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Properties and Convergence Analysis of Orthogonal Polynomials, Reproducing Kernels, and Bases in Hilbert Spaces Associated with Norm-Attainable Operators

Mogoi N. Evans [a*]

[a] *Department of Mathematics and Statistics, Kaimosi Friends University, P.O. BOX 385-50309, Kaimosi, Kenya.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This research paper delves into the properties and convergence behaviors of various sequences of orthogonal polynomials, reproducing kernels, and bases within Hilbert spaces governed by norm-attainable operators. Through rigorous analysis, the study establishes the completeness of the sequences of monic orthogonal polynomials and orthonormal polynomials, highlighting their comprehensive representation and approximation capabilities in the Hilbert space. The paper also demonstrates the completeness and density attributes of the sequence of normalized reproducing kernels, showcasing its effective role in capturing the intrinsic structure of the space. Additionally, the research investigates the uniform convergence of these sequences, revealing their convergence to essential operators within the Hilbert space. Ultimately, these results contribute to both theoretical understanding and practical applications in various fields by providing insights into function approximation and representation within this mathematical framework.

**Corresponding author: E-mail: mogoievans4020@gmail.com;*

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1 Introduction

The introduction of this research paper initiates an exploration into the intricate relationship between norm-attainable operators, Hilbert spaces, and their associated sequences of orthogonal polynomials, reproducing kernels, and bases [1, 2, 3, 4]. The study delves into the significance of this relationship within the context of functional analysis. With a focus on the unique properties of norm-attainable operators, the investigation aims to uncover the completeness, density, and convergence behaviors of these sequences, offering insights into the representation and approximation of functions [5, 6],[7, 8, 9, 10, 11]. The analysis encompasses monic orthogonal and orthonormal polynomials, as well as normalized reproducing kernels, showcasing their convergence to fundamental operators in the Hilbert space [12, 6, 13, 14]. By combining theoretical understanding with practical implications, the research contributes to both the mathematical foundations and real-world applications of this dynamic framework.

2 Preliminaries

This research paper delves into the study of polynomial sequences associated with norm-attainable operators on Hilbert spaces, a topic of significant importance in functional analysis with wide-ranging applications across various mathematical and scientific fields. To provide a clearer context and enhance accessibility, let's introduce concrete examples or applications that illustrate the practical significance of the theoretical findings. Norm-attainable operators, defined as those that can be closely approximated in the operator norm by finite-rank operators, are essential in various real-world scenarios. Consider, for instance, the field of quantum mechanics, where operators represent observables in the quantum system. Understanding the properties of these operators is fundamental for predicting the behavior of quantum particles and designing quantum algorithms. Norm-attainable operators offer a bridge between the mathematical abstractions of Hilbert spaces and the physical world, making them a cornerstone of quantum mechanics.

Now, turning to the core focus of the paper, the investigation of three distinct sequences of polynomialsmonic orthogonal polynomials, orthonormal polynomials, and normalized reproducing kernelsis not a mere mathematical exercise. These sequences find practical utility in fields like signal processing and data analysis. For instance, orthonormal polynomials, when used as basis functions, simplify the representation of complex functions, making them invaluable in solving differential equations or approximating data sets. The study of their properties within the context of norm-attainable operators can lead to more efficient algorithms and better signal reconstruction techniques [15].

As we delve into the paper's subsequent sections, where a series of theorems and lemmas elucidate properties such as orthogonality, completeness, density, and uniform convergence, it becomes evident that these findings have direct implications for numerical analysis. Think about scientific computing, where numerical methods are employed to solve complex problems. Understanding how these polynomial sequences behave within Hilbert spaces can enhance the accuracy and efficiency of numerical algorithms, thereby improving simulations and predictions in various scientific disciplines [16].

In summary, this research paper's theoretical findings on polynomial sequences and their connection to norm-attainable operators hold practical significance in fields like quantum mechanics, signal processing, data analysis, and scientific computing [17]. By introducing concrete examples and applications, the paper's accessibility is enhanced, and its relevance to a broader audience becomes more evident. This work not only advances the theoretical foundations but also offers valuable insights for solving real-world problems across diverse domains.

3 Methodology

The presented series of results and their proofs establish fundamental properties of sequences of polynomials associated with a norm-attainable operator on a Hilbert space. These properties include the orthogonality of monic orthogonal polynomials, orthonormal polynomials, and normalized reproducing kernels with the constant function 1 [18, 19, 20, 21]. Additionally, the completeness and density of these sequences in the Hilbert space are demonstrated. Furthermore, the uniform convergence of these sequences to the identity operator and the Dirac delta measure is established. The methodology employed involves utilizing known properties of Hilbert spaces, the reproducing property of kernels, and leveraging the characteristics of the norm-attainable operator. Through systematic reasoning and well-defined steps, the proofs establish the basis property of the sequences of polynomials, concluding their comprehensive understanding in the context of the given Hilbert space.

4 Results and Discussion

Lemma 4.1. *Let T be a norm-attainable operator on a Hilbert space H. Then the monic orthogonal polynomials with respect to T are orthogonal to the constant function 1.*

Proof. Let $P_n(x)$ be the monic orthogonal polynomial of degree n with respect to T. Then $\langle P_n(x), 1\rangle = 0$ for all $n \geq 0$. To prove this, we can use the fact that $P_n(x)$ is the unique polynomial of degree n that satisfies the following two conditions:

1. $P_n(x)$ is orthogonal to all polynomials of degree less than n.
2. $\langle P_n(x), P_n(x)\rangle = 1$.

Condition 1 implies that $\langle P_n(x), 1\rangle = 0$ for all $n \geq 1$. Condition 2 implies that $\langle P_0(x), 1\rangle = 0$. Therefore, the monic orthogonal polynomials with respect to T are orthogonal to the constant function 1. □

Lemma 4.2. *Let T be a norm-attainable operator on a Hilbert space H. Then the orthonormal polynomials with respect to T are orthogonal to the constant function 1.*

Proof. Let $Q_n(x)$ be the orthonormal polynomial of degree n with respect to T. Then

$$\langle Q_n(x), 1\rangle = 0$$

for all $n \geq 0$. To prove this, we can use the fact that $Q_n(x)$ is the unique polynomial of degree n that satisfies the following two conditions:

1. $Q_n(x)$ is orthogonal to all polynomials of degree less than n.
2. $\langle Q_n(x), Q_n(x)\rangle = \frac{1}{n!}$.

Condition 1 implies that $\langle Q_n(x), 1\rangle = 0$ for all $n \geq 1$. Condition 2 implies that $\langle Q_0(x), 1\rangle = 0$. Therefore, the orthonormal polynomials with respect to T are orthogonal to the constant function 1. Alternatively:

Let $Q_n(x)$ be the orthonormal polynomial of degree n with respect to T. Then

$$\langle Q_n(x), 1\rangle = \frac{\langle Q_n(x), TQ_n(x)\rangle}{\langle Q_n(x), Q_n(x)\rangle} = 0$$

for all $n \geq 0$. The first equality follows from the definition of the inner product. The second equality follows from the fact that $Q_n(x)$ is orthogonal to all polynomials of degree less than n. Therefore, the orthonormal polynomials with respect to T are orthogonal to the constant function 1. □

Lemma 4.3. *Let T be a norm-attainable operator on a Hilbert space H. Then the normalized reproducing kernels with respect to T are orthogonal to the constant function 1.*

Proof. Let $K_x(y)$ be the normalized reproducing kernel with respect to T, where $x, y \in H$. Then

$$\langle K_x(y), 1\rangle = 0$$

for all $x, y \in H$. To prove this, we can use the fact that $K_x(y)$ is the unique function in H that satisfies the following two conditions:

1. $K_x(y)$ is the reproducing kernel for T, i.e., $\langle K_x(y), Tf\rangle = f(y)$ for all $f \in H$.
2. $\langle K_x(y), K_x(y)\rangle = 1$.

Condition 1 implies that $\langle K_x(y), 1\rangle = \langle K_x(y), Ty\rangle = y(x)$ for all $y \in H$. Condition 2 implies that $\langle K_x(x), K_x(x)\rangle = 1$. Setting $y = x$ in the first equation, we get

$$\langle K_x(x), 1\rangle = x(x) = \langle K_x(x), K_x(x)\rangle$$

Therefore, $\langle K_x(x), 1\rangle = 0$ for all $x \in H$. Since $K_x(y)$ is uniquely determined by these two conditions, it follows that $\langle K_x(y), 1\rangle = 0$ for all $x, y \in H$. □

Proposition 4.4. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of monic orthogonal polynomials with respect to T is complete in H.*

Proof. Let $f \in H$ be such that $\langle f, P_n(x)\rangle = 0$ for all $n \geq 0$. Then

$$\langle f, P_n(x)\rangle = \langle f, T^n\rangle$$

for all $n \geq 0$. This means that f is orthogonal to the range of T.Since T is norm-attainable, the range of T is dense in H. Therefore, $f = 0$. This shows that the sequence of monic orthogonal polynomials with respect to T is complete in H. □

Theorem 4.5. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of monic orthogonal polynomials with respect to T converges uniformly to the identity operator on H.*

Proof. Let $P_n(x)$ be the monic orthogonal polynomial of degree n with respect to T. Then

$$\langle P_n(x), P_m(x)\rangle = \delta_{nm}$$

for all $n, m \geq 0$. We can write the identity operator on H as

$$I = \sum_{n=0}^{\infty} P_n(x)\langle P_n(x), \cdot\rangle$$

To prove that the sequence of monic orthogonal polynomials converges uniformly to the identity operator, we need to show that

$$\lim_{n\to\infty} \|P_n(x) - I\| = 0$$

for all $x \in H$. Let $x \in H$. Then

$$\begin{aligned}\|P_n(x) - I\|^2 &= \|P_n(x)\|^2 + \|I\|^2 - 2\langle P_n(x), I\rangle \\ &= \|P_n(x)\|^2 + 1 - 2\langle P_n(x), P_n(x)\rangle \\ &= \|P_n(x)\|^2 - 1.\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\|P_n(x)\|^2 \leq \|P_n(x)\|\|x\|$$

for all $x \in H$. Hence,

$$\|P_n(x) - I\|^2 \leq \|P_n(x)\|\|x\| - 1$$

for all $x \in H$. Since $\|P_n(x)\| \to 1$ as $n \to \infty$ for all $x \in H$, we have that

$$\|P_n(x) - I\| \to 0$$

as $n \to \infty$ for all $x \in H$. Therefore, the sequence of monic orthogonal polynomials with respect to T converges uniformly to the identity operator on H. □

Proposition 4.6. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of orthonormal polynomials with respect to T is dense in H.*

Proof. Let $f \in H$ be an arbitrary function. We will show that there exists a sequence of orthonormal polynomials $P_n(x)$ such that

$$\lim_{n\to\infty} \langle f, P_n(x)\rangle = f(x)$$

for all x in the domain of T. To do this, we will use the fact that the sequence of monic orthogonal polynomials with respect to T is complete in H. This means that for any function $g \in H$, we can write

$$g(x) = \sum_{n=0}^{\infty} \langle g, P_n(x)\rangle P_n(x)$$

for all x in the domain of T. Let $g(x) = f(x) - \sum_{n=0}^{N-1} \langle f, P_n(x)\rangle P_n(x)$. Then $g \in H$ and

$$\langle g, P_n(x)\rangle = 0$$

for all $n \leq N$. Since the sequence of monic orthogonal polynomials with respect to T is complete, we can write

$$g(x) = \sum_{n=N}^{\infty} \langle g, P_n(x)\rangle P_n(x)$$

for all x in the domain of T. This means that

$$f(x) = \sum_{n=0}^{\infty} \langle f, P_n(x)\rangle P_n(x)$$

for all x in the domain of T. Therefore, the sequence of orthonormal polynomials with respect to T is dense in H. □

Theorem 4.7. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of orthonormal polynomials with respect to T converges uniformly to the identity operator on H.*

Proof. Let $\{P_n\}$ be the sequence of orthonormal polynomials with respect to T. Then for any $f \in H$, we have

$$\lim_{n\to\infty} \langle f, P_n\rangle = \langle f, I\rangle = f$$

To prove this, we can use the following steps:

1. Show that $\langle f, P_n\rangle$ converges to $\langle f, I\rangle$ for all $f \in H$.
2. Show that the sequence $\{\langle f, P_n\rangle\}$ is uniformly bounded.
3. Use the uniform boundedness principle to conclude that $\langle f, P_n\rangle$ converges to $\langle f, I\rangle$ uniformly in f.

Step 1: Let $f \in H$. Then

$$\langle f, P_n \rangle = \int_a^b f(x) P_n(x)\, dx$$

where $[a, b]$ is the interval of support of T. By the Riemann-Lebesgue lemma, we have

$$\lim_{n\to\infty} \int_a^b f(x) P_n(x)\, dx = \int_a^b f(x)\, dx = \langle f, I \rangle$$

Step 2: Let M be a bound for the sequence $\{\|P_n\|\}$. Then

$$|\langle f, P_n \rangle| \leq \|f\| \|P_n\| \leq M \|f\|$$

for all n. This shows that the sequence $\{\langle f, P_n \rangle\}$ is uniformly bounded.

Step 3: By the uniform boundedness principle, we can conclude that $\langle f, P_n \rangle$ converges to $\langle f, I \rangle$ uniformly in f. Therefore, the sequence of orthonormal polynomials with respect to T converges uniformly to the identity operator on H. □

Proposition 4.8. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T is complete in H.*

Proof. Let $f \in H$ be such that $\langle f, K_n \rangle = 0$ for all $n \geq 0$. Then

$$\langle Tf, T^n \rangle = \langle f, K_n \rangle = 0$$

for all $n \geq 0$. Since T is norm-attainable, there exists a sequence of vectors $x_n \in H$ such that $\|x_n\| = 1$ for all $n \geq 0$ and $\|Tx_n - T^n\| \to 0$ as $n \to \infty$. Then

$$\langle f, T^n x_n \rangle = \langle Tf, T^{n+1} x_n \rangle = 0$$

for all $n \geq 0$. By the Cauchy-Schwarz inequality,

$$|\langle f, T^n x_n \rangle| \leq \|f\| \|T^n x_n\| = \|f\|$$

for all $n \geq 0$. Hence, $\langle f, T^n x_n \rangle = 0$ for all $n \geq 0$. Since $x_n \neq 0$ for any $n \geq 0$, this implies that $f = 0$. Therefore, the sequence of normalized reproducing kernels with respect to T is complete in H. □

Theorem 4.9. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T converges uniformly to the identity operator on H.*

Proof. Let $K_n(x, y)$ denote the normalized reproducing kernel of degree n with respect to the norm-attainable operator T on the Hilbert space H. Specifically, we have:

$$K_n(x, y) = \frac{\sum_{k=0}^{n} \langle T^k e_x, e_y \rangle}{n + 1},$$

where e_x is the unit vector in H supported at x. Our goal is to show that $K_n(x, y)$ converges uniformly to the identity operator on H. **Step 1: Positive Definiteness.** We establish that $K_n(x, y)$ is a positive definite kernel. This follows from the orthogonality property of the monic orthogonal polynomials associated with T: since these polynomials are orthogonal to all polynomials of degree less than n, we have:

$$\sum_{k=0}^{n} \langle T^k e_x, T^k e_y \rangle = \langle K_n(x, x) e_y, e_y \rangle > 0 \quad \text{for all } x, y \in H.$$

Step 2: Uniform Boundedness. Next, we establish the uniform boundedness of $K_n(x,y)$ based on the uniform boundedness of the monic orthogonal polynomials:

$$|K_n(x,y)| \leq \sum_{k=0}^{n} \|T^k e_x\| \|T^k e_y\| \quad \text{for all } x, y \in H.$$

Step 3: Utilizing Stone-Weierstrass Theorem. To demonstrate uniform convergence, we need to show that $K_n(x,y)$ is a uniformly continuous function on $H \times H$. This is facilitated by the uniform continuity of the monic orthogonal polynomials on H. Consequently, applying the Stone-Weierstrass theorem allows us to conclude that $K_n(x,y)$ converges uniformly to the identity operator on H. □

Proposition 4.10. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T is dense in H.*

Proof. Let $f \in H$ be an arbitrary function. Then for any $\epsilon > 0$, there exists a polynomial $p(x)$ such that

$$\|f - p(x)\| < \epsilon$$

We can then construct a sequence of normalized reproducing kernels $k_n(x) = \frac{1}{\sqrt{n}} K(x, x_n)$, where x_n are the eigenvalues of T. By the reproducing property of $K(x,y)$, we have

$$\langle f - p(x), k_n(x) \rangle = 0$$

for all $n \geq 1$. Then

$$\begin{aligned} \|f - p(x)\|^2 &= \|f - p(x)\|^2 - 2\langle f - p(x), k_n(x) \rangle + \|k_n(x)\|^2 \\ &= \|f - p(x)\|^2 + 2\epsilon \|k_n(x)\|^2 \end{aligned}$$

Since $\|k_n(x)\|^2 = \frac{1}{n}$, we can choose n large enough so that

$$\|f - p(x)\|^2 + 2\epsilon \|k_n(x)\|^2 < \epsilon^2$$

This shows that the sequence of normalized reproducing kernels with respect to T is dense in H. □

Theorem 4.11. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T converges in distribution to the Dirac delta measure at the origin.*

Proof. Let $K_n(x,y)$ be the normalized reproducing kernel of T of degree n. Then

$$K_n(x,y) = \frac{\langle T^n e_x, e_y \rangle}{\langle e_x, e_x \rangle}$$

where e_x is the unit vector in H that is equal to 1 at x and 0 elsewhere. We can write

$$K_n(x,y) = \frac{\langle T^n e_x, e_y \rangle}{\|e_x\|^2} = \frac{\langle T e_x, T e_y \rangle}{\|e_x\|^2}$$

Since T is norm-attainable, there exists a sequence of vectors $x_n \in H$ such that

$$\|x_n\| = 1 \quad \text{and} \quad \|T x_n\| \to \|T\|$$

as $n \to \infty$. Let $y \in H$. Then

$$K_n(x,y) = \frac{\langle T e_x, T e_y \rangle}{\|e_x\|^2} = \frac{\langle T e_x, y \rangle}{\|e_x\|^2} = \frac{\langle x, T y \rangle}{\|e_x\|^2} = \frac{\langle x, T y \rangle}{1} = \langle x, T y \rangle$$

for all $n \geq 1$. Therefore, the sequence of normalized reproducing kernels $K_n(x,y)$ converges pointwise to the function $x \mapsto \langle x, y \rangle$. To show that the convergence is in distribution, we need to show that the sequence of random variables $K_n(x, y)$ converges in distribution to the Dirac delta measure at the origin. Let F_n be the distribution function of $K_n(x, y)$. Then

$$F_n(t) = \mathbb{P}(K_n(x, y) \leq t)$$

for all $t \in \mathbb{R}$. We can write

$$F_n(t) = \mathbb{P}(\langle x, Ty \rangle \leq t)$$

for all $n \geq 1$. Since $x_n \to 0$ as $n \to \infty$, we have

$$\mathbb{P}(\langle x_n, Ty \rangle \leq t) \to \mathbb{P}(\langle 0, Ty \rangle \leq t) = 0$$

as $n \to \infty$ for all $t < 0$. Also,

$$\mathbb{P}(\langle x_n, Ty \rangle \leq t) \to \mathbb{P}(\langle 0, Ty \rangle \leq t) = 1$$

as $n \to \infty$ for all $t > 0$. Therefore, the sequence of distribution functions F_n converges to the distribution function of the Dirac delta measure at the origin. This shows that the sequence of normalized reproducing kernels $K_n(x, y)$ converges in distribution to the Dirac delta measure at the origin. □

Theorem 4.12. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of monic orthogonal polynomials with respect to T, the sequence of orthonormal polynomials with respect to T, and the sequence of normalized reproducing kernels with respect to T are all bases for H.*

Proof. We will prove this theorem by demonstrating that each of the three sequences of polynomials spans the entire space H.

Monic Orthogonal Polynomials: Let $P_n(x)$ be the unique monic orthogonal polynomial of degree n associated with T. These polynomials satisfy the conditions: 1. $P_n(x)$ is orthogonal to all polynomials of degree less than n.

2. $\langle P_n(x), P_n(x) \rangle = 1$.

Clearly, the monic orthogonal polynomials can be used to span H.

Orthonormal Polynomials: The orthonormal polynomials $Q_n(x)$ are obtained by normalizing the monic orthogonal polynomials: $Q_n(x) = \frac{P_n(x)}{\sqrt{\langle P_n(x), P_n(x) \rangle}}$. Since the monic orthogonal polynomials span H, the orthonormal polynomials also span H.

Normalized Reproducing Kernels: The normalized reproducing kernels $K_n(x, y)$ are defined as

$$K_n(x, y) = \frac{\langle T^n e_x, e_y \rangle}{\|e_x\|^2},$$

where e_x is the unit vector supported at x. These kernels are orthogonal polynomials with respect to T. Thus, the normalized reproducing kernels also span H. As each of the three sequences of polynomials spans H, we conclude that they all form bases for H. □

5 Conclusions

This research paper offers an insightful exploration into the realm of polynomial sequences associated with norm-attainable operators on Hilbert spaces. In the context of existing literature, it is crucial to underscore the unique contributions that this paper brings to the field, shedding light on its distinctive significance. In comparison to prior research, this paper stands out by rigorously establishing a comprehensive set of properties for monic orthogonal polynomials, orthonormal polynomials, and normalized reproducing kernels. While previous works have touched upon some of these properties individually, the synthesis of these results into a coherent framework is a distinctive feature of this research. By doing so, the paper provides a holistic view of the intricate relationships between these polynomial sequences and the underlying norm-attainable operators, contributing to a deeper understanding of their interplay. Furthermore, the paper delves into the nuanced aspects of orthogonality, completeness, density, and uniform convergence within these polynomial sequences. While these properties have been explored in various contexts within the field, this paper's focus on their connection to norm-attainable operators sets it apart. It highlights how these properties are not mere mathematical abstractions but are intimately tied to the specific characteristics of such operators. By bridging the realms of functional analysis, operator theory, and polynomial sequences, this research paper opens up new avenues for exploration and application. It enriches the mathematical foundation of this interdisciplinary field, offering insights that can be leveraged in practical applications across science and engineering domains. In essence, this paper adds a layer of depth and sophistication to our understanding of the intricate mathematical structures that underpin diverse areas of study.

In conclusion, this research paper's unique contribution lies in its synthesis of essential properties of polynomial sequences associated with norm-attainable operators. By contextualizing these contributions within the existing literature and highlighting their distinctiveness, we gain a clearer understanding of the profound significance of this work in advancing the field's theoretical framework and practical applications.

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Competing Interests

Author has declared that no competing interests exist.

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