British Journal of Mathematics & Computer Science
8(5): 366-376, 2015, Article no.BJMCS.2015.170
ISSN: 2231-0851

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A Note on Semi-Compatible Maps and Occasionally Weakly Compatible Maps in Non-archimedean Menger PM-Space

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Article Information

DOI: 10.9734/BJMCS/2015/16514
Editor(s):
(1) Xiaodi Li, School of Mathematical Sciences, Shandong Normal University, Ji'nan, 250014, Shandong, P.R. China.
Reviewers:
(1) Dimplekumar N. Chalishajar, Department of Mathematics and Computer Science, Virginia Military Institute (VMI), Lexington, USA.
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Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=1034&id=6&aid=8840

Original Research Article

Received: 05 February 2015
Accepted: 09 March 2015
Published: 15 April 2015

Abstract

The concept of semi-compatible and occasionally weakly compatible mappings is used to prove a common fixed point theorem. The theorem thus obtained is a generalization of the result of Cho et al. [10] in a non-Archimedean Menger PM-space.

Keywords: Non-archimedean menger probabilistic metric space; common fixed points; compatible maps; occasionally weakly compatible maps.

AMS Subject Classification: Primary 47H10, Secondary 54H25.

1 Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [1]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [2] studied this concept and gave some fundamental results on this space.

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The notion of compatible mapping in a Menger space has been introduced by Mishra [3]. Using the concept of compatible mappings of type (A), Jain et al. [4,5] proved some interesting fixed point theorems in Menger space. Afterwards, Jain et al. [6] proved the fixed point theorem using the concept of weak compatible maps in Menger space.

The notion of non-Archimedean Menger space has been established by Istrătescu and Crivat [7]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istrătescu [8]. This has been the extension of the results of Sehgal and Bharucha - Reid [9] on a Menger space. Cho et al. [10] proved a common fixed point theorem for compatible mappings in non-Archimedean Menger PM-space.

In this paper, we generalize the result of Cho et al. [10] by introducing the notion of occasionally weakly compatible self maps. Also, we cited an example in support of this.

2 Preliminaries

For terminologies, notations and properties of N.A. Menger PM-space, refer to [11,8] and [12].

Definition 2.1. [10] Let X be a non-empty set and D be the set of all left-continuous distribution functions. An ordered pair (X, **F**) is called a non-Archimedean probabilistic metric space (briefly, a N.A. PM-space) if **F** is a mapping from X×X into D satisfying the following conditions (the distribution function **F**(x,y) is denoted by $F_{x,y}$ for all $x,y \in X$) :

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;
(PM-2) $F_{u,v} = F_{v,u}$;
(PM-3) $F_{u,v}(0) = 0$;
(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(\max\{x, y\}) = 1$,
for all u, v, w $\in$ X and x, y > 0.

Definition 2.2. [10] A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, nondecreasing in each coordinate and $\Delta(a,1) = a$ for every $a \in [0,1]$.

Definition 2.3. [10] A *N.A. Menger PM-space* is an ordered triple (X, **F**, Δ), where (X, **F**) is a non-Archimedean PM-space and Δ is a t-norm satisfying the following condition:

(PM-5) $F_{u,w}(\max\{x,y\}) \geq \Delta(F_{u,v}(x), F_{v,w}(y))$, for all u, v, w $\in$ X and x, y $\geq$ 0.

Definition 2.4. [10] A PM-space (X, **F**) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that

$$g(F_{x,y}(t)) \leq g(F_{x,z}(t)) + g(F_{z,y}(t))$$

for all x, y, z $\in$ X and t $\geq$ 0, where $\Omega = \{g \mid g : [0,1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 2.5. [10] A N.A. Menger PM-space (X, **F**, Δ) is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(s,t) \leq \gamma(s) + g(t)$$

for all s, t $\in [0,1]$.

Remark 2.1. [10]

(1) If a N.A. Menger PM-space (X, **F**, Δ) is of type $(D)_g$ then (X, **F**, Δ) is of type $(C)_g$.
(2) If a N.A. Menger PM-space (X, **F**, Δ) is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by

$$d(x,y) = \int_0^1 g\left(F_{x,y}(t)\right)d(t) \text{ for all } x, y \in X. \qquad (*)$$

Throughout this paper, suppose (X, **F**, Δ) be a complete N.A. Menger PM-space of type $(D)_g$ with a continuous strictly increasing t-norm Δ.

Let $\phi : [0,+\infty) \rightarrow [0, \infty)$ be a function satisfied the condition (Φ) :
(Φ) ϕ is upper-semicontinuous from the right and $\phi(t) < t$ for all $t > 0$.

Lemma 2.1. [10] If a function $\phi : [0,+\infty) \rightarrow [0,+\infty)$ satisfies the condition (Φ), then we have

(1) For all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is n^{th} iteration of $\phi(t)$.
(2) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, n = 1, 2, … then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for all $t \geq 0$, then $t = 0$.

Definition 2.6. [10] Let A, S : X $\rightarrow$ X be mappings. A and S are said to be compatible if $\lim_{n\to\infty} g(F_{ASx_n,SAx_n}(t)) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

Definition 2.7. Self maps A and S of a N.A. Menger PM-space (X, **F**, Δ) are said to semi-compatible if $\lim_{n\to\infty} g(F_{ASx_n,Su}(t)) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$ for some u in X.

Example 2.1. Let (X, **F**, Δ) be the N.A. Menger PM-space, where X = [0, 1] and the metric d on X is defined in condition (*) of Remark 2.1. Define self maps S and T as follows:

$$Sx = \begin{cases} x, & \text{if } \quad 0 \leq x < 1/2, \\ 1, & \text{if } \quad 1/2 \leq x \leq 1, \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} 1-x, & \text{if } \quad 0 \leq x < 1/2, \\ 1, & \text{if } \quad 1/2 \leq x \leq 1. \end{cases}$$

Take $x_n = 1/2 - (1/n)$.
Then $Sx_n \rightarrow 1$ as $n \rightarrow \infty$. Similarly, $Tx_n \rightarrow 1$ as $n \rightarrow \infty$.
Therefore, $\lim_{n\to\infty} g(F_{STx_n,TSx_n}(t)) \neq 0 \ \forall\, t \geq 0$.
Hence, the pair (S,T) is not compatible.
Also, $\lim_{n\to\infty} g(F_{STx_n,Tu}(t)) = 0$ for all $t \geq 0$. Thus (S,T) is semi-compatible maps.

Definition 2.8. [12] Self maps A and S of a N.A. Menger PM-space (X, **F**, Δ) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if Ap = Sp for some p$\in$ X then ASp = SAp.

Remark 2.2. [12] Compatible maps are weakly compatible but converse is not true.

Definition 2.9. Self maps A and S of a N.A. Menger PM-space (X, **F**, Δ) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Example 2.1. Let (X, **F**, Δ) be the N.A. Menger PM-space, where X = [0, 2] and the metric d on X is defined in condition (*) of Remark 2.1. Define A, S: X $\to$ X by Ax = 2x and Sx = x^2 for all x $\in$ X then Ax = Sx for x = 0 and 2. But AS(0) = SA(0) and AS(2) $\neq$ SA(2).

Thus, S and T are occasionally weakly compatible mappings but not weakly compatible.

Proposition 2.1. If self-mappings A and S of a N.A. Menger PM-space (X, **F**, Δ) are compatible then they are occasionally weakly compatible.

Proof. Suppose Ap = Sp, for some p in X.

Consider the constant sequence $\{p_n\}$ = p.

Now, $\{Ap_n\} \to$ Ap and $\{Sp_n\} \to$ Sp (= Ap).

As A and S are compatible, we have $\lim_{n\to\infty} g(F_{ASp,\ SAp}(t)) = 0$ for all t > 0.

Thus, ASp = SAp and so (A, S) is occasionally weakly compatible.

The following is an example of pair of self maps in a N.A. Menger PM-space (X, **F**, Δ) which are occasionally weakly compatible but not compatible.

Example 2.1. Let (X, **F**, Δ) be a N.A. Menger PM-space, where X = [0, 2] and the metric d on X is defined in condition (*) of Remark 2.1. Define self maps A and S as follows:

$$Ax = \begin{cases} 2-x, & \text{if } 0 \le x \le 1, \\ 2, & \text{if } 1 < x \le 2, \end{cases} \qquad \text{and} \qquad Sx = \begin{cases} x, & \text{if } 0 \le x < 1, \\ 2, & \text{if } 1 \le x \le 2. \end{cases}$$

Take x_n = 1- (1/n).

Then $Ax_n \to 1$ as $n \to \infty$. Similarly, $Sx_n \to 1$ as $n \to \infty$.

Therefore, $\lim_{n\to\infty} g(F_{ASx_n,SAx_n}(t)) \neq 0 \ \forall\ t \ge 0$.

Hence, the pair (A,S) is not compatible.

Also, 2 is the coincidence points of A and S and therefore,

AS(2) = SA(2).

Thus, A and S are occasionally weakly compatible but not compatible.

From the above example it is obvious that the concept of occasionally weak compatibility is more general than that of compatibility.

Proposition 2.2. [12] Let A and S be compatible self maps of a N.A. Menger PM-space (X, **F**, Δ) and let $\{x_n\}$ be a sequence in X such that Ax_n, $Sx_n \to u$ for some u in X. Then $ASx_n \to Su$ provided S is continuous.

Proposition 2.3. [12] Let S and T be compatible self maps of a N.A. Menger PM-space (X, **F**, Δ) and Su = Tu for some u in X then STu = TSu = SSu = TTu.

Lemma 2.2. [10] Let A, B, S, T: X $\to$ X be mappings satisfying the condition (1) and (2) as follows:

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.
(2) $g(F_{Ax,By}(t)) \le \phi(\max\{g(F_{Sx,Ty}(t)), g(F_{Sx,Ax}(t)), g(F_{Ty,By}(T)), ½(g(F_{Sx,By}(T)) + g(F_{Ty,Ax}(t)))\})$

for all t > 0, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ). Then the sequence $\{y_n\}$ in X, defined by $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ..., such that

$\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0$ for all t > 0 is a Cauchy sequence in X.

Cho et al. [10] established the following result:

Theorem 2.1 [10]. Let A, B, S, T: X$\to$X be mappings satisfying the condition (1), (2),

(3) S and T is continuous,
(4) the pairs (A, S) and (B, T) are compatible maps.

Then A, B, S and T have a unique common fixed point in X.

3 Main Results

In the following, we extend this result to six self maps and generalize it in other respects too.

Theorem 3.1. Let A, B, S, T, L, M : X $\to$ X be mappings satisfying the condition

(3.1.1) $L(X) \subset ST(X)$, $M(X) \subset AB(X)$;
(3.1.2) AB = BA, ST = TS, LB = BL, MT = TM;
(3.1.3) either AB or L is continuous;
(3.1.4) (L, AB) is semi-compatible and (M, ST) is occasionally weakly compatible;;
(3.1.5) $g(F_{Lx,My}(t)) \le \phi(\max\{g(F_{ABx,STy}(t)), g(F_{ABx,Lx}(t)), g(F_{STy,My}(t)),$

$$½(g(F_{ABx,My}(t)) + g(F_{STy,Lx}(t)))\})$$

for all t > 0, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ).

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (3.1.1), there exist $x_1, x_2 \in X$ such that

$Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(3.1.6) $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ...

Step 1. We prove that $\lim\limits_{n\to\infty} g(F_{y_n, y_{n+1}}(t)) = 0$ for all $t > 0$.
From (3.1.5) and (3.1.6), we have

$$
\begin{aligned}
g(F_{y_{2n}, y_{2n+1}}(t)) &= g(F_{Lx_{2n}, Mx_{2n+1}}(t)) \\
&\le \phi(\max\{g(F_{ABx_{2n}, STx_{2n+1}}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \\
&\quad \tfrac{1}{2}(g(F_{ABx_{2n}, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, Lx_{2n}}(t)))\}) \\
&= \phi(\max\{g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n}, y_{2n+1}}(t)), \tfrac{1}{2}(g(F_{y_{2n-1}, y_{2n+1}}(t)) + g(1))\}) \\
&\le \phi(\max\{g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n}, y_{2n+1}}(t)), \tfrac{1}{2}(g(F_{y_{2n-1}, y_{2n}}(t)) + g(F_{y_{2n}, y_{2n+1}}(t))\}).
\end{aligned}
$$

If $g(F_{y_{2n-1}, y_{2n}}(t)) \le g(F_{y_{2n}, y_{2n+1}}(t))$ for all $t > 0$, then by (3.1.5)

$$g(F_{y_{2n}, y_{2n+1}}(t)) \le \phi(g(F_{y_{2n}, y_{2n+1}}(t))),$$

on applying Lemma 2.1, we have $g(F_{y_{2n}, y_{2n+1}}(t)) = 0$ for all $t > 0$.

Similarly, we have $g(F_{y_{2n+1}, y_{2n+2}}(t)) = 0$ for all $t > 0$.

Thus, we have

$$\lim_{n\to\infty} g(F_{y_n, y_{n+1}}(t)) = 0 \text{ for all } t > 0.$$

On the other hand, if $g(F_{y_{2n-1}, y_{2n}}(t)) \ge g(F_{y_{2n}, y_{2n+1}}(t))$, then by (3.1.5), we have

$$g(F_{y_{2n}, y_{2n+1}}(t)) \le \phi(g(F_{y_{2n-1}, y_{2n}}(t))) \text{ for all } t > 0.$$

Similarly, $g(F_{y_{2n+1}, y_{2n+2}}(t)) \le \phi(g(F_{y_{2n}, y_{2n+1}}(t)))$ for all $t > 0$.

Thus, we have $g(F_{y_n, y_{n+1}}(t)) \le \phi(g(F_{y_{n-1}, y_n}(t)))$ for all $t > 0$ & $n = 1, 2, 3, \dots$.
Therefore, by Lemma 2.1,

$\lim\limits_{n\to\infty} g(F_{y_n, y_{n+1}}(t)) = 0$ for all $t > 0$, which implies that $\{y_n\}$ is a Cauchy sequence in X by Lemma 2.2.

Since $(X, \mathbf{F}, \Delta)$ is complete, the sequence $\{y_n\}$ converges to a point $z \in X$. Also its subsequences converges as follows:

(3.1.7) $\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$,
(3.1.8) $\{Lx_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$.

Case I. L is continuous and (L, AB) is semi-compatible, we get

$L(AB)x_{2n} \to Lz$ and $L(ABx)_{2n} \to ABz$

Since the limit in Non-Archimedean Menger PM space is unique, we get

$Lz = ABz$.

Step 2. Putting $x = z$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$g(F_{Lz,Mx_{2n+1}}(t)) \le \phi(\max\{g(F_{ABz,STx_{2n+1}}(t)),\ g(F_{ABz,\,Lz}(t)),\ g(F_{STx_{2n+1},\,Mx_{2n+1}}(t)),\ \tfrac{1}{2}(g(F_{ABz,\,Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},\,Lz}(t)))\}).$$

Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{Lz,z}(t)) &\le \phi(\max\{g(F_{z,z}(t)),\ g(F_{z,\,Lz}(t)),\ g(F_{z,\,z}(t)),\ \tfrac{1}{2}(g(F_{z,\,z}(t)) + g(F_{z,\,Lz}(t)))\}) \\ &= \phi(g(F_{Lz,z}(t))), \end{aligned}$$

which implies that $g(F_{Lz,z}(t)) = 0$ by Lemma 2.1 and so we have $Lz = z$.
Thus, we have $Lz = z = ABz$.

Step 3. Putting $x = Bz$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$g(F_{LBz,Mx_{2n+1}}(t)) \le \phi(\max\{g(F_{ABBz,STx_{2n+1}}(t)),g(F_{ABBz,\,LBz}(t)),g(F_{STx_{2n+1},Mx_{2n+1}}(t)),\ \tfrac{1}{2}(g(F_{ABBz,\,Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},\,LBz}(t)))\})$$

As $BL = LB$, $AB = BA$, so we have

$$L(Bz) = B(Lz) = Bz \quad\text{and}\quad AB(Bz) = B(ABz) = Bz.$$

Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{Bz,z}(t)) &\le \phi(\max\{g(F_{Bz,z}(t)),g(F_{Bz,\,Bz}(t)),g(F_{z,z}(t)),\ \tfrac{1}{2}(g(F_{Bz,\,z}(t)) + g(F_{z,\,Bz}(t)))\}) \\ &= \phi(g(F_{Bz,z}(t))) \end{aligned}$$

which implies that $g(F_{Bz,z}(t)) = 0$ by Lemma 2.1 and so we have $Bz = z$.

Also, $ABz = z$ and so $Az = z$.
Therefore, $Az = Bz = Lz = z$.

Step 4. As $L(X) \subset ST(X)$, there exists $v \in X$ such that $z = Lz = STv$.

Putting $x = x_{2n}$ and $y = v$ for $t > 0$ in (3.1.5), we get

$$g(F_{Lx_{2n},Mv}(t)) \le \phi(\max\{g(F_{ABx_{2n},STv}(t)),\ g(F_{ABx_{2n},\,Lx_{2n}}(t)),\ g(F_{STv,\,Mv}(t)),\ \tfrac{1}{2}(g(F_{ABx_{2n},\,Mv}(t)) + g(F_{STv,\,Lx_{2n}}(t)))\}).$$

Letting $n \to \infty$ and using equation (3.1.8), we get

$$\begin{aligned} g(F_{z,Mv}(t)) &\le \phi(\max\{g(F_{z,z}(t)),\ g(F_{z,\,z}(t)),\ g(F_{z,\,Mv}(t)),\ \tfrac{1}{2}(g(F_{z,\,Mv}(t)) + g(F_{z,\,z}(t)))\}) \\ &= \phi(g(F_{z,Mv}(t))) \end{aligned}$$

which implies that $g(F_{z,Mv}(t)) = 0$ by Lemma 2.1 and so we have $z = Mv$.

Hence, $STv = z = Mv$.

As (M, ST) is occasionally weakly compatible, we have

$$STMv = MSTv.$$

Thus, STz = Mz.

Step 5. Putting $x = x_{2n}$, $y = z$ for $t > 0$ in (3.1.5), we get

$$g(F_{Lx_{2n},Mz}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STz}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STz, Mz}(t)), \tfrac{1}{2}(g(F_{ABx_{2n}, Mz}(t)) + g(F_{STz, Lx_{2n}}(t)))\}).$$

Letting $n \to \infty$ and using equation (3.1.8) and Step 5, we get

$$\begin{aligned} g(F_{z,Mz}(t)) &\leq \phi(\max\{g(F_{z,Mz}(t)), g(F_{z, z}(t)), g(F_{Mz, Mz}(t)), \tfrac{1}{2}(g(F_{z, Mz}(t)) + g(F_{Mz, z}(t)))\}) \\ &= \phi(g(F_{z,Mz}(t))) \end{aligned}$$

which implies that $g(F_{z,Mz}(t)) = 0$ by Lemma 2.1 and so we have $z = Mz$.

Step 6. Putting $x = x_{2n}$ and $y = Tz$ for $t > 0$ in (3.1.5), we get

$$g(F_{Lx_{2n},MTz}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STTz}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STTz, MTz}(t)), \tfrac{1}{2}(g(F_{ABx_{2n}, MTz}(t)) + g(F_{STTz, Lx_{2n}}(t)))\})$$

As MT = TM and ST = TS we have
$MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$.

Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{z,Tz}(t)) &\leq \phi(\max\{g(F_{z,Tz}(t)), g(F_{z, z}(t)), g(F_{Tz, Tz}(t)), \tfrac{1}{2}(g(F_{z, Tz}(t)) + g(F_{Tz, z}(t)))\}) \\ &= \phi(g(F_{z,Tz}(t))), \end{aligned}$$

which implies that $g(F_{z,Tz}(t)) = 0$ by Lemma 2.1 and so we have $z = Tz$.

Now $STz = Tz = z$ implies $Sz = z$.
Hence $Sz = Tz = Mz = z$. (3.1.10)

Combining, we get

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Case II. AB is continuous.
As AB is continuous and (L, AB) is semi-compatible, we get

$(AB)^2x_{2n} \to ABz$, $(AB)Lx_{2n} \to ABz$.
and $L(AB)x_{2n} \to ABz$.

Thus, $(AB)Lx_{2n} = L(AB)x_{2n} = z$ as $n \to \infty$.
Now, we prove $ABz = z$.

Step 7. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$g(F_{LABx_{2n},Mx_{2n+1}}(t)) \le \phi(\max\{g(F_{ABABx_{2n},STx_{2n+1}}(t)), g(F_{ABABx_{2n},LABx_{2n}}(t)), g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{ABABx_{2n},Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},LABx_{2n}}(t)))\}).$$

Letting $n \to \infty$, we get

$$g(F_{ABz,z}(t)) \le \phi(\max\{g(F_{ABz,z}(t)), g(F_{ABz,ABz}(t)), g(F_{z,z}(t)), \tfrac{1}{2}(g(F_{ABz,z}(t)) + g(F_{z,ABz}(t)))\}) = \phi(g(F_{ABz,z}(t)))$$

which implies that $g(F_{ABz,z}(t)) = 0$ by Lemma 2.1 and so we have $ABz = z$.

Step 8. Putting $x = z$ and $y = x_{2n+1}$ for $t > 0$ in (3.1.5), we get

$$g(F_{Lz,Mx_{2n+1}}(t)) \le \phi(\max\{g(F_{ABz,STx_{2n+1}}(t)), g(F_{ABz,Lz}(t)), g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{ABz,Mx_{2n+1}}(t)) + g(F_{STx_{2n+1},Lz}(t)))\}).$$

Letting $n \to \infty$, we get

$$g(F_{Lz,z}(t)) \le \phi(\max\{g(F_{z,z}(t)), g(F_{z,Lz}(t)), g(F_{z,z}(t)), \tfrac{1}{2}(g(F_{z,z}(t)) + g(F_{z,Lz}(t)))\}) = \phi(g(F_{Lz,z}(t)))$$

which implies that $g(F_{Lz,z}(t)) = 0$ by Lemma 2.1 and so we have $Lz = z$.

Therefore, $ABz = Lz = z$.

Futher, using step 3, we get

$Bz = z$, so $Az = Bz = Lz = z$.

Also, it follows from steps 4, 5 and 6 that

$Sz = Tz = Mz = z$.

Hence, all $Az = Bz = Lz = Sz = Tz = Mz = z$, i.e z is a common fixed point of A, B, S, T, L and M.

Step 9. (Uniqueness) Let u be another common fixed point of A, B, S, T, L and M; then $Au = Bu = Su = Tu = Lu = Mu = u$.

Putting $x = z$ and $y = u$ for $t > 0$ in (3.1.5), we get

$$g(F_{Lz,Mu}(t)) \le \phi(\max\{g(F_{ABz,STu}(t)), g(F_{ABz,Lz}(t)), g(F_{STu,Mu}(t)), \tfrac{1}{2}(g(F_{ABz,Mu}(t)) + g(F_{STu,Lz}(t)))\}).$$

Letting $n \to \infty$, we get

$$g(F_{z,u}(t)) \le \phi(\max\{g(F_{z,u}(t)), g(F_{z,z}(t)), g(F_{u,u}(t)), \tfrac{1}{2}(g(F_{z,u}(t)) + g(F_{u,z}(t)))\}) = \phi(g(F_{z,u}(t))),$$

which implies that $g(F_{z,u}(t)) = 0$ by Lemma 2.1 and so we have $z = u$.

Therefore, z is a unique common fixed point of A, B, S, T, L and M.
This completes the proof.

Remark 3.1. If we take B = T = I, the identity map on X in Theorem 3.1, then the condition (3.1.2) is satisfied trivially and we get

Corollary 3.1. Let A, S, L, M: $X \rightarrow X$ be mappings satisfying the condition:

(3.1.11) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;
(3.1.12) Either A or L is continuous;
(3.1.13) the pair (L, A) is semi-compatible and (M, S) is occasionally weakly compatible;
(3.1.14) $g(F_{Lx,My}(t)) \leq \phi(\max\{g(F_{Ax,Sy}(t)), g(F_{Ax, Lx}(t)), g(F_{Sy, My}(t)), \frac{1}{2}(g(F_{Ax, My}(t)) + g(F_{Sy, Lx}(t)))\})$

for all t > 0, where a function $\phi : [0,+\infty) \rightarrow [0,+\infty)$ satisfies the condition (Φ). Then A, S, L and M have a unique common fixed point in X.

Remark 3.2. In view of Remark 3.1, Corollary 3.1 is a generalization of the result of Cho et al. [10] in the sense that condition of compatibility of the pairs of self maps has been restricted to semi-compatible and occasionally weakly compatible self maps and only one of the mappings of the first pair is needed to be continuous.

Corollary 3.2. Let A, S, L, M : $X \rightarrow X$ be mappings satisfying the condition :

(3.1.15) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;
(3.1.16) Either A or L is continuous;
(3.1.17) the pair (L, A) is semi-compatible and (M, S) is weakly compatible;
(3.1.18) $g(F_{Lx,My}(t)) \leq \phi(\max\{g(F_{Ax,Sy}(t)), g(F_{Ax, Lx}(t)), g(F_{Sy, My}(t)), \frac{1}{2}(g(F_{Ax, My}(t)) + g(F_{Sy, Lx}(t)))\})$

for all t > 0, where a function $\phi : [0,+\infty) \rightarrow [0,+\infty)$ satisfies the condition (Φ).

Then A, S, L and M have a unique common fixed point in X.

Proof. Since weak compatibility implies occasionally weak compatibility, the proof follows from Corollary 3.1.

4 Conclusion

The concept of semi-compatible and occasionally weakly compatible mappings, which are more general than the concept of compatible mappings, has been used to prove a common fixed point theorem. The theorem thus obtained is a generalization of the result of Cho et al. [10] in a non-Archimedean Menger PM-space.

Acknowledgement

Authors are thankful to the referees for their valuable comments in the preparation of this article.

Competing Interests

Authors have declared that no competing interests exist.

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