 dx + M\int_{\mathbb{S}} m^2 dx \\ - 2\lambda\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx + c\|m\|_{H^1}^2 \\ \le (5M - 2\lambda + c)\|m\|_{H^1}^2.\end{aligned} \tag{38}$$

By using Gronwall's inequality, we get

$$\|m\|_{H^1}^2 \le e^{(5M-2\lambda+c)t}\|m_0\|_{H^1}^2 \le e^{(5M-2\lambda+c)T}\|m_0\|_{H^1}^2. \tag{39}$$

Therefore,

$$\|u\|_{H^3}^2 \le \|m\|_{H^1}^2 \le e^{(5M-2\lambda+c)T}\|m_0\|_{H^1}^2 \le 3e^{(5M-2\lambda+c)T}\|u_0\|_{H^1}^2, \tag{40}$$

which implies the H^3-norm of the solution $u(t,x)$ to (1) does not blow up in finite time.

On the other hand, if

$$\liminf_{t\longrightarrow T}\left\{\inf_{x\in\mathbb{S}} u_x(t,x)\right\} = -\infty, \tag{41}$$

then solution $u(t,x)$ to (1) will blow up in finite time. This completes the proof of the theorem. □

Remark 7. Similar to the case of nonperiodic, we prove the following result: Let $u_0(x) \in H^s, s > 3/2$ be given and $\lambda > 0$. And u be the corresponding solution $u(t,x)$ to problem (1). Assume that $u_x > -k$ for some positive constant k. Then, we obtain that $\|u\|_{H^s}(\mathbb{S}) \le e^{\sigma t}\|u_0\|_{H^s}(\mathbb{S})$, for a certain positive constant σ.

3. Blow-up Solutions and Blow-up Rate

In this section, we establish some sufficient conditions for the breaking of waves for the initial-value problem (1). To this end, we need the following lemma.

Lemma 8. *Let $T > 0$ and $v(t,x) \in C^1([0,T);H^2)$ be a given function. Then, for any $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{S}$ such that*

$$m(t) = \inf_{x\in\mathbb{S}} v_x(t,x) = v_x(t,\xi(t)), \tag{42}$$

and the function $m(t)$ is almost everywhere differentiable in $[0,T)$, with

$$m'(t) = v_{tx}(t,\xi(t)), a.e. on [0,T). \tag{43}$$

Now we are in position to state the following that provide a case that wave breaks in finite time.

Theorem 9. *Let $u_0 \in H^s$ with $s > 3/2$ and $\lambda > 0$. Assume $k := \max\{|\alpha+\Gamma|, |\beta/3|, (|\gamma|/4)\}$ and $U_0 := \max\{\|u_0\|_{H^1}, \|u_0\|_{H^1}^4\}$. If there exists some $x_0 \in \mathbb{S}$ such that*

$$u_{0'}(x_0) < -\lambda - \sqrt{\lambda^2 + 2\left(1 + \frac{3}{2}\cdot\frac{e+1}{e-1}\right)(1+k)U_0}. \tag{44}$$

Then, the corresponding solution $u(t,x)$ to (1) blows up in finite time in the following sense: there exists a T_1 with

$$\begin{aligned}0 < T_1 < \frac{1}{\sqrt{\lambda^2 + 2(1+3/2\cdot(e+1)/(e-1))(1+k)U_0}} \\ \cdot \ln\frac{m(0) + \lambda - \sqrt{\lambda^2 + 2(1+3/2\cdot(e+1)/(e-1))(1+k)U_0}}{m(0) + \lambda + \sqrt{\lambda^2 + 2(1+3/2\cdot(e+1)/(e-1))(1+k)U_0}},\end{aligned} \tag{45}$$

such that

$$\liminf_{t\uparrow T_1}\left\{\inf_{x\in\mathbb{S}} u_x(t,x)\right\} = -\infty. \tag{46}$$

Proof. . Differentiating the first equation in (11) with respect to x, we get

$$u_{xt} + \frac{u_x^2}{2} + (u+\Gamma)u_{xx} + \lambda u_x = u^2 - \Lambda^{-2}\left(u^2 + u_x^2\right) - h(u) + \Lambda^{-2}h(u), \tag{47}$$

where we have used the relation $\partial_x^2\Lambda^{-2} = \Lambda^{-2} - 1$.

According to Lemma 8 and the local well-posedness theorem, there is at least one point $\xi(t) \in \mathbb{S}$ satisfying $m(t) = \inf_{x\in\mathbb{S}} u_x(t,x) = u_x(t,\xi(t))$. Hence,

$$u_{xx}(t,\xi(t)) = 0, a.e.t \in [0,T). \tag{48}$$

Thus, we get

$$m'(t) = -\frac{1}{2}m^2(t) - \lambda m(t) + f(t,\xi(t)), \tag{49}$$

where $f(t,\xi(t))$ is given by

$$f(t,\xi(t)) = u^2(t,\xi(t)) - G * \left(u^2 + \frac{u_x^2}{2}\right)(t,\xi(t)) - h(u)(t,\xi(t)) + G * h(u)(t,\xi(t)). \tag{50}$$

Note that

$$u^2(t,\xi(t)) \le \frac{e+1}{2(e-1)}\|u\|_{H^1}^2 \le \frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2,$$

$$\left\|G * \left(u^2 + u_x^2\right)\right\|_{L^\infty} \le \|G\|_{L^\infty}\left\|u^2 + u_x^2\right\|_{L^1} \le \frac{e+1}{2(e-1)}\|u\|_{H^1}^2 \le \frac{e+1}{2(e-1)}\|u_0\|_{H^1}^2,$$

$$|G * h(u)| \le \|G\|_{L^\infty}\|h(u)\|_{L^\infty} \le \frac{e+1}{2(e-1)}\|h(u)\|_{H^1}, \tag{51}$$

then

$$|G * h(u) - h(u)| \le \left(1 + \frac{e+1}{2(e-1)}\right)\|h(u)\|_{H^1}. \tag{52}$$

Let $U_0 := \max\left\{\|u_0\|_{H^1}, \|u_0\|_{H^1}^4\right\}$, it follows from (9) that

$$\|h(u)\|_{H^1} \le k \max\left\{\|u_0\|_{H^1}, \|u_0\|_{H^1}^4\right\} \le kU_0, \tag{53}$$

with $k := \max\{|\alpha+\Gamma|, |\beta/3|, |\gamma|/4\}$ is a positive constant. Therefore, we obtain

$$|f| \le \frac{e+1}{e-1}\|u_0\|_{H^1}^2 + \left(1 + \frac{e+1}{2(e-1)}\right)kU_0 \le \left(1 + \frac{3}{2}\cdot\frac{e+1}{e-1}\right)(1+k)U_0. \tag{54}$$

Combining (49) and (54), we have

$$m'(t) \le -\frac{1}{2}m^2(t) - \lambda m(t) + \left(1 + \frac{3(e+1)}{2(e-1)}\right)(1+k)U_0 \doteq -\frac{1}{2}m^2(t) - \lambda m(t) + c = -\frac{1}{2}\left(m(t) + \lambda + \sqrt{\lambda^2 + 2c}\right) \cdot \left(m(t) + \lambda - \sqrt{\lambda^2 + 2c}\right) \; a.e.on\,[0,T), \tag{55}$$

with

$$c = \left(1 + \frac{3}{2}\cdot\frac{e+1}{e-1}\right)(1+k)U_0. \tag{56}$$

According to assumption (44), we have $m(0) < -\lambda - \sqrt{\lambda^2 + 2c}$, hence $m'(0) < 0$. Considering the continuity of $m(t)$ with respect to t, we can obtain that for any $t \in [0,T)$, $m'(t) < 0$ and $m(t) < -\lambda - \sqrt{\lambda^2 + 2c}$.

Then by solving the inequality (54), it follows that

$$\frac{m(0) + \lambda + \sqrt{\lambda^2 + 2c}}{m(0) + \lambda - \sqrt{\lambda^2 + 2c}} e^{\sqrt{\lambda^2+2c}t} \le \frac{m(t) + \lambda + \sqrt{\lambda^2 + 2c}}{m(t) + \lambda - \sqrt{\lambda^2 + 2c}}, \tag{57}$$

then

$$\frac{m(0) + \lambda + \sqrt{\lambda^2 + 2c}}{m(0) + \lambda - \sqrt{\lambda^2 + 2c}} e^{\sqrt{\lambda^2+2c}t} - 1 \le \frac{2\sqrt{\lambda^2 + 2c}}{m(t) + \lambda - \sqrt{\lambda^2 + 2c}} \le 0. \tag{58}$$

Notice that $0 < m(0) + \lambda + \sqrt{\lambda^2 + 2c}/m(0) + \lambda - \sqrt{\lambda^2 + 2c} < 1$, there exists

$$0 < T_1 \le \frac{1}{\sqrt{\lambda^2 + 2c}} \ln \frac{m(0) + \lambda - \sqrt{\lambda^2 + 2c}}{m(0) + \lambda + \sqrt{\lambda^2 + 2c}} \tag{59}$$

such that

$$\liminf_{t\uparrow T_1} m(t) = -\infty, \tag{60}$$

which demonstrates that the solution $u(t,x)$ blows up at a time $0 < T \le T_1$. □

Theorem 10. *If $T < \infty$ is the blow-up time of the solution to (1) with initial data $u_0 \in H^s, s > 3/2$ satisfying the assumption of Theorem 9. Then,*

$$\liminf_{t\uparrow T}\left\{\inf_{x\in\mathbb{S}} u_x(t,x)(T-t)\right\} = -2. \tag{61}$$

Proof. From (49) and (54), we know that

$$-c \le m'(t) + \frac{1}{2}m^2(t) + \lambda m(t) \le c, \quad a.e.on\,[0,T), \tag{62}$$

therefore,

$$-c-\frac{1}{2}\lambda^2 \le m'(t)+\frac{1}{2}(m(t)+\lambda)^2 \le c+\frac{1}{2}\lambda^2, a.e.on\,[0,T). \tag{63}$$

Choose $0<\varepsilon<1/2$. Since $\liminf_{t\uparrow T} m(t)=\infty$, we get $\liminf_{t\uparrow T}(m(t)+\lambda)=-\infty$; there is some point $t_0\in(0,T)$ such that

$$m(t_0)+\lambda<0, \quad (m(t_0)+\lambda)^2>\frac{1}{\varepsilon}\left(c+\frac{1}{2}\lambda^2\right). \tag{64}$$

Since $m(t)$ is absolutely continuous on $[0,T)$. By the above differential inequality, it follows that $m(t)$ is strictly decreasing on $[t_0,T)$ and hence

$$(m(t)+\lambda)^2>\frac{1}{\varepsilon}\left(c+\frac{1}{2}\lambda^2\right), t\in[t_0,T). \tag{65}$$

Combining (63) and (63), we get

$$-\frac{1}{2}-\varepsilon\le\frac{m'(t)}{(m(t)+\lambda)^2}\le-\frac{1}{2}+\varepsilon, a.e.on\, t\in[t_0,T). \tag{66}$$

Since $\varepsilon\in(0,1/2)$ is arbitrary, the above relation implies

$$\liminf_{t\uparrow T}(m(t)+\lambda)(T-t)=-2, \tag{67}$$

namely, $\liminf_{t\uparrow T}(m(t)+\lambda)(T-t)=-2$. This implies $\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{S}}u_x(t,x)(T-t)\}=-2$ by in view of the definition of $m(t)$. □

4. Global Existence

In this section, we turn our attention to existence of the global solution of system (1). We begin with the related results as follows.

Lemma 11. *Let $u\in C^1([0,T),H^s)$ be the solution of problem (1) with $u_0\in H^s, s>3/2$, and $T>0$ be the maximal time of existence. Then, the problem*

$$\begin{aligned} q_t(t,x)&=u(t,q)+\Gamma, \quad (t,x)\in[0,T)\times\mathbb{S},\\ q(0,x)&=x, \quad x\in\mathbb{S}. \end{aligned} \tag{68}$$

has a unique solution $q(t,x)\in C^1([0,T)\times\mathbb{R},\mathbb{R})$ and $q(t,\cdot)$ is an increasing diffeomorphism of the line with

$$q_x(t,x)=\exp\left(\int_0^t u_x(s,q(s,x))ds\right)>0, (t,x)\in[0,T)\times\mathbb{R}. \tag{69}$$

Furthermore,

$$m(t,q(t,x))q_x^2(t,x)=m_0(x)e^{-\lambda t}+\int_0^t e^{\lambda(s-t)}q_x^2(s,x)\partial_x h(u(s,q))ds, \tag{70}$$

where q is the solution of the problem (72) and $h(u)$ is the function given in (44).

Proof. The proof of Lemma 11 is similar as for the classic CH Equation (2), see ref. [36] for details. □

Theorem 12. *Let $u_0\in H^s, s>3/2$, and $m_0(x)=u_0(x)-u_0''(x)$. Assume that*
$m_0(x)\ge 0$ on $\mathbb{S}$ or $m_0(x)\ge 0$ on $\mathbb{S}$ does not change sign

$$\mathrm{sgn}\,(m_0)=\mathrm{sgn}\,(m). \tag{71}$$

Then, the solution $u(t,x)$ of (1) possesses bounded from blow x-derivative, which implies the global existence of the solution $u(t,x)$ in time t.

Proof. Since $u=G*m$, with $G(x)=\cosh(x-[x]-1/2)/2\sinh(1/2), x\in\mathbb{R}$, we get

$$\begin{aligned} u(t,x)&=\frac{1}{2\sinh(1/2)}\int_{\mathbb{S}}\cosh(x-\xi-[x-\xi]-1/2)m(t,\xi)d\xi\\ &=\frac{e^x}{4\sinh(1/2)}\int_0^x e^{-\xi-1/2}m(t,\xi)d\xi+\frac{e^x}{4\sinh(1/2)}\\ &\quad\cdot\int_x^1 e^{-\xi+1/2}m(t,\xi)d\xi+\frac{e^{-x}}{4\sinh(1/2)}\int_0^x e^{\xi+1/2}m(t,\xi)d\xi\\ &\quad+\frac{e^{-x}}{4\sinh(1/2)}\int_x^1 e^{\xi-1/2}m(t,\xi)d\xi. \end{aligned} \tag{72}$$

Differentiating this representation of u with respect to x gives

$$\begin{aligned} u_x(t,x)&=\frac{e^x}{4\sinh(1/2)}\int_0^x e^{-\xi-1/2}m(t,\xi)d\xi+\frac{e^x}{4\sinh(1/2)}\\ &\quad\cdot\int_x^1 e^{-\xi+1/2}m(t,\xi)d\xi-\frac{e^{-x}}{4\sinh(1/2)}\int_0^x e^{\xi+1/2}m(t,\xi)d\xi\\ &\quad-\frac{e^{-x}}{4\sinh(1/2)}\int_x^1 e^{\xi-1/2}m(t,\xi)d\xi. \end{aligned} \tag{73}$$

By combining (72) and (73), we have

$$\begin{aligned} u(t,x)+u_x(t,x)&=\frac{e^x}{2\sinh(1/2)}\int_0^x e^{-\xi-1/2}m(t,\xi)d\xi\\ &\quad+\frac{e^x}{2\sinh(1/2)}\int_x^1 e^{-\xi+1/2}m(t,\xi)d\xi, \end{aligned} \tag{74}$$

$$u_x(t,x) - u(t,x) = -\frac{e^{-x}}{2\sinh(1/2)}\int_0^x e^{\xi+1/2}m(t,\xi)d\xi - \frac{e^{-x}}{2\sinh(1/2)}\int_x^1 e^{\xi-1/2}m(t,\xi)d\xi. \quad (75)$$

Since that sgn $(m) = \text{sgn}(m_0)$, it follows from (72), (74), and (75) that if $m_0(x) \geq 0$, then $u(t,x) \geq 0$ and $u(t,x) \geq -u_x(t,x)$ or if $m_0(x) \leq 0$, then $u(t,x) \leq 0$ and $u_x(t,x) \geq u(t,x)$.

The above facts are enough to ensure that

$$u_x(t,x) \geq -\|u(t,x)\|_{L^\infty}, (t,x) \in [0,T) \times \mathbb{S}. \quad (76)$$

Consider that $\|u\|_{L^\infty} \leq 1/\sqrt{2}\|u\|_{H^1} \leq 1/\sqrt{2}\|u_0\|_{H^1}$, we conclude that $u_x(t,x) \geq -1/\sqrt{2}\|u_0\|_{H^1}$. The above inequality and Theorem 10 imply $T = \infty$, which shows the solution $u(t,x) \geq 0$ or $u(t,x) \leq 0$ exists globally in time. □

Remark 13. It is observed that if the conditions of Theorem 12 are satisfied and if $m_0 \in H^1(\mathbb{S})$, then u does not blow up in finite time by considering Theorem 9. This is similar to the previous results on the line [33].

Remark 14. From Lemma 11, we can observe that the presence of function $h(u)$ is the main obstacle of our investigation of existence of the global solutions. In fact, if $h(u) = 0$, it follows from Lemma 11 that the second condition of Theorem 12 would hold automatically as a consequence of its first condition. Therefore, the sign invariance of m is essential to prove the global existence of solutions.

Remark 15. Similarly for the Camassa-Holm equation and other analogous equations (see [4, 5, 7, 25, 33]), (1) admits the wave-breaking phenomenon, but differently from the Camassa-Holm equation, we cannot assure the global existence of solutions through the way we followed here [30, 33].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. Furthermore, all data in the paper are available.

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