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Cross Evaluation of Detection Schemes for Sparse Signals

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Authors' contributions

This work was carried out in collaboration with all authors. Author PPK designed the study and carried the mathematical analysis, jointly with author ATB. Authors PPK and ATB jointly performed the literature search. Author SS performed the numerical evaluations and the word processing. All authors checked the final results as well as read and approved the final manuscript.

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ABSTRACT

We consider environments where sparse signals are embedded in additive white noise. We consider specific signal models and cross-evaluate previously derived parametrically optimal, robust and tree-search policies for the detection of signal presence, in terms of the a posteriori probabilities of correct detection they induce. We specifically present numerical results for the case of a constant signal embedded in additive white Gaussian noise and the signal presence per observation being generated independently by a Bernoulli variable, in both the presence and the absence of data outliers.

Keywords: Sparse signals; detection of signal presence; parametrically optimal; robust and tree-search detection; white noise.

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1. INTRODUCTION

Sparse signals have received considerable attention the last few years, where sparsity has been defined only in vague terms. In addition, pertinent existing research has focus on the extraction of such signals via linear transformations [1,2], while no attention has been given to their localization.

In this paper, we define sparsity by an upper bound on the percentage of signal-containing observations within a given observations set. We then undertake the task of detecting the location the sparse signals, extending the results first presented in [3]. In particular, special attention is given to the case where a constant signal is sparsely embedded in additive white Gaussian noise, while data outliers may also be present. We then extend parametric, robust and tree-searching detection techniques, first developed in [3], to incorporate Bernoulli signal-generating models. We evaluate the performance of the extended techniques via studying the probabilities of correct detection they induce. The latter studies include derivations of bounds as well as numerical evaluations for various levels of signal sparsity and various values of signal-to-noise-ratios.

The organization of the paper is as follows: In Section 2, we state the fundamental general problem and summarize the previously found results. In Section 3, we develop a posteriori probability of correct detection expressions for the Bernoulli signal-presence model and subsequently present and discuss numerical results. In Section 4, we present conclusions

2. PROBLEM STATEMENT AND SUMMARY OF PREVIOUS RESULTS

We consider a sequence of observations generated by mutually independent random variables, a small percentage of which represent signal embedded in noise, while the remaining percentage represent just noise. Let it be known that the percentage of observations representing signal presence is bounded from above by a given value α. We assume additive, zero mean and white Gaussian noise; thus, that the random variables representing the noise are zero mean Gaussian, independent and identically distributed. We assume that the signal is independently distributed over the set of observations. We denote by $x_1,\dots x_n$, a sequence of n such observations, while we denote by $X_1,\dots,X_n$, the sequence of mutually independent random variables whose realization is the sequence $x_1,\dots x_n$. We also denote by $f_1(.)$ the probability density function (pdf) of the variables which represent signal presence, while we denote by $f_0(.)$ the pdf of the variables which represent just noise. Given the observation sequence $x_1,\dots x_n$ and assuming $f_1(.)$, $f_0(.)$ and α known, the objective is to identify the locations of the signal presence; that is, which ones of the $x_1,\dots x_n$ observations originated from the $f_1(.)$ pdf.

The approach to the problem solution in [3] is Maximum Likelihood (ML); which is equivalent to that of the Bayesian minimization of error probability approach, when all signal locations and their number are equally probable [4]. That is, given the sequence $x_1,\dots x_n$, the optimal detector decides in favor of the $i_1,\dots,i_m$; $0 \le m \le \alpha n$ signal locations if:

$$\sum_{1\le j\le m} \log f_1(x_j) + \sum_{k\ne\{ij\}} log f_0(x_k) = \max \left(\sum_{(j)} \log f_1(x_j) + \sum_{k\ne\{ij\}} log f_0(x_k)\right) \quad (1)$$

Let us define:

$$g(x) \equiv \log\left(\frac{f_1(x)}{f_0(x)}\right) \quad (2)$$

From [3], we then express the optimal ML detector as follows:

2.1 Optimal ML Detector

(a) Given the sequence $x_1,\dots x_n$, compute all $g(x_j)$; $1 \le j \le n$
(b) If $g(x_k) \le 0$; for all k, $1 \le k \le n$, then, decide that no observation contains signal.
(c) If $\exists$ a set of integers $\{i_1,\dots,i_m\}$; $1 \le m \le \alpha n$: $g(x_k) > 0$; for all $k \in \{i_1,\dots,i_m\}$ and $g(x_k) \le 0$; for all $k \notin \{i_1,\dots,i_m\}$, then decide that the observations containing the signal are all those with indices in the set $\{i_1,\dots,i_m\}$.

(d) If ∃ a set of integers $\{i_1,\dots,i_m\}$; $m > \alpha n$: $g(x_k) > 0$; for all $k \in \{i_1,\dots,i_m\}$ and $g(x_k) \le 0$; for all $k \notin \{i_1,\dots,i_m\}$, then decide that the observations containing the signal are those whose indices k are contained in the set $\{i_1,\dots,i_m\}$and whose $g(x_k)$ values are the αn highest in the set.

We consider now the special case where the signal is a known constant $\theta > 0$ and the noise is zero mean white Gaussian with standard deviation σ per observation, G(0, σ). From [3] we then have that the optimal ML detector described above takes here the following form:

2.2 Optimal ML Detector for Constant Signal θ and G(0,σ) White Noise

(a) Given the sequence $x_1,\dots x_n$, compute all $(x_j - \theta/2)$; $1 \le j \le n$
(b) If $(x_k - \theta/2) \le 0$; for all k, $1 \le k \le n$, then, decide that no observation contains signal.
(c) If ∃ a set of integers $\{i_1,\dots,i_m\}$; $1 \le m \le \alpha n$: $(x_k - \theta/2) > 0$; for all $k \in \{i_1,\dots,i_m\}$ and $(x_k - \theta/2) \le 0$; for all $k \notin \{i_1,\dots,i_m\}$, then decide that the observations containing the signal are all those with indices in the set $\{i_1,\dots,i_m\}$.
(d) If ∃ a set of integers $\{i_1,\dots,i_m\}$; $m > \alpha n$: $(x_k - \theta/2) > 0$; for all $k \in \{i_1,\dots,i_m\}$ and $(x_k - \theta/2) \le 0$; for all $k \notin \{i_1,\dots,i_m\}$, then decide that the observations containing the signal are those whose indices k are contained in the set $\{i_1,\dots,i_m\}$and whose $(x_k - \theta/2)$ values are the αn highest in the set.

The complexity of the above detector is of order n*l*ogn [3]. Denoting by $\varphi(x)$ and $\Phi(x)$, respectively, the pdf and the cumulative distribution function (cdf) of the zero mean and unit variance Gaussian random variable and assuming αn as an integer, we have from [3] that the conditional probabilities of correct detection, conditioned on the number of observations containing signal, are given in this case by the expressions below where.

$$P_d(\{i_1,\dots,i_m\}) = \left[\Phi\left(\frac{\theta}{2\sigma}\right)\right]^n ; 0 \le m \le \alpha n$$
$$P_d(\{i_1,\dots,i_m\}) = \alpha n \int_{\frac{-\theta}{2\sigma}}^{\infty} dw\varphi(w)[\Phi(-w)]^{\alpha n-1}\left[\Phi\left(w+\frac{\theta}{\sigma}\right)\right]^{(1-\alpha)n}$$
$$= \alpha n \int_{\infty}^{\frac{-\theta}{2\sigma}} dw\varphi(w)[\Phi(w)]^{\alpha n-1}\left[\Phi\left(-w+\frac{\theta}{\sigma}\right)\right]^{(1-\alpha)n} ; m = \alpha n \quad (3)$$

We can also find the following Lemma in [3], pertinent to small percentage of signal presence and small signal-to-noise ratios.

<u>Lemma 1</u>

Let $\alpha \to 0$ and $\theta/\sigma \to 0$. Then, the probability of correct detection in (4) is of order $\alpha n\, \Phi^n(\theta/\sigma)$.

2.3 The Outlier Resistant Detector for Constant Signal θ and G(0,σ) White Outlier Contaminated Noise

This case represents the occasional presence of extreme data outliers which may be contaminating the Gaussian environment. Then, instead of white and Gaussian, the noise environment is modeled as white with pdf belonging to a class *F* of density functions, defined as follows, for some given value ε in (0, 0.5), where ε represents the outlier contamination level:

$F = \{f;\ f = (1-\varepsilon)\, f_0 + \varepsilon\, h,\ f_0$ is the Gaussian zero mean and standard deviation σ pdf, h is any pdf$\}$

The outlier resistant robust detector is then found based on the *least favorable* density f^* in class *F* above, where the Kullback-Leibler number between f^* and its shifted by location parameter θ version attains the infimum among the Kullback-Leibler numbers realized by all pdfs in *F*, [4-6]. Then, the log likelihood ratio in (2) is a truncated version of that used in 2.2. As a result, for θ>0, the ML robust detector is operating as follows:

Robustl ML Detector

(a) Given the sequence $x_1,\ldots x_n$, compute all $[z(x_j) - \theta/2]\ ;\ 1 \le j \le n$,
; where,

$$z(x) = \begin{Bmatrix} d; & x \ge d \\ x; & -d+\theta < x < d \\ -d+\theta; & x \le -d+\theta \end{Bmatrix} \quad (4)$$

$$d: (1-\varepsilon)\left\{\Phi\left(\frac{-d+\theta}{\sigma}\right) + exp\left\{\frac{\theta d}{\sigma^2} - \frac{\theta^2}{2\sigma^2}\right\}\Phi\left(\frac{d}{\sigma}\right)\right\} = 1 \quad (5)$$

(b) *If* $[z(x_k) - \theta/2] \le 0$; for all k, $1 \le k \le n$, then, decide that no observation contains signal.

(c) If ∃ a set of integers $\{i_1,\ldots,i_m\}$; $1 \le m \le \alpha n$: $[z(x_k) - \theta/2] > 0$; for all $k \in \{i_1,\ldots,i_m\}$ *and* $[z(x_k) - \theta/2] \le 0$; for all $k \notin \{i_1,\ldots,i_m\}$, then decide that the observations containing the signal are all those with indices in the set $\{i_1,\ldots,i_m\}$.

(d) If ∃ a set of integers $\{i_1,\ldots,i_m\}$; $m > \alpha n$: $[z(x_k) - \theta/2] > 0$; for all $k \in \{i_1,\ldots,i_m\}$ and $[z(x_k) - \theta/2] \le 0$; for all $k \notin \{i_1,\ldots,i_m\}$, then decide that the observations containing the signal are those whose indices k are contained in the set $\{i_1,\ldots,i_m\}$and whose $[z(x_k) - \theta/2]$ values are the αn highest in the set.

Denoting by $P^r_{do}(\{i_1,\ldots,i_m\})$ the probability of correct detection induced by the robust ML detector, given that the noise is Gaussian containing no outliers and given that the signal occurs at the observation indices $\{i_1,\ldots,i_m\}$, the following expressions have been derived in [3], assuming again that αn is an integer:

$$P^r_{do}(\{i_1,\ldots,i_m\}) = [\Phi]^n\left(\frac{\theta}{2\sigma}\right); 0 \le m \le \alpha n$$

$$P^r_{do}(\{i_1,\ldots,i_m\}) = \alpha n \int_{\frac{-\theta}{2\sigma}}^{\frac{d-\theta}{\sigma}} dw\varphi(w)[\Phi(-w)]^{\alpha n-1} \times \left[\Phi\left(w+\frac{\theta}{\sigma}\right)\right]^{(1-\alpha)n} + \Phi^{\alpha n}\left(\frac{-d+\theta}{\sigma}\right)\Phi^{(1-\alpha)n}\left(\frac{d}{\sigma}\right): m = \alpha n \quad (6)$$

Comparing expressions (3) and (6), we notice that the robust detector induces lower probability of correct detection at the nominal Gaussian model; for the case of m=αn, where the difference of the two probabilities decreases monotonically with decreasing contamination level ε. As found in [3], this loss of performance of the robust detector at the nominal Gaussian model is at the gain of, frequently dramatic, performance improvement in the presence of outliers.

Let there exist a small positive value ς, such that the noise per observation is zero mean Gaussian; with probability *1-ς* and is an infinite positive value y; with probability ς. We express below the probabilities $P_{d\varsigma}(\{i_1,\ldots,i_m\})$ and $P^r_{d\varsigma}(\{i_1,\ldots,i_m\})$ induced by this outlier model and the optimal ML detector in 2.2 versus the robust detector, respectively, as expressed in [3].

$$P_{d\zeta}(\{i_1,\ldots.i_m\}) = P^r_{d\zeta}(\{i_1,\ldots.i_m\}) = (1-\zeta)^{n-m} \times \left[(1-\zeta)\Phi\left(\frac{\theta}{2\sigma}\right)+\zeta\right]^m \Phi^{n-m}\left(\frac{\theta}{2\sigma}\right); 0 \le m < \alpha n \quad (7)$$

$$P_{d\zeta}(\{i_1,\ldots.i_m\}) = (1-\zeta)^{1+(1-\alpha)n}\alpha n \times \int_{\frac{-\theta}{2\sigma}}^{\infty} dw\varphi(w)[(1-\zeta)\Phi(-w)+\zeta]^{\alpha n-1}\left[\Phi\left(w+\frac{\theta}{\sigma}\right)\right]^{(1-\alpha)n} + \zeta^{\alpha n}(1-\zeta)^{(1-\alpha)n};\ m = \alpha n \quad (8)$$

$$P^r_{d\zeta}(\{i_1,\ldots,i_m\}) = (1-\zeta)^{1+(1-\alpha)n}\alpha n \times \int_{\frac{-\theta}{2\sigma}}^{\frac{d-\theta}{\sigma}} dw\varphi(w)[(1-\zeta)\Phi(-w)+\zeta]^{\alpha n-1}\left[\Phi\left(w+\frac{\theta}{\sigma}\right)\right]^{(1-\alpha)n} + (1-\zeta)^{(1-\alpha)n}\left[(1-\zeta)\Phi\left(\frac{-d+\theta}{\sigma}\right)+\right.$$
$$\zeta \alpha n\, \Phi 1{-}\alpha n d\sigma; m{=}\alpha n \quad (9)$$

Comparison between expressions (8) and (9) reveals that, in the presence of outliers, the robust detector attains probability of correct detection higher than that attained by the detector in 2.2, where this performance improvement increases monotonically with increasing ς value.

2.4 Tree-Search Detector

This detector is proposed for the special case where the components of the sparse signal are spread relatively evenly across the n members of the observation set. Then, the objective of the detector is to identify observation clusters containing at most a single signal-presence observation. In this case, for $f_1(.)$ and $f_0(.)$ respectively being the pdfs of signal-presence versus signal -absence observations and *for* $g(x) \equiv \log (f_1(x)/f_0(x))$, a tree-search detector developed in [3] operates as follows, where the size N of the observation set is assumed to be a power of 2:

Tree–Search Detector

(a) Given the sequence $x_1,\ldots x_N$, compute all $g(x_j);\ 1 \le j \le N = 2^n$.

(b) Utilize a sequence $\{\beta_k\}$ of given algorithmic constants as:

(i) If $\sum_{k=1}^{N} g(x_k) \le \beta_n$, then, decide that at most a single signal component is contained in the sequence $x_1,\ldots x_N$ and stop.

(ii) If $\sum_{k=1}^{N} g(x_k) > \beta_n$, then, create the two partial sums $\sum_{k=1}^{\frac{N}{2}} g(x_k)$ and $\sum_{k=\frac{N}{2}+1}^{N} g(x_k)$

(iii) Test each of the two sums in (ii) against the constant β_{n-1} and go back to steps (i) and (ii).

(c) In general, the observation set $x_1,\ldots x_N$ is sequentially subdivided in powers of 2 number of portions, until the subdivision stops. If, during the algorithmic process, the observations with indices $\{i_1,\ldots,i_m\};\ m= 2^l$ are tested, then,

(i) If $\sum_{k=1}^{m} g(x_{i_k}) \le \beta_l$, then, decide that at most a single signal component is contained in the sequence $x_1,\ldots x_N$ and stop.

(ii) If $\sum_{k=1}^{m} g(x_{i_k}) > \beta_l$, then, create the two partial sums $\sum_{k=1}^{\frac{m}{2}} g(x_{i_k})$ and $\sum_{k=\frac{m}{2}+1}^{m} g(x_{i_k})$.

(iii) Test each of the two sums in (ii) against the constant β_{l-1} and go back to steps (i) and (ii).

For m signal-containing observations within a total of N observations, the complexity of the tree-search detector is of order $[\log m]N$. Focusing on the constant signal and additive, zero mean, white Gaussian noise model in (ii), where the function g(x) in the description of the tree–search detector equals $x–\theta/2$, the Lemma below was proven in [3].

Lemma 2

Let a constant signal θ be additively embedded in white, zero mean Gaussian noise with standard deviation σ per observation. Let $m = 2^l$ be the number of signal components, given that they are spread uniformly across a total of $N = 2^n$ observations, where $n - l > 1$. Let the constants $\{\beta_k\}$ used by the tree–search detector be such that: $\beta_k < 2\beta_{k-1}$; for all $2 \le k \le n$. Then, the conditional probability of correct detection, $P_d(l,n)$, conditioned on l and n, as induced by the tree-search detector, is given by the following expression, where this probability is also conditioned on the above uniform signal spreading assumption.

$$P_d(l,n) = \left[\Phi(c_{ln})\big(\Phi(c_{ln}) - \Phi(d_{ln})\big) - \int_{d_{ln}}^{c_{ln}} dx\varphi(x)\Phi\,(d_{ln} + c_{ln} - x)\right]^{2^{l-1}}$$

$;\ n-1 \ge n - l \ge 2$

$$P_d(0,n) = \Phi(c_{0n}) \qquad (10)$$

Where,

$$c_{ln} \equiv \left[\beta_{n-l} + (2^{n-l-1} - 1)\frac{\theta^2}{\sigma^2}\right]\frac{\sigma}{\theta}2^{-\frac{n-l}{2}} \tag{11}$$

$$d_{ln} \equiv \left[\beta_{n-l+1} - \beta_{n-l} + (2^{n-l-1} - 1)\frac{\theta^2}{\sigma^2}\right]\frac{\sigma}{\theta}2^{-\frac{n-l}{2}} \tag{12}$$

For $n - l \geq 2$ and $(\theta/\sigma)^2$ values above $\beta_{n-l}\, [2^{n-l-1}-1]^{-1}$, the probability of correct decision in (10) is increasing with increasing signal-to-nose ratio θ/σ, as well as with increasing difference $n - l$. Also, the β_{n-l} and β_{n-l+1} values should be of $2^{-(n-l)}$ order; for asymptotically large values of the difference $n - l$. We may select the specific values of the constants β_{n-l} and β_{n-l+1} based on a maximization of correct detection criterion, as stated in Lemma 3 below found in [3].

Lemma 3

Let a constant signal θ be additively embedded in white zero mean Gaussian noise with standard deviation per observation, σ. Let the signal θ occur with probability q per observation, independently across all $N = 2^n$ observations. Let $P_d\,(n\text{-}l,q)$ denote then the probability of correctly distinguishing between at most one and at least two signal components in 2^{n-l} observations, via the use of the tree-search constant β_{n-l}. The constant β_{n-l} may be selected as that which maximizes the probability $P_d\,(n\text{-}l,q)$, where the latter is given by the following expression.

$$P_d(n-l,q) = (1-q)^{2^{n-l}}\Phi(c_{nl0}) + 2^{n-l}q(1-q)^{2^{n-l}-1}\Phi(c_{nl1}) + \sum_{k=2}^{2^{n-l}}\binom{2^{n-l}}{k}q^k(1-q)^{2^{n-l}-k}\Phi(-c_{nlk}) \tag{13}$$

Where,

$$c_{nlk} \triangleq \lambda_{n-l} + (2^{n-l-1} - k)\mu_{n-l} \tag{14}$$

$$\lambda_{n-l} \triangleq \frac{\sigma\beta_{n-l}}{\theta 2^{\frac{n-l}{2}}} \tag{15}$$

$$\mu_{n-l} \triangleq \frac{\theta}{\sigma 2^{\frac{n-l}{2}}} \tag{16}$$

For signal-to-noise *ratio* θ/σ asymptotically small and q less than 0.5, the constant β_{n-l} which maximizes the probability $P_d\,(n\text{-}l,q)$ in (13) is given by equation (17) below, where it can be then shown that $\beta_{n-l+1} < 2\,\beta_{n-l}$.

$$\beta_{n-l} = \left[1 + \frac{2q[1-2(1-q)^{2^{n-l}-1}]}{-1+2(1-q)^{2^{n-l}}+2^{n-l+1}q(1-q)^{2^{n-l}-1}}\right]^{-1} \tag{17}$$

We note that we may “robustify” the tree-search detector at the Gaussian nominal model, by using, instead, $g(x) = z(x) - \theta/2$; for $z(x)$ as in (4). The error performance of the robust tree–search detector may be then studied asymptotically. We will not include such asymptotic study in this paper.

3. THE BERNOULLI SIGNAL–PRESENCE MODEL AND NUMERICAL EVALUATIONS

In this section, we consider the model of constant signal $\theta>0$ embedded in additive zero mean white Gaussian noise, with standard deviation per sample σ. We select values of the various parameters involved and cross-evaluate probabilities of correct detection, as induced by the detectors summarized in Section 2. For the parametric optimal detector in Section 2(ii) and the robust detector in Section 2.3, we select in addition a Bernoulli model for signal generation as follows: Given n and α, the signal θ>0 occurs per observation with probability p, given by the expression below, where q is a constant in (0, 0.5) which is closer to 0 for sparse signals:

$$p = \frac{q}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}} \tag{18}$$

We then Denote by P_d, $P^r_{d\,0}$, $P_{d\,\zeta}$ and $P^r_{d\,\zeta}$ the a posteriori probabilities of correct detection relating to the conditional probabilities of correct detection in (3), (6), (8) and (9), respectively, where the signal occurrence model in (17) is adopted. We can express these a posteriori probabilities as follows, assuming αn integer:

$$P_d = \Phi\left(\frac{\theta}{2\sigma}\right)^n \left[\frac{\sum_{m=0}^{\propto n-1}\binom{n}{m}q^m(1-q)^{n-m}}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}}\right] + \left[\frac{\binom{n}{\alpha n}q^{\propto n}(1-q)^{n-\propto n}}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}}\right] P_d(\{i_1,\dots,i_m\})(from\ (3)) \tag{19}$$

$$P^r{}_{d0} = \Phi\left(\frac{\theta}{2\sigma}\right)^n \left[\frac{\sum_{m=0}^{\propto n-1}\binom{n}{m}q^m(1-q)^{n-m}}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}}\right] + \left[\frac{\binom{n}{\alpha n}q^{\propto n}(1-q)^{n-\propto n}}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}}\right] P^r{}_{d0}(\{i_1,\dots,i_m\})\big(from\ (6)\big) \tag{20}$$

$$P_{d\varsigma} = P_{dj}(\{i_1,\dots,i_m\})(from\ (7))\left[\frac{\sum_{m=0}^{\propto n-1}\binom{n}{m}q^m(1-q)^{n-m}}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}}\right] + P_{d\varsigma}(\{i_1,\dots,i_m\})\big(from\ (8)\big)\left[\frac{\binom{n}{\alpha n}q^{\propto n}(1-q)^{n-\propto n}}{\sum_{m=0}^{\propto n-1}\binom{n}{m}q^m(1-q)^{n-m}}\right] \tag{21}$$

$$P^r_{d\varsigma} = P_{d\varsigma}(\{i_1,\dots,i_m\})(from\ (7))\left[\frac{\sum_{m=0}^{\propto n-1}\binom{n}{m}q^m(1-q)^{n-m}}{\sum_{m=0}^{\propto n}\binom{n}{m}q^m(1-q)^{n-m}}\right] + P^r_{d\varsigma}(\{i_1,\dots,i_m\})(from\ (9))\left[\frac{\binom{n}{\alpha n}q^{\propto n}(1-q)^{n-\propto n}}{\sum_{m=0}^{\propto n-1}\binom{n}{m}q^m(1-q)^{n-m}}\right] \tag{22}$$

For the parametric and robust detectors, we numerically evaluated the a posteriori probabilities in (19), (20), (21) and (22). For the tree-search detector, we first evaluated the a posteriori probability of correct detection $P_d\,(n\text{-}l,q)$ in expression (10). We subsequently evaluated the probability of correct detection $P_d(n, q)$, when a signal per observation is present with probability q, the signal is a constant $\theta > 0$ embedded in white Gaussian noise, the signal-spreading of Lemma 2 holds and the tree–search is performed. For $P_d\,(l,n)$ as in (10), the probability $P_d(n, q)$ is given by the following expression, whose value decreases with decreasing Bernoulli parameter q:

$$P_d(n,q) = \left[\sum_{l=0}^{n-2} 2^{(n-l)2^l} q^{2^l}(1-q)^{2^n-2^l}\right]^{-1} \sum_{l=0}^{n-2} P_d(l,n)\, 2^{(n-l)2^l} q^{2^l}(1-q)^{2^n-2^l} \tag{23}$$

In our numerical evaluations, we adopted the parameter values given below. For the tree-search detector, we used the β_{n-l} expressions in (17).

- σ= 4.5E-4
- q = 0.05 for the parametric and robust detectors; 0.05, 0.1, 0.2 and 0.3 for tree –search detector
- ε = 0.05
- ς = 0.05, 0.1
- α = 0.05, 0.1, 02, 0.3
- θ/σ=0.5, 1.0, 4.5, 7.0 for the parametric and robust detectors; 7.0, 0.07 and 0.007 for tree- search detector

Results from our numerical comparisons between the parametric and the robust detectors are exhibited in Figs. 1 to 6. Numerical evaluations for the tree-search detector are shown in Figs 7 and 8.

From Figs. 1 to 6, we observe the quantitative performance superiority of the robust detector in the presence of outliers, as compared to that of the parametric detector. The later performance superiority occurs at the expense of reduced performance, when outliers are absent. A satisfactory performance tradeoff, in the absence versus the presence of outliers, may be attained by the robust detector whose design parameter ε takes a small value. As viewed from Figs 5 and 6, for example, the robust detector designed for outlier probability ε=0.05 outperforms significantly the parametric detector when the actual outlier probability is ζ = 0.1. From Figs. 1 to 6, we also observe the quantitative performance increase, for both the parametric and the robust detectors, as the signal-to-noise ratio and the assumed in the detector design percentage α of signal-containing observations increase (in contrast to the actual percentage represented by the Bernoulli parameter q). In all cases, performance decreases exponentially with the number of observations, for sufficiently large sample sizes.

Figs. 7 and 8 exhibit the behavior of the probability $P_d(n, q)$ of correct detection in (23), as induced by the tree-search detector at various signal-to-noise ratios, for q=0.01 and q=0.05, respectively. In the figures, the horizontal axis depicts the changing n values; the signal-to-noise ratios (SNRs) selected are within the range (in Lemma 2) guaranteeing monotone increase of the probability in (10) with increasing SNR. From these figures, we observe the profound performance reduction, when the Bernoulli parameter q decreases from 0.05 to 0.01; we also observe the significant impact of the signal-to-noise ratio on the $P_d(n, q)$ performance.

Comparing Figs. 2 and 6, we note that, in the absence of outliers and the presence of the Bernoulli signal-generating model with parameter q=0.05, the performance of the tree-search detector is comparable to that of the optimal parametric detector.

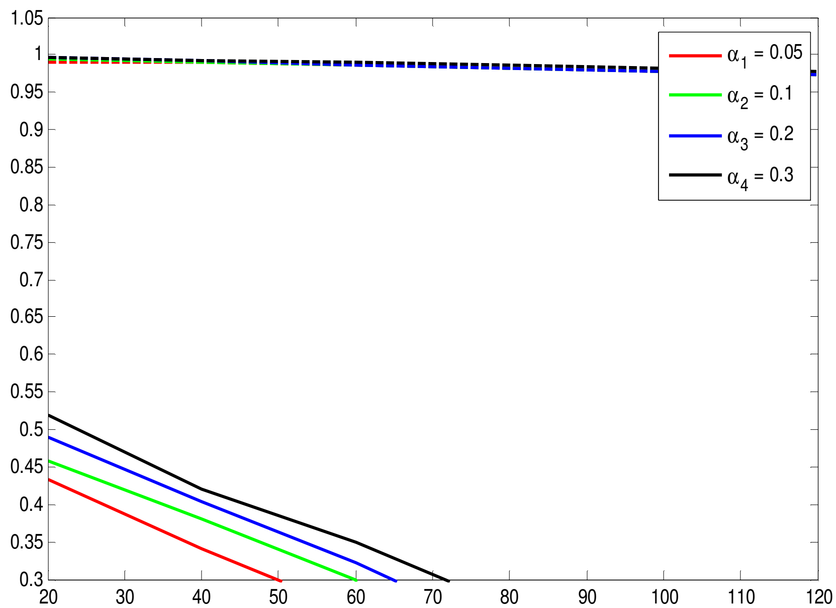

Fig. 1. Shows the comparison between the parametric and the robust in the absence of outliers, probabilities in (19) and (20), for q=0.05 and SNR=7, when ε=0.05

------ Parametric detector; x-axis: number of observations

——— Robust detector; y-axis: P_d value

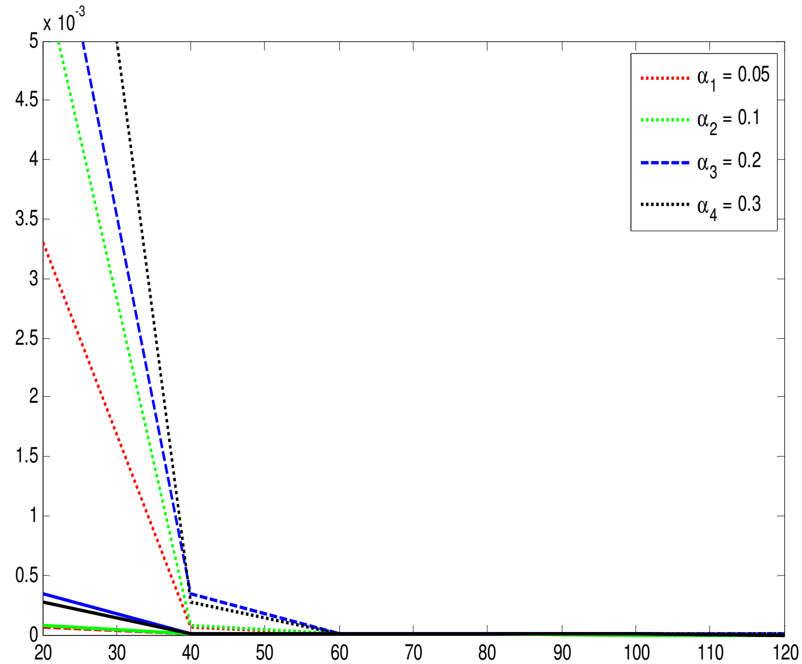

Fig. 2. Shows the comparison between the parametric and robust detector model in the absence of outliers, probabilities in (19) and (20), for *q*=0.05 and SNR=0.5, when *ε*=0.05

-------- *Parametric detector; x-axis: number of observations*

———— *Robust detector; y-axis: P_d value*

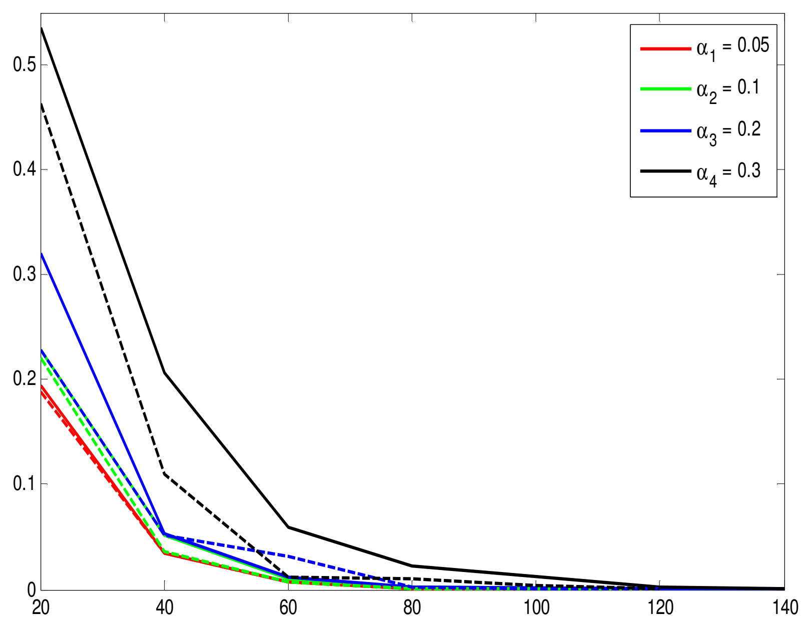

Fig. 3. Shows the comparison between parametric and robust detector in the presence of outliers, probabilities in (21) and (22), for *q*=0.05 and SNR=7, when *ζ*=0.05 and *ε*=0.05

-------- *Parametric detector; x-axis: number of observations*

———— *Robust detector; y-axis: P_d value*

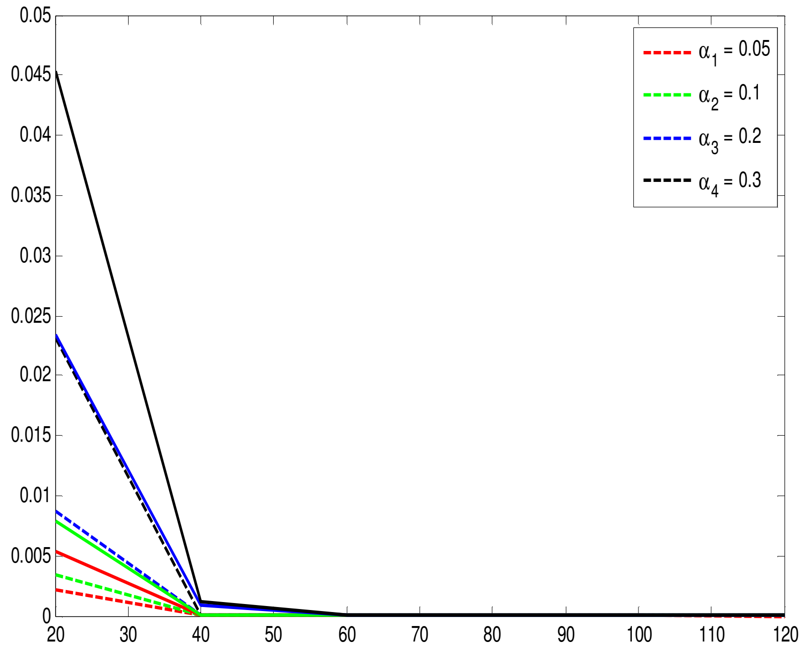

Fig. 4. Shows the comparison between parametric and robust detector in the presence of outliers, probabilities in (21) and (22), for *q*=0.05 and SNR= 0.5,when *ζ*=0.05 and *ε*=0.05

-------- *Parametric detector; x-axis: number of observations*

——— *Robust detector; y-axis: P_d value*

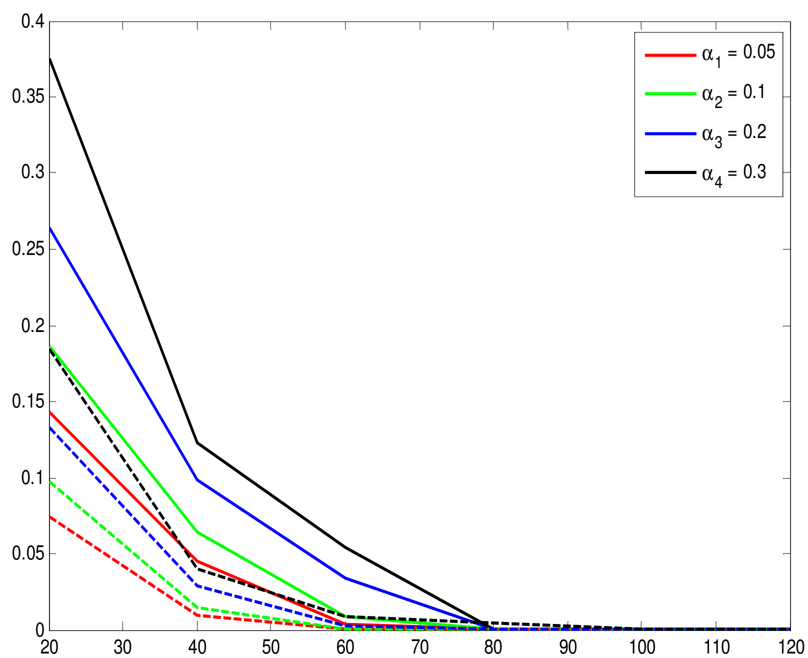

Fig. 5. Shows the comparison between parametric and robust detector in the presence of outliers, probabilities in (21) and (22), for *q*=0.05 and SNR=7, when *ζ*=0.1 and *ε*=0.05

-------- *Parametric detector; x-axis: number of observations*

——— *Robust detector; y-axis: P_d value*

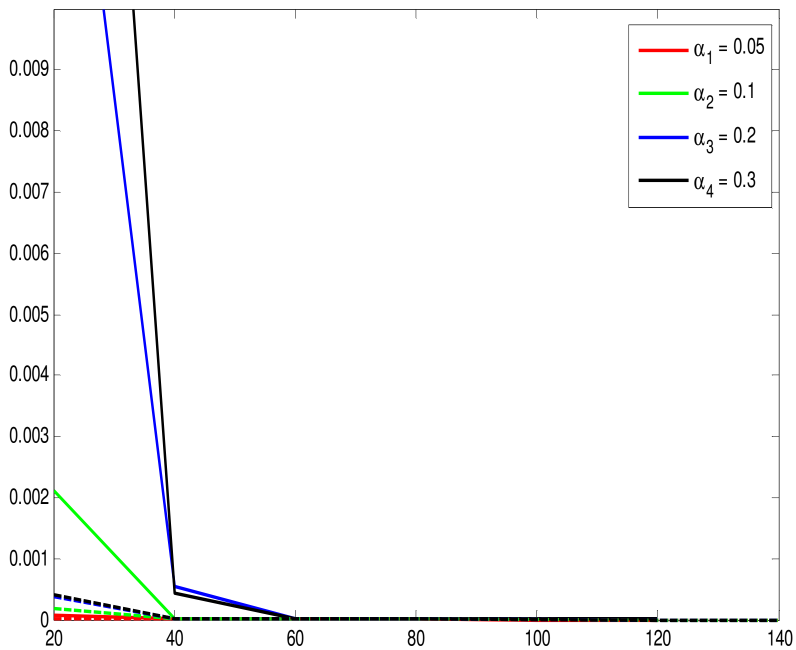

Fig. 6. Shows the comparison between parametric and robust detector in the presence of outliers, probabilities in (21) and (22), for *q*=0.05 and SNR=0.5, when *ζ*=0.1 and *ε*=0.05

-------- Parametric detector; x-axis: number of observations

———— Robust detector; y-axis: P_d value

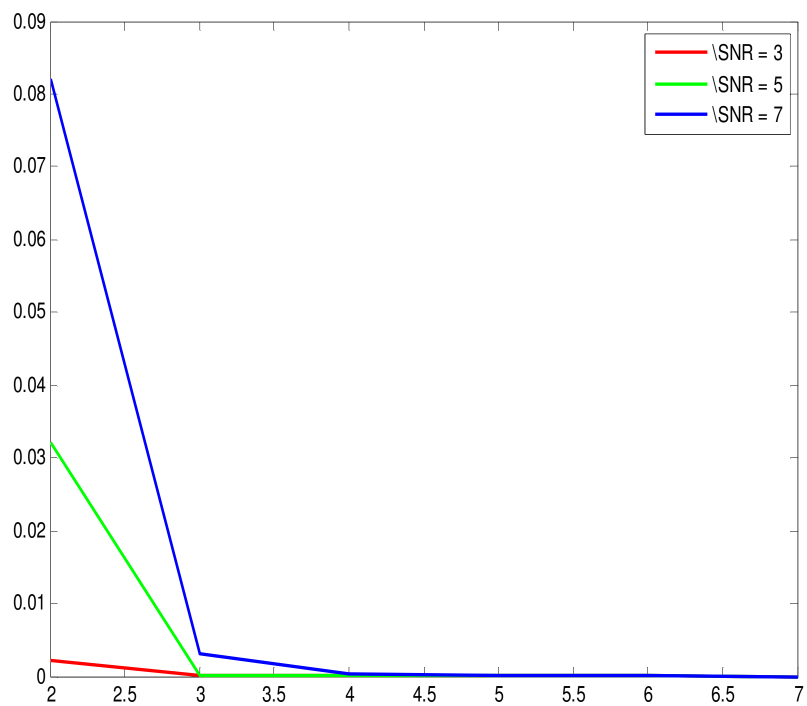

Fig. 7. The probability of correct detection $P_d(n, q)$ in (23) for the tree-search detector, as function of *n*, for *q*=0.01

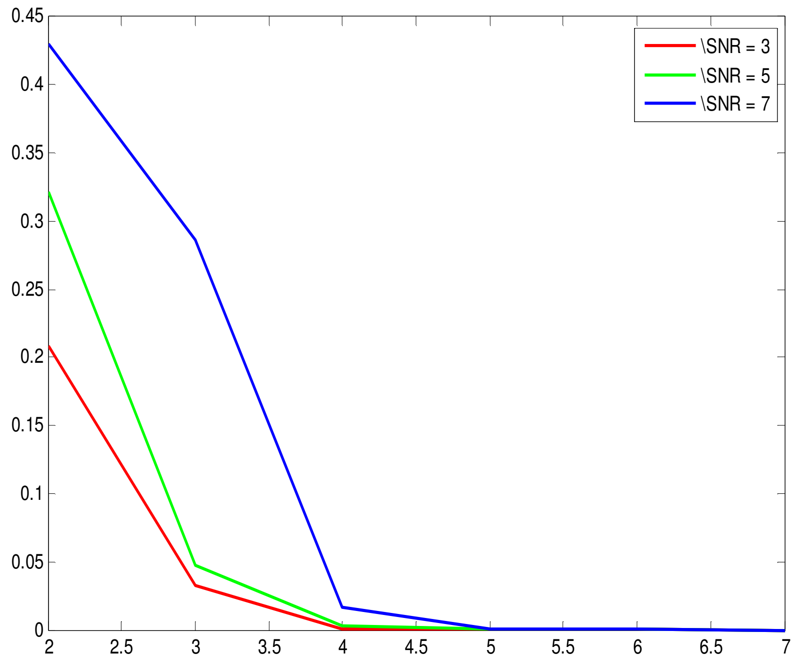

Fig. 8. The probability of correct detection $P_d(n, q)$ in (23) for the tree- search detector, as function of n, for q = 0.05

4. CONCLUSION

We considered the case where a constant signal is sparsely generated by a Bernoulli variable and is subsequently embedded in additive Gaussian white noise. We then evaluated the performance of three sparse signal detectors-parametrically optimal, robust and tree-search-in terms of the probability of correct detection they induce. The parametrically optimal and robust detectors were designed based on an assumed known maximum percentage of signal containing observations, while the tree-search detector has been addressing an assumed uniform signal spreading across all observations without any a priori knowledge of percentages regarding signal containing observations. The parametrically optimal detector has been designed around the assumption that no data outliers may ever occur, while the robust detector's design incorporates an assumed maximum percentage of data outliers. All three detectors have been evaluated for the same value 0.05 of the signal-generating Bernoulli parameter, while the tree-search detector has also been evaluated for the Bernoulli parameter value 0.01. The performed evaluations have led to the following general qualitative conclusions: (a) As the spreading of the sparse signals increases, so does the difficulty of their localization, where such difficulty decreases with increasing signal-to-noise ratio; (b) The highest the assumed percentage of signal containing observations in the design of the parametrically optimal and the robust detectors, the better their performance, in both the presence and the absence of outliers; (c) In the presence of outlier data, the robust detector may significantly outperform the parametrically optimal detector, at the expense of relative insignificant performance reduction in the absence of outlier data.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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