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Fixed Point and Coincidence Point Theorems in Dualistic Partial Metric Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, motivated by Fulga and Proca [1], we define the notion of dualistic E-contraction, generalized dualistic E-contraction, and Dass-Gupta dualistic rational E-contraction. We establish some new fixed-point theorems for E-contraction, generalized dualistic E-contraction, and Dass-Gupta dualistic rational E-contraction in a DPM space. Also, we define dualistic E_Δ-contraction, generalized dualistic E_Δ-contraction, and Dass-Gupta dualistic rational E_Δ-contraction. We establish some common fixed-point theorems for E_Δ-contraction, generalized dualistic E_Δ-contraction and Dass-Gupta dualistic rational E_Δ-contraction in the setting of DPM spaces. Our results extend and generalize some well-known results of [1] and [2]. We also provide an example that shows the usefulness of these contractions.

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1 Introduction

During the past twenty years, one of the most active areas of study has been fixed point theory. Novel and captivating outcomes are attained, primarily in two aspects: altering the framework (the composition of the abstract space, such as the b-metric, delta symmetric quasi-metric, or non-symmetric metric space, among others) or modifying the characteristics of the operators.

The common idea of a metric space has been numerous times generalized. A partial metric (PM) space, which Matthews developed and examined, is one such generalization [3]. He verified the exact correspondence between the so-called weightable quasi-metric spaces and PM spaces. PM space has certain generalizations. One major modification to Matthews' definition of the PM, for instance, was suggested by O'Neill [4] and involved moving their range from $[0,\infty)$ to $(-\infty,\infty)$. Dualistic partial metric (DPM) is the term used to refer to the PM space in the O'Neill sense, and a pair $(\mathcal{C},\tau^*)$ here $\mathcal{C}$ is a nonempty set and τ^* is a DPM on $\mathcal{C}$ is called a DPM space, according to [4]. O'Neill established multiple links between PM space and the topological features of domain theory in this manner. Studying Banach's contraction principle is the first step in creating contractual requirements. Several fixed-point theorems for some generalized metric space have made use of these criteria. Fulga and Proca [1] established the idea of E-contraction. Several writers have since refined this idea, including [5, 6].

A coincidence point in mathematics is the location at which two or more functions coincide, or intersect, indicating that they have the same value at that particular position. Numerous areas of mathematics, such as algebra, differential equations, and calculus, are interested in coincidence points. They can be used to solve equations, comprehend the behavior of mathematical models, and solve optimization problems. A key finding in the theory of fixed-point theorems is the Coincidence Point Theorem, which establishes the circumstances in which two mappings share a fixed point. The Kakutani-Ky Fan Coincidence Theorem is another name for it, and it bears the names of the mathematicians Ky Fan and Shizuo Kakutani who independently proved it in the 1940s.

The theorem states the following:

Let $\mathcal{C}$ be a topological space and $\mathcal{A}$ a convex subset of a Hausdroff vector space. Let $\Upsilon,\Delta:\mathcal{C}\to\mathcal{C}$ be mappings, where Υ is upper semi-continuous and Δ is compact and continuous. If there exists a point $\theta\in\mathcal{C}$ such that $\Upsilon(\theta)\subseteq\Delta(\theta)$, then there exists a point $\vartheta\in\mathcal{C}$ such that $\Upsilon(\vartheta)=\Delta(\vartheta)$.

Aydi *et al*. [7] proved some coincidence and common fixed-point results in partially ordered cone metric spaces. Since then, there have been many results related to coincidence and common fixed-point, we refer to ([8], [9], [10], [11]) and references therein. Fixed point theorems for generalized contractions on partial metric spaces were established by Altun *et al*. [12]. Some fixed-point theorems in ordered dualistic partial metric spaces were introduced by Arshad *et al*. [13]. Certain fixed-point results for dualistic rational contractions were established by Nazam *et al*. [14]. Certain fixed-point outcomes in ordered dualistic partial metric spaces were proven by Nazam and Arshad [15]. Fixed point theorems for contractions with rational inequalities in the extended b-metric space were proved in 2019 by Alqahtani *et al*. [16]. Fixed point theorems for rational type contractions in extended b-metric spaces were demonstrated by Huang *et al*. [17]. Within the context of super metric spaces, Karapinar and Fulga [18] established fixed point theorems for contraction in rational forms.

In this paper, we shall propose three types of contraction, namely, E-contraction, generalized dualistic E-contraction and Dass-Gupta dualistic rational E-contraction, which combine the dualistic contraction approach, E-contraction and rational contraction setting and establish some fixed-point results in the framework of DPM spaces. Also, prove some common fixed-point results via dualistic E_Δ-contraction, generalized dualistic E_Δ-contraction and Dass-Gupta dualistic rational E_Δ-contraction in a DPM space. Our result extends and generalizes some well-known results of [2] and [1]. Also, we verify our results with an example.

2 Preliminaries

We recall some mathematical basics and definitions to make this paper self-sufficient.

***Definition* 2.1** (see [3]) Let $\mathcal{C}$ be a non-empty set. A partial metric (PM) on $\mathcal{C}$ is a function $\tau: \mathcal{C} \times \mathcal{C} \to [0, \infty)$ complying with following axioms, for all $\theta, \vartheta, \omega \in \mathcal{C}$

(τ_1) $\theta = \vartheta \Leftrightarrow \tau(\theta, \vartheta) = \tau(\theta, \theta) = \tau(\vartheta, \vartheta)$;
(τ_2) $\tau(\theta, \theta) \leq \tau(\theta, \vartheta)$;
(τ_3) $\tau(\theta, \vartheta) = \tau(\vartheta, \theta)$;
(τ_4) $\tau(\theta, \vartheta) \leq \tau(\theta, \omega) + \tau(\omega, \vartheta) - \tau(\omega, \omega)$

The pair $(\mathcal{C}, \tau)$ is called a partial metric space (PM space).

***Definition* 2.2** (see [4]) Let $\mathcal{C}$ be a non-empty set. A dualistic partial metric (DPM) on $\mathcal{C}$ is a function $\tau^*: \mathcal{C} \times \mathcal{C} \to (-\infty, \infty)$ satisfying the following axioms, for all $\theta, \vartheta, \omega \in \mathcal{C}$

(τ_1^*) $\theta = \vartheta \Leftrightarrow \tau^*(\theta, \vartheta) = \tau^*(\theta, \theta) = \tau^*(\vartheta, \vartheta)$;
(τ_2^*) $\tau^*(\theta, \theta) \leq \tau^*(\theta, \vartheta)$;
(τ_3^*) $\tau^*(\theta, \vartheta) = \tau^*(\vartheta, \theta)$;
(τ_4^*) $\tau^*(\theta, \omega) + \tau^*(\vartheta, \vartheta) \leq \tau^*(\theta, \vartheta) + \tau^*(\vartheta, \omega)$

The pair $(\mathcal{C}, \tau^*)$ is called a dualistic partial metric space (DPM space).

***Remark* 2.3** Noting that each PM is a DPM, but the converse is false. Indeed, define τ^* on $(-\infty, \infty)$ as $\tau^*(\theta, \vartheta) = \max\{\theta, \vartheta\}, \forall \theta, \vartheta \in (-\infty, \infty)$. Obviously, τ^* is a DPM on $(-\infty, \infty)$. Since $\tau^*(\theta, \vartheta) < 0 \notin [0, \infty), \forall\, \theta, \vartheta \in (-\infty, 0)$ and then τ^* is not a PM on $(-\infty, \infty)$. This confirms our remark.

***Example* 2.4** (see [19], [4])

(1) Define $\tau_d^*: \mathcal{C} \times \mathcal{C} \to (-\infty, \infty)$ by $\tau_d^*(\theta, \vartheta) = d(\theta, \vartheta) + b$, where d is a metric on a nonempty set $\mathcal{C}$ and $b \in (-\infty, \infty)$ is arbitrary constant, then it is easy to check that τ_d^* verifies axioms $(\tau_1^*) - (\tau_4^*)$ and hence $(\mathcal{C}, \tau^*)$ is a DPM space.
(2) Let τ be a PM defined on a non-empty set $\mathcal{C}$. The function $\tau^*: \mathcal{C} \times \mathcal{C} \to (-\infty, \infty)$ defined by $\tau^*(\theta, \vartheta) = \tau(\theta, \vartheta) - \tau(\theta, \theta) - \tau(\vartheta, \vartheta)$ satisfies the axioms $(\tau_1^*) - (\tau_4^*)$ and so it defines a DPM on $\mathcal{C}$. Note that $\tau^*(\theta, \vartheta)$ may have negative values.
(3) Let $\mathcal{C} = (-\infty, \infty)$. Define $\tau^*: \mathcal{C} \times \mathcal{C} \to (-\infty, \infty)$ by $\tau^*(\theta, \vartheta) = |\theta - \vartheta|$ if $\theta \neq \vartheta$ and $\tau^*(\theta, \vartheta) = -\gamma$ if $\theta = \vartheta$ and $\gamma > 0$. We can easily see that τ^* is a DPM on $\mathcal{C}$.

O'Neill [4] established that each DPM τ^* on $\mathcal{C}$ generates a $\mathcal{T}_0$ topology $\tau(\tau^*)$ on $\mathcal{C}$ having a base, the family of τ^*-balls $\{\mathcal{B}_{\tau^*}(\theta, \epsilon) \,|\, \theta \in \mathcal{C}, \epsilon > 0\}$, where

$$\mathcal{B}_{\tau^*}(\theta, \epsilon) = \{\vartheta \in \mathcal{C} \,|\, \tau^*(\theta, \vartheta) < \tau^*(\theta, \theta) + \epsilon\}. \tag{2.1}$$

If $(\mathcal{C}, \tau^*)$ is a DPM space, then the function $d_{\tau^*}: \mathcal{C} \times \mathcal{C} \to [0, \infty)$ defined by

$$d_{\tau^*}(\theta, \vartheta) = \tau^*(\theta, \vartheta) - \tau^*(\theta, \theta) \tag{2.2}$$

defines a quasi-metric on $\mathcal{A}$ such that $\tau(\tau^*) = \tau(d_{\tau^*})$ and

$$d_{\tau^*}^s(\theta, \vartheta) = \max\{d_{\tau^*}(\theta, \vartheta), d_{\tau^*}(\vartheta, \theta)\} \tag{2.3}$$

defines a metric on $\mathcal{C}$.

Definition 2.5 (see [20]) Let $(\mathcal{C}, \tau^*)$ be a DPM space.

1. A sequence $\{\theta_i\}$ in $\mathcal{C}$ is said to converge or to be convergent if there is a $\theta \in \mathcal{C}$ such that

$$\lim_{i\to\infty} \tau^*(\theta_i, \theta) = \tau^*(\theta, \theta).$$

θ is called the *limit* of $\{\theta_i\}$ and we write $\theta_i \to \theta$.

2. A sequence $\{\theta_i\}$ in $\mathcal{C}$ is said to be Cauchy sequence if $\lim_{i,j\to\infty} \tau^*(\theta_n, \theta_m)$ exists, finite.

3. A DPM space $\mathcal{C} = (\mathcal{C}, \tau^*)$ is said to be complete if every Cauchy sequence $\{\theta_i\}$ in $\mathcal{C}$ converges, with respect to $\tau(\tau^*)$, to a point $\theta \in \mathcal{C}$ such that

$$\tau^*(\theta, \theta) = \lim_{i,j\to\infty} \tau^*\big(\theta_i, \theta_j\big).$$

Remark 2.6 For a sequence, convergence with respect to metric space may not imply convergence with respect to DPM space. Indeed, if we take $\gamma = 1$ and $\left\{\theta_i = \frac{1-i}{i} : i \geq 1\right\}_{i\in\mathbb{N}} \subset \mathcal{C}$ as in Example 2.4 (3). Mention that $\lim_{i\to\infty} d(\theta_i, -1) = -1$ and therefore, $\theta_i \to -1$ with respect to d. On the other hand, we make a conclusion that $\theta_i \nrightarrow -1$ with respect to τ^* because $\lim_{i\to\infty} \tau^*(\theta_i, -1) = \lim_{i\to\infty} \tau^*|\theta_i - (-1)| = \lim_{i\to\infty} \left|\frac{1-i}{i} + 1\right| = 0$ and $\tau^*(-1, -1) = -1$.

Lemma 2.7 (see [20]) Let $(\mathcal{C}, \tau^*)$ be a DPM space.

(1) Every Cauchy sequence in $(\mathcal{C}, d^s_{\tau^*})$ is also a Cauchy sequence in $(\mathcal{C}, \tau^*)$.
(2) A DPM $(\mathcal{C}, \tau^*)$ is complete if and only if the induced metric space $(\mathcal{C}, d^s_{\tau^*})$ is complete.
(3) A sequence $\{\theta_i\}$ in $\mathcal{C}$ converges to a point $\theta \in \mathcal{C}$ with respect to $\tau(d^s_{\tau^*})$ if and only if

$$\tau^*(\theta, \theta) = \lim_{i\to\infty} \tau^*(\theta_i, \theta) = \lim_{i,j\to\infty} \tau^*\big(\theta_i, \theta_j\big).$$

***Definition* 2.8** (see [11]) Let $(\mathcal{C}, \mathfrak{d})$ be a metric space. A mapping and $\Delta: \mathcal{C} \to \mathcal{C}$ is said to be an E-contraction if there exists a real number $\mathfrak{r} \in [0,1)$ such that

$$\mathfrak{d}(\Delta\theta, \Delta\vartheta) \leq \mathfrak{r}[\mathfrak{d}(\theta, \vartheta) + |\mathfrak{d}(\theta, \Delta\theta) - \mathfrak{d}(\vartheta, \Delta\vartheta)|]$$
for all $\theta, \vartheta \in \mathcal{C}$.

***Definition* 2.9** (see [2]) Let $(\mathcal{C}, \mathfrak{d})$ be a metric space. A mapping and $\Delta: \mathcal{C} \to \mathcal{C}$ is said to be a Dass-Gupta Rational contraction if there exist real numbers $\mathfrak{r}_1, \mathfrak{r}_2 \in [0,1)$ with $\mathfrak{r}_1 + \mathfrak{r}_2 < 1$ such that

$$\mathfrak{d}(\Delta\theta, \Delta\vartheta) \leq \mathfrak{r}_1 \frac{[1+\mathfrak{d}(\theta, \Delta\theta)]\mathfrak{d}(\vartheta, \Delta\vartheta)}{1+\mathfrak{d}(\theta, \vartheta)} + \mathfrak{r}_2 \mathfrak{d}(\theta, \vartheta)$$
for all $\theta, \vartheta \in \mathcal{C}$.

In DPM space, we define dualistic E-contraction, generalized dualistic E-contraction and Dass-Gupta dualistic rational E-contraction. We establish some new fixed-point theorems for E-contraction, generalized dualistic E-contraction and Dass-Gupta dualistic rational E-contraction defined on a DPM space. Also, we define dualistic E_Δ-contraction, generalized dualistic E_Δ-contraction and Dass-Gupta dualistic rational E_Δ-contraction in DPM space. We establish some coincidence and common fixed-point theorems for E_Δ-contraction, generalized dualistic E_Δ-contraction and Dass-Gupta dualistic rational E_Δ-contraction defined on DPM spaces. These theorems expand and generalize several intriguing findings from metric fixed-point theory to the dualistic partial metric setting. We also provide an example to support our results.

3 Main Results

This section contains some fixed-point theorems for dualistic E-contraction, dualistic rational E-contraction and generalized dualistic E-contraction, an illustrative example and deductions. We begin with the following definitions.

***Definition* 3.1** Let $(\mathcal{C}, \tau^*)$ be a DPM space. A mapping $\Upsilon: \mathcal{C} \to \mathcal{C}$ is called dualistic E-contraction, if for all distinct $\theta, \vartheta \in \mathcal{C}$, there exists a number $\lambda \in [0,1)$ such that

$$|\tau^*(\Upsilon\theta, \Upsilon\vartheta)| \leq \lambda\big[|\tau^*(\theta, \vartheta)| + \big||\tau^*(\theta, \Upsilon\theta)| - |\tau^*(\vartheta, \Upsilon\vartheta)|\big|\big]. \tag{3.1}$$

***Definition* 3.2** Let $(\mathcal{C}, \tau^*)$ be a DPM space. A mapping $\Upsilon: \mathcal{C} \to \mathcal{C}$ is called dualistic rational E-contraction, if for all distinct $\theta, \vartheta \in \mathcal{C}$, there exist numbers $\lambda_1, \lambda_2 \in [0,1)$ with $\lambda_1 + \lambda_2 < 1$ such that

$$|\tau^*(\Upsilon\theta, \Upsilon\vartheta)| \leq \lambda_1\big[|\tau^*(\theta, \vartheta)| + \big||\tau^*(\theta, \Upsilon\theta)| - |\tau^*(\vartheta, \Upsilon\vartheta)|\big|\big] + \lambda_2 \frac{[1+|\tau^*(\theta,\Upsilon\theta)|]|\tau^*(\vartheta,\Upsilon\vartheta)|}{1+|\tau^*(\theta,\vartheta)|}. \tag{3.2}$$

***Definition* 3.3** Let $(\mathcal{C}, \tau^*)$ be a DPM space. A mapping $\Upsilon: \mathcal{C} \to \mathcal{C}$ is called generalized dualistic E-contraction, if for all distinct $\theta, \vartheta \in \mathcal{C}$, there exists a number $\lambda \in [0,1)$ such that

$$|\tau^*(\Upsilon\theta, \Upsilon\vartheta)| \leq \lambda \max\Big\{|\tau^*(\theta, \vartheta)| + \big||\tau^*(\theta, \Upsilon\theta)| - |\tau^*(\vartheta, \Upsilon\vartheta)|\big|, \frac{|\tau^*(\theta,\Upsilon\theta)|+|\tau^*(\vartheta,\Upsilon\vartheta)|}{2}\Big\}. \tag{3.3}$$

Our first result as follows.

***Theorem* 3.4** Let $(\mathcal{C}, \tau^*)$ be a complete DPM space and $\Upsilon: \mathcal{C} \to \mathcal{C}$ be a dualistic E-contraction. Then Υ has a unique fixed point.

Proof. Let $\{\theta_i\}$ be the sequence in $\mathcal{C}$ defined as $\theta_1 = \Upsilon\theta_0, \theta_i = \Upsilon\theta_{i-1}$, for any $i \in \mathbb{N}$, where θ_0 is an arbitrary fixed point in $\mathcal{C}$. If there exists $r \in \mathbb{N}$ such that $\theta_{r+1} = \theta_r$, then θ_r is a fixed point of Υ and $\tau^*(\theta_r, \theta_r) = 0$. Suppose that $\theta_{i+1} \neq \theta_i$ for any $i \in \mathbb{N}$. By (3.1), we have

$$\begin{aligned} |\tau^*(\theta_i, \theta_{i+1})| &= |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \\ &\leq \lambda\big[|\tau^*(\theta_{i-1}, \theta_i)| + \big||\tau^*(\theta_{i-1}, \Upsilon\theta_{i-1})| - |\tau^*(\theta_i, \Upsilon\theta_i)|\big|\big] \\ &= \lambda\big[|\tau^*(\theta_{i-1}, \theta_i)| + \big||\tau^*(\theta_{i-1}, \theta_i)| - |\tau^*(\theta_i, \theta_{i+1})|\big|\big] \end{aligned} \tag{3.4}$$

If $|\tau^*(\theta_{i-1}, \theta_i)| < |\tau^*(\theta_i, \theta_{i+1})|$ for some i, from (3.4), we have

$$|\tau^*(\theta_i, \theta_{i+1})| \leq \lambda[|\tau^*(\theta_{i-1}, \theta_i)| - |\tau^*(\theta_{i-1}, \theta_i)| + |\tau^*(\theta_i, \theta_{i+1})|] = \lambda|\tau^*(\theta_i, \theta_{i+1})|,$$

which is a contradiction. Hence, $|\tau^*(\theta_{i-1}, \theta_i)| \geq |\tau^*(\theta_i, \theta_{i+1})|$ and so from (3.4), we have

$$\begin{aligned} |\tau^*(\theta_i, \theta_{i+1})| &\leq \lambda[|\tau^*(\theta_{i-1}, \theta_i)| + |\tau^*(\theta_{i-1}, \theta_i)| - |\tau^*(\theta_i, \theta_{i+1})|] \\ &= \lambda[2|\tau^*(\theta_{i-1}, \theta_i)| - |\tau^*(\theta_i, \theta_{i+1})|]. \end{aligned}$$

The last inequality gives

$$|\tau^*(\theta_i, \theta_{i+1})| \leq \frac{2\lambda}{1+\lambda}|\tau^*(\theta_{i-1}, \theta_i)| = c|\tau^*(\theta_{i-1}, \theta_i)|.$$

where $c = \frac{2\lambda}{1+\lambda}$. From this, we can write,

$$|\tau^*(\theta_i, \theta_{i+1})| \leq c|\tau^*(\theta_{i-1}, \theta_i)| \leq c^2|\tau^*(\theta_{i-2}, \theta_{i-1})| \leq \cdots \leq c^i|\tau^*(\theta_0, \theta_1)|. \tag{3.5}$$

Now, consider the self-distance

$$\begin{aligned} |\tau^*(\theta_i, \theta_i)| &= |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_{i-1})| \\ &\leq \lambda\big[|\tau^*(\theta_{i-1}, \theta_{i-1})| + \big||\tau^*(\theta_{i-1}, \Upsilon\theta_{i-1})| - |\tau^*(\theta_{i-1}, \Upsilon\theta_{i-1})|\big|\big] \\ &= \lambda|\tau^*(\theta_{i-1}, \theta_{i-1})| \end{aligned} \tag{3.6}$$

Similarly,

$$|\tau^*(\theta_{i-1}, \theta_{i-1})| \leq \lambda|\tau^*(\theta_{i-2}, \theta_{i-2})|$$

Inequality (3.6) implies that

$$|\tau^*(\theta_i,\theta_i)| \le \lambda^2|\tau^*(\theta_{i-2},\theta_{i-2})|$$

Proceeding further in a similar way, we get

$$|\tau^*(\theta_i,\theta_i)| \le \lambda^i|\tau^*(\theta_0,\theta_0)|$$

Equation implies (2.2) that

$$\begin{aligned} d_{\tau^*}(\theta_i,\theta_{i+1}) &\le |\tau^*(\theta_i,\theta_{i+1})| - \tau^*(\theta_i,\theta_i) \\ &\le |\tau^*(\theta_i,\theta_{i+1})| + |\tau^*(\theta_i,\theta_i)| \\ &\le c^i|\tau^*(\theta_0,\theta_1)| + \lambda^i|\tau^*(\theta_0,\theta_0)| \end{aligned}$$

Now, for $j > i$, we have

$$\begin{aligned} d_{\tau^*}(\theta_i,\theta_j) &\le d_{\tau^*}(\theta_i,\theta_{i+1}) + d_{\tau^*}(\theta_{i+1},\theta_{i+2}) + \cdots + d_{\tau^*}(\theta_{j-1},\theta_j) \\ &\le c^i|\tau^*(\theta_0,\theta_1)| + \lambda^i|\tau^*(\theta_0,\theta_0)| + c^{i+1}|\tau^*(\theta_0,\theta_1)| + \lambda^{i+1}|\tau^*(\theta_0,\theta_0)| \\ &+ \cdots + c^{j-1}|\tau^*(\theta_0,\theta_1)| + \lambda^{j-1}|\tau^*(\theta_0,\theta_0)| \\ &= (c^i + c^{i+1} + \cdots + c^{j-1})|\tau^*(\theta_0,\theta_1)| \\ &+(\lambda^i + \lambda^{i+1} + \cdots + \lambda^{j-1})|\tau^*(\theta_0,\theta_0)| \\ &\le \tfrac{c^i}{1-c}|\tau^*(\theta_0,\theta_1)| + \tfrac{\lambda^i}{1-\lambda}|\tau^*(\theta_0,\theta_0)| \end{aligned}$$

The last inequality gives

$$d_{\tau^*}(\theta_i,\theta_j) \le \tfrac{c^i}{1-c}|\tau^*(\theta_0,\theta_1)| + \tfrac{\lambda^i}{1-\lambda}|\tau^*(\theta_0,\theta_0)|$$

We conclude that $\lim_{i,j\to\infty} d^s_{\tau^*}(\theta_i,\theta_j) = \lim_{i,j\to\infty} \max\{d_{\tau^*}(\theta_i,\theta_j), d_{\tau^*}(\theta_j,\theta_i)\} = 0$, thus, $\{\theta_i\}$ is a Cauchy sequence in $(\mathcal{C}, d^s_{\tau^*})$. Since $(\mathcal{C},\tau^*)$ is a complete DPM space, by Lemma 2.7(2), $(\mathcal{C}, d^s_{\tau^*})$ is a complete metric space. Thus, there exists $\omega \in (\mathcal{C}, d^s_{\tau^*})$ such that $\theta_i \to \omega$ as $i \to \infty$, that is $\lim_{i\to\infty} d_{\tau^*}(\theta_i,\omega) = 0$ and by Lemma 2.7(3), we know that

$$\tau^*(\omega,\omega) = \lim_{i\to\infty} \tau^*(\theta_i,\omega) = \lim_{i,j\to\infty} \tau^*(\theta_i,\theta_j). \tag{3.7}$$

Since, $\lim_{i\to\infty} d_{\tau^*}(\theta_i,\omega) = 0$, by (2.2) and (3.7), we have

$$\tau^*(\omega,\omega) = \lim_{i\to\infty} \tau^*(\theta_i,\omega) = \lim_{i,j\to\infty} \tau^*(\theta_i,\theta_j) = 0. \tag{3.8}$$

This shows that $\{\theta_i\}$ is a Cauchy sequence converging to $\omega \in (\mathcal{C},\tau^*)$. We shall show that ω is a fixed point of Υ. From condition (3.1), we have

$$\begin{aligned} |\tau^*(\theta_{i+1},\Upsilon\omega)| &= |\tau^*(\Upsilon\theta_i,\Upsilon\omega)| \\ &\le \lambda\big[|\tau^*(\theta_i,\omega)| + \big||\tau^*(\theta_i,\Upsilon\theta_i)| - |\tau^*(\omega,\Upsilon\omega)|\big|\big] \\ &= \lambda\big[|\tau^*(\theta_i,\omega)| + \big||\tau^*(\theta_i,\theta_{i+1})| - |\tau^*(\omega,\Upsilon\omega)|\big|\big] \end{aligned}$$

Applying limit as $i \to \infty$ and using equation (3.8), we have

$$|\tau^*(\omega,\Upsilon\omega)| \le \lambda|\tau^*(\omega,\Upsilon\omega)|,$$

which implies that $|\tau^*(\omega,\Upsilon\omega)| = 0$, because $\lambda < 1$ and then $\tau^*(\omega,\Upsilon\omega) = 0$. Again from (3.1), we have

$$|\tau^*(\Upsilon\omega,\Upsilon\omega)| \le \lambda\big[|\tau^*(\omega,\omega)| + \big||\tau^*(\omega,\Upsilon\omega)| - |\tau^*(\omega,\Upsilon\omega)|\big|\big] = \lambda|\tau^*(\omega,\omega)|$$

Since $\lambda < 1$, and $\tau^*(\omega,\omega) = 0$, we get $|\tau^*(\Upsilon\omega,\Upsilon\omega)| = 0$. Hence, $\tau^*(\Upsilon\omega,\Upsilon\omega) = 0$, and

$$\tau^*(\omega,\omega) = \tau^*(\Upsilon\omega,\Upsilon\omega) = \tau^*(\omega,\Upsilon\omega) \tag{3.9}$$

By using axiom (τ_1^*), we have $\omega = \Upsilon\omega$. This shows that ω is a fixed point of Υ.

To prove the uniqueness of ω, suppose that ω^* is another fixed point of Υ, then $\Upsilon\omega^* = \omega^*$ and $\tau^*(\omega^*,\omega^*) = 0$. By (3.1), we obtain

$$\begin{aligned}|\tau^*(\omega,\omega^*)| &= |\tau^*(\Upsilon\omega,\Upsilon\omega^*)|\\ &\le \lambda\big[|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\Upsilon\omega)| - |\tau^*(\omega^*,\Upsilon\omega^*)|\big|\big]\\ &= \lambda\big[|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\omega)| - |\tau^*(\omega^*,\omega^*)|\big|\big]\\ &= \lambda|\tau^*(\omega,\omega^*)|\end{aligned}$$

which implies that $(1-\lambda)|\tau^*(\omega,\omega^*)| \le 0$. This is possible only when $|\tau^*(\omega,\omega^*)| = 0$, since $\lambda < 1$. Hence, $\tau^*(\omega,\omega^*) = 0$ and then,

$$\tau^*(\omega,\omega^*) = \tau^*(\omega,\omega) = \tau^*(\omega^*,\omega^*)$$

By (τ_1^*), we have $\omega = \omega^*$. Consequently, Υ has unique fixed point ω.

***Theorem* 3.5** Let $(\mathcal{C},\tau^*)$ be a complete DPM space and $\Upsilon:\mathcal{C}\to\mathcal{C}$ be a dualistic rational E-contraction. Then Υ has a unique fixed point.

Proof. Following the steps of proof of Theorem 3.1, we construct the sequence $\{\theta_i\}$ by iterating

$$\theta_1 = \Upsilon\theta_0, \theta_i = \Upsilon\theta_{i-1}, \text{ for any } i \in \mathbb{N}.$$

where $\theta_0 \in \mathcal{C}$ is arbitrary point. Then, by (3.2), we have

$$\begin{aligned}|\tau^*(\theta_i,\theta_{i+1})| &= |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|\\ &\le \lambda_1\big[|\tau^*(\theta_{i-1},\theta_i)| + \big||\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})| - |\tau^*(\theta_i,\Upsilon\theta_i)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|)|\tau^*(\theta_i,\Upsilon\theta_i)|}{1+|\tau^*(\theta_{i-1},\theta_i)|}\\ &= \lambda_1\big[|\tau^*(\theta_{i-1},\theta_i)| + \big||\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|\big|\big] + \lambda_2\frac{(1+|\tau^*(\theta_{i-1},\theta_i)|)|\tau^*(\theta_i,\theta_{i+1})|}{1+|\tau^*(\theta_{i-1},\theta_i)|}\\ &= \lambda_1\big[|\tau^*(\theta_{i-1},\theta_i)| + \big||\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|\big|\big] + \lambda_2|\tau^*(\theta_i,\theta_{i+1})|\end{aligned} \tag{3.10}$$

If $|\tau^*(\theta_{i-1},\theta_i)| < |\tau^*(\theta_i,\theta_{i+1})|$ for some i, from (3.10), we have

$$\begin{aligned}|\tau^*(\theta_i,\theta_{i+1})| &\le \lambda_1[|\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_{i-1},\theta_i)| + |\tau^*(\theta_i,\theta_{i+1})|] + \lambda_2|\tau^*(\theta_i,\theta_{i+1})|\\ &= (\lambda_1+\lambda_2)|\tau^*(\theta_i,\theta_{i+1})|,\end{aligned}$$

which is a contradiction. Hence, $|\tau^*(\theta_{i-1},\theta_i)| \ge |\tau^*(\theta_i,\theta_{i+1})|$and so from (3.10), we have

$$\begin{aligned}|\tau^*(\theta_i,\theta_{i+1})| &\le \lambda_1[|\tau^*(\theta_{i-1},\theta_i)| + |\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|] + \lambda_2|\tau^*(\theta_i,\theta_{i+1})|\\ &= \lambda_1[2|\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|] + \lambda_2|\tau^*(\theta_i,\theta_{i+1})|.\end{aligned}$$

The last inequality gives

$$|\tau^*(\theta_i,\theta_{i+1})| \le \frac{2\lambda_1}{1+\lambda_1-\lambda_2}|\tau^*(\theta_{i-1},\theta_i)| = c|\tau^*(\theta_{i-1},\theta_i)|.$$

where $c = \frac{2\lambda_1}{1+\lambda_1-\lambda_2}$. From this, we can write,

$$|\tau^*(\theta_i,\theta_{i+1})| \le c|\tau^*(\theta_{i-1},\theta_i)| \le c^2|\tau^*(\theta_{i-2},\theta_{i-1})| \le \cdots \le c^i|\tau^*(\theta_0,\theta_1)|. \tag{3.11}$$

Now, consider the self-distance

$$\begin{aligned}|\tau^*(\theta_i,\theta_i)| &= |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_{i-1})| \\ &\le \lambda_1\big[|\tau^*(\theta_{i-1},\theta_{i-1})| + \big||\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})| - |\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|\big|\big] + \lambda_2\frac{(1+|\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|)|\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|}{1+|\tau^*(\theta_{i-1},\theta_{i-1})|} \\ &= \lambda_1\big[|\tau^*(\theta_{i-1},\theta_{i-1})| + \big||\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_{i-1},\theta_i)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\theta_{i-1},\theta_i)|)|\tau^*(\theta_{i-1},\theta_i)|}{1+|\tau^*(\theta_{i-1},\theta_{i-1})|} \\ &= \lambda_1|\tau^*(\theta_{i-1},\theta_{i-1})| + \lambda_2|\tau^*(\theta_{i-1},\theta_i)| \\ &\le (\lambda_1+\lambda_2)|\tau^*(\theta_{i-1},\theta_i)| = \lambda|\tau^*(\theta_{i-1},\theta_i)| \end{aligned} \tag{3.12}$$

where $\lambda = \lambda_1 + \lambda_2$. Using inequality (3.11), we have

$$|\tau^*(\theta_i,\theta_i)| \le \lambda c^{i-1}|\tau^*(\theta_0,\theta_1)|.$$

Equation implies (2.2) that

$$\begin{aligned} d_{\tau^*}(\theta_i,\theta_{i+1}) &\le |\tau^*(\theta_i,\theta_{i+1})| - \tau^*(\theta_i,\theta_i) \\ &\le |\tau^*(\theta_i,\theta_{i+1})| + |\tau^*(\theta_i,\theta_i)| \\ &\le c^i|\tau^*(\theta_0,\theta_1)| + \lambda c^{i-1}|\tau^*(\theta_0,\theta_1)| \end{aligned}$$

Now, for $j > i$, we have

$$\begin{aligned} d_{\tau^*}(\theta_i,\theta_j) &\le d_{\tau^*}(\theta_i,\theta_{i+1}) + d_{\tau^*}(\theta_{i+1},\theta_{i+2}) + \cdots + d_{\tau^*}(\theta_{j-1},\theta_j) \\ &\le c^i|\tau^*(\theta_0,\theta_1)| + \lambda c^{i-1}|\tau^*(\theta_0,\theta_1)| + c^{i+1}|\tau^*(\theta_0,\theta_1)| + \lambda c^i|\tau^*(\theta_0,\theta_1)| \\ &+ \cdots + c^{j-1}|\tau^*(\theta_0,\theta_1)| + \lambda c^{j-2}|\tau^*(\theta_0,\theta_1)| \\ &= (c^i + c^{i+1} + \cdots + c^{j-1})|\tau^*(\theta_0,\theta_1)| \\ &+\lambda(c^{i-1} + c^i + \cdots + c^{j-2})|\tau^*(\theta_0,\theta_1)| \\ &\le \frac{c^i}{1-c}|\tau^*(\theta_0,\theta_1)| + \lambda\frac{c^{i-1}}{1-c}|\tau^*(\theta_0,\theta_1)| \end{aligned}$$

The last inequality gives

$$d_{\tau^*}(\theta_i,\theta_j) \le \frac{c^i}{1-c}|\tau^*(\theta_0,\theta_1)| + \lambda\frac{c^{i-1}}{1-c}|\tau^*(\theta_0,\theta_1)|$$

We conclude that $\lim_{i,j\to\infty} d^s_{\tau^*}(\theta_i,\theta_j) = 0$, thus, $\{\theta_i\}$ is a Cauchy sequence in $(\mathcal{C}, d^s_{\tau^*})$. Since $(\mathcal{C},\tau^*)$ is a complete DPM space, by Lemma 2.7(2), $(\mathcal{C}, d^s_{\tau^*})$ is a complete metric space. Thus, there exists $\omega \in (\mathcal{C}, d^s_{\tau^*})$ such that $\theta_i \to \omega$ as $i \to \infty$, that is $\lim_{i\to\infty} d_{\tau^*}(\theta_i,\omega) = 0$ and by Lemma 2.7(3), we know that

$$\tau^*(\omega,\omega) = \lim_{i\to\infty}\tau^*(\theta_i,\omega) = \lim_{i,j\to\infty}\tau^*(\theta_i,\theta_j). \tag{3.13}$$

Since, $\lim_{i\to\infty} d_{\tau^*}(\theta_i,\omega) = 0$, by (2.2) and (3.7), we have

$$\tau^*(\omega,\omega) = \lim_{i\to\infty}\tau^*(\theta_i,\omega) = \lim_{i,j\to\infty}\tau^*(\theta_i,\theta_j) = 0. \tag{3.14}$$

This shows that $\{\theta_i\}$ is a Cauchy sequence converging to $\omega \in (\mathcal{C},\tau^*)$. We shall show that ω is a fixed point of Υ. From condition (3.2), we have

$$\begin{aligned} |\tau^*(\theta_{i+1},\Upsilon\omega)| &= |\tau^*(\Upsilon\theta_i,\Upsilon\omega)| \\ &\le \lambda_1\big[|\tau^*(\theta_i,\omega)| + \big||\tau^*(\theta_i,\Upsilon\theta_i)| - |\tau^*(\omega,\Upsilon\omega)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\theta_i,\Upsilon\theta_i)|)|\tau^*(\omega,\Upsilon\omega)|}{1+|\tau^*(\theta_i,\omega)|} \\ &\le \lambda_1\big[|\tau^*(\theta_i,\omega)| + \big||\tau^*(\theta_i,\theta_{i+1})| - |\tau^*(\omega,\Upsilon\omega)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\theta_i,\theta_{i+1})|)|\tau^*(\omega,\Upsilon\omega)|}{1+|\tau^*(\theta_i,\omega)|} \end{aligned}$$

Applying limit as $i \to \infty$ and using equation (3.14), we have

$$|\tau^*(\omega,\Upsilon\omega)| \le (\lambda_1+\lambda_2)|\tau^*(\omega,\Upsilon\omega)|,$$

which implies that $|\tau^*(\omega,\Upsilon\omega)| = 0$, because $\lambda_1+\lambda_2 < 1$ and then $\tau^*(\omega,\Upsilon\omega) = 0$. Again from (3.2), we have

$$\begin{aligned}|\tau^*(\Upsilon\omega,\Upsilon\omega)| &\le \lambda_1\big[|\tau^*(\omega,\omega)| + \big||\tau^*(\omega,\Upsilon\omega)| - |\tau^*(\omega,\Upsilon\omega)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\omega,\Upsilon\omega)|)|\tau^*(\omega,\Upsilon\omega)|}{1+|\tau^*(\omega,\omega)|}\\ &= \lambda_1|\tau^*(\omega,\omega)|\end{aligned}$$

Since $\lambda_1 < 1$, and $\tau^*(\omega,\omega) = 0$, we get $|\tau^*(\Upsilon\omega,\Upsilon\omega)| = 0$. Hence, $\tau^*(\Upsilon\omega,\Upsilon\omega) = 0$, and

$$\tau^*(\omega,\omega) = \tau^*(\Upsilon\omega,\Upsilon\omega) = \tau^*(\omega,\Upsilon\omega)$$

By using axiom (τ_1^*), we have $\omega = \Upsilon\omega$. This shows that ω is a fixed point of Υ. To prove the uniqueness of ω, suppose that ω^* is another fixed point of Υ, then $\Upsilon\omega^* = \omega^*$ and $\tau^*(\omega^*,\omega^*) = 0$. By (3.2), we obtain

$$\begin{aligned}|\tau^*(\omega,\omega^*)| &= |\tau^*(\Upsilon\omega,\Upsilon\omega^*)|\\ &\le \lambda_1\big[|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\Upsilon\omega)| - |\tau^*(\omega^*,\Upsilon\omega^*)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\omega,\Upsilon\omega)|)|\tau^*(\omega,\Upsilon\omega)|}{1+|\tau^*(\omega,\omega)|}\\ &\le \lambda_1\big[|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\omega)| - |\tau^*(\omega^*,\omega^*)|\big|\big] + \lambda_2\frac{(1+|\tau^*(\omega,\omega)|)|\tau^*(\omega,\omega)|}{1+|\tau^*(\omega,\omega)|}\\ &= (\lambda_1+\lambda_2)|\tau^*(\omega,\omega^*)|\end{aligned}$$

which implies that $\big(1-(\lambda_1+\lambda_2)\big)|\tau^*(\omega,\omega^*)| \le 0$. This is possible only when $|\tau^*(\omega,\omega^*)| = 0$, since $\lambda_1+\lambda_2 < 1$. Hence, $\tau^*(\omega,\omega^*) = 0$ and then,

$$\tau^*(\omega,\omega^*) = \tau^*(\omega,\omega) = \tau^*(\omega^*,\omega^*)$$

By (τ_1^*), we have $\omega = \omega^*$. Consequently, Υ has unique fixed point ω.

***Theorem* 3.6** Let $(\mathcal{C},\tau^*)$ be a complete DPM space and $\Upsilon:\mathcal{C}\to\mathcal{C}$ be a generalized dualistic E-contraction. Then Υ has a unique fixed point.

Proof Following the steps of proof of Theorem 3.1, we construct the sequence $\{\theta_i\}$ by iterating

$$\theta_1 = \Upsilon\theta_0, \theta_i = \Upsilon\theta_{i-1}, \text{ for any } i \in \mathbb{N}.$$

where $\theta_0 \in \mathcal{C}$ is arbitrary point. Then, by (3.3), we have

$$\begin{aligned}|\tau^*(\theta_i,\theta_{i+1})| &= |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|\\ &\le \lambda\max\left\{|\tau^*(\theta_{i-1},\theta_i)| + \big||\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})| - |\tau^*(\theta_i,\Upsilon\theta_i)|\big|, \frac{|\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|+|\tau^*(\theta_i,\Upsilon\theta_i)|}{2}\right\}\\ &= \lambda\max\left\{|\tau^*(\theta_{i-1},\theta_i)| + \big||\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|\big|, \frac{|\tau^*(\theta_{i-1},\theta_i)|-|\tau^*(\theta_i,\theta_{i+1})|}{2}\right\} \qquad (3.15)\end{aligned}$$

If $|\tau^*(\theta_{i-1},\theta_i)| < |\tau^*(\theta_i,\theta_{i+1})|$ for some i, from (3.15), we have

$$\begin{aligned}|\tau^*(\theta_i,\theta_{i+1})| &\le \lambda\max\left\{|\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_{i-1},\theta_i)| + |\tau^*(\theta_i,\theta_{i+1})|, \frac{|\tau^*(\theta_{i-1},\theta_i)|+|\tau^*(\theta_i,\theta_{i+1})|}{2}\right\},\\ &= \lambda\max\left\{|\tau^*(\theta_i,\theta_{i+1})|, \frac{|\tau^*(\theta_{i-1},\theta_i)|+|\tau^*(\theta_i,\theta_{i+1})|}{2}\right\}\\ &\le \lambda\max\left\{|\tau^*(\theta_i,\theta_{i+1})|, \frac{|\tau^*(\theta_i,\theta_{i+1})|+|\tau^*(\theta_i,\theta_{i+1})|}{2}\right\}\\ &= \lambda|\tau^*(\theta_i,\theta_{i+1})|\end{aligned}$$

which is a contradiction. Hence, $|\tau^*(\theta_{i-1},\theta_i)| \ge |\tau^*(\theta_i,\theta_{i+1})|$ and so from (3.15), we have

$$\begin{aligned}|\tau^*(\theta_i,\theta_{i+1})| &\le \lambda\max\left\{|\tau^*(\theta_{i-1},\theta_i)| + |\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|, \frac{|\tau^*(\theta_{i-1},\theta_i)|+|\tau^*(\theta_i,\theta_{i+1})|}{2}\right\}\\ &\le \lambda\max\{2|\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|, |\tau^*(\theta_{i-1},\theta_i)|\}\\ &= \lambda\{2|\tau^*(\theta_{i-1},\theta_i)| - |\tau^*(\theta_i,\theta_{i+1})|\}.\end{aligned}$$

The last inequality gives

$$|\tau^*(\theta_i,\theta_{i+1})| \le \frac{2\lambda}{1+\lambda}|\tau^*(\theta_{i-1},\theta_i)| = c|\tau^*(\theta_{i-1},\theta_i)|. \tag{3.16}$$

where $c = \frac{2\lambda}{1+\lambda}$. From this, we can write,

$$|\tau^*(\theta_i,\theta_{i+1})| \le c|\tau^*(\theta_{i-1},\theta_i)| \le c^2|\tau^*(\theta_{i-2},\theta_{i-1})| \le \cdots \le c^i|\tau^*(\theta_0,\theta_1)|. \tag{3.17}$$

Now, consider the self-distance

$$\begin{aligned}
|\tau^*(\theta_i,\theta_i)| &= |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_{i-1})| \\
&\le \lambda \max\left\{|\tau^*(\theta_{i-1},\theta_{i-1})| + \left||\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})| - |\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|\right|, \frac{|\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|+|\tau^*(\theta_{i-1},\Upsilon\theta_{i-1})|}{2}\right\} \\
&= \lambda \max\{|\tau^*(\theta_{i-1},\theta_{i-1})|, |\tau^*(\theta_{i-1},\theta_i)|\} \\
&\le \lambda \max\{|\tau^*(\theta_{i-1},\theta_i)|, |\tau^*(\theta_{i-1},\theta_i)|\} \\
&= \lambda|\tau^*(\theta_{i-1},\theta_i)| \\
&\le \lambda c^{i-1}|\tau^*(\theta_0,\theta_1)|
\end{aligned} \tag{3.18}$$

The equation implies (2.2) that

$$\begin{aligned}
d_{\tau^*}(\theta_i,\theta_{i+1}) &\le |\tau^*(\theta_i,\theta_{i+1})| - \tau^*(\theta_i,\theta_i) \\
&\le |\tau^*(\theta_i,\theta_{i+1})| + |\tau^*(\theta_i,\theta_i)| \\
&\le c^i|\tau^*(\theta_0,\theta_1)| + \lambda c^{i-1}|\tau^*(\theta_0,\theta_1)| \\
&= (c^i + \lambda c^{i-1})|\tau^*(\theta_0,\theta_1)|
\end{aligned}$$

Now, for $j > i$, we have

$$\begin{aligned}
d_{\tau^*}(\theta_i,\theta_j) &\le d_{\tau^*}(\theta_i,\theta_{i+1}) + d_{\tau^*}(\theta_{i+1},\theta_{i+2}) + \cdots + d_{\tau^*}(\theta_{j-1},\theta_j) \\
&\le (c^i + \lambda c^{i-1})|\tau^*(\theta_0,\theta_1)| + (c^{i+1} + \lambda c^i)|\tau^*(\theta_0,\theta_1)| \\
&\quad + \cdots + (c^{j-1} + \lambda c^{j-2})|\tau^*(\theta_0,\theta_1)| \\
&= [(c^i + c^{i+1} + \cdots + c^{j-1}) + \lambda(c^{i-1} + c^i + \cdots + c^{j-2})]|\tau^*(\theta_0,\theta_1)| \\
&= (c^i + \lambda c^{i-1})(1 + c + c^2 \ldots + c^{j-i-1})|\tau^*(\theta_0,\theta_1)| \\
&\le \frac{(c^i+\lambda c^{i-1})}{1-c}|\tau^*(\theta_0,\theta_1)|
\end{aligned}$$

The last inequality gives

$$d_{\tau^*}(\theta_i,\theta_j) \le \frac{c^i}{1-c}|\tau^*(\theta_0,\theta_1)| + \lambda\frac{c^{i-1}}{1-c}|\tau^*(\theta_0,\theta_0)|$$

We conclude that $\lim_{i,j\to\infty} d^s_{\tau^*}(\theta_i,\theta_j) = \lim_{i,j\to\infty} \max\{d_{\tau^*}(\theta_i,\theta_j), d_{\tau^*}(\theta_j,\theta_i)\} = 0$, thus, $\{\theta_i\}$ is a Cauchy sequence in $(\mathcal{C}, d^s_{\tau^*})$. Since $(\mathcal{C},\tau^*)$ is a complete DPM space, by Lemma 2.7(2), $(\mathcal{C}, d^s_{\tau^*})$ is a complete metric space. Thus, there exists $\omega \in (\mathcal{C}, d^s_{\tau^*})$ such that $\theta_i \to \omega$ as $i \to \infty$, that is $\lim_{i\to\infty} d_{\tau^*}(\theta_i,\omega) = 0$ and by Lemma 2.7(3), we know that

$$\tau^*(\omega,\omega) = \lim_{i\to\infty}\tau^*(\theta_i,\omega) = \lim_{i,j\to\infty}\tau^*(\theta_i,\theta_j). \tag{3.19}$$

Since, $\lim_{i\to\infty} d_{\tau^*}(\theta_i,\omega) = 0$, by (2.2) and (3.19), we have

$$\tau^*(\omega,\omega) = \lim_{i\to\infty}\tau^*(\theta_i,\omega) = \lim_{i,j\to\infty}\tau^*(\theta_i,\theta_j) = 0. \tag{3.20}$$

This shows that $\{\theta_i\}$ is a Cauchy sequence converging to $\omega \in (\mathcal{C},\tau^*)$. We shall show that ω is a fixed point of Υ. From condition (3.3), we have

$$|\tau^*(\theta_{i+1},\Upsilon\omega)| = |\tau^*(\Upsilon\theta_i,\Upsilon\omega)|$$

$$\leq \lambda \max\left\{|\tau^*(\theta_i,\omega)| + \big||\tau^*(\theta_i,\Upsilon\theta_i)| - |\tau^*(\omega,\Upsilon\omega)|\big|, \frac{|\tau^*(\theta_i,\Upsilon\theta_i)|+|\tau^*(\omega,\Upsilon\omega)|}{2}\right\}$$
$$= \lambda \max\left\{|\tau^*(\theta_i,\omega)| + \big||\tau^*(\theta_i,\theta_{i+1})| - |\tau^*(\omega,\Upsilon\omega)|\big|, \frac{|\tau^*(\theta_i,\theta_{i+1})|+|\tau^*(\omega,\Upsilon\omega)|}{2}\right\}$$

Applying limit as $i \to \infty$ and using equation (3.20), we have

$$|\tau^*(\omega,\Upsilon\omega)| \leq \lambda \max\left\{|\tau^*(\omega,\Upsilon\omega)|, \frac{|\tau^*(\omega,\Upsilon\omega)|}{2}\right\} = \lambda|\tau^*(\omega,\Upsilon\omega)|$$

which implies that $|\tau^*(\omega,\Upsilon\omega)| = 0$, because $\lambda < 1$ and then $\tau^*(\omega,\Upsilon\omega) = 0$. Again from (3.3), we have

$$|\tau^*(\Upsilon\omega,\Upsilon\omega)| \leq \lambda \max\left\{|\tau^*(\omega,\omega)| + \big||\tau^*(\omega,\Upsilon\omega)| - |\tau^*(\omega,\Upsilon\omega)|\big|, \frac{|\tau^*(\omega,\Upsilon\omega)|+|\tau^*(\omega,\Upsilon\omega)|}{2}\right\}$$
$$= \lambda \max\{|\tau^*(\omega,\omega)|, |\tau^*(\omega,\Upsilon\omega)|\}$$

Since $\lambda < 1$, and $\tau^*(\omega,\omega) = 0, \tau^*(\omega,\Upsilon\omega) = 0$ we get $|\tau^*(\Upsilon\omega,\Upsilon\omega)| = 0$. Hence, $\tau^*(\Upsilon\omega,\Upsilon\omega) = 0$ and then

$$\tau^*(\omega,\omega) = \tau^*(\Upsilon\omega,\Upsilon\omega) = \tau^*(\omega,\Upsilon\omega)$$

By using axiom (τ_1^*), we have $\omega = \Upsilon\omega$. This shows that ω is a fixed point of Υ. To prove the uniqueness of ω, suppose that ω^* is another fixed point of Υ, then $\Upsilon\omega^* = \omega^*$ and $\tau^*(\omega^*, \omega^*) = 0$. By (3.3), we obtain

$$\begin{aligned}|\tau^*(\omega,\omega^*)| &= |\tau^*(\Upsilon\omega,\Upsilon\omega^*)| \\ &\leq \lambda \max\left\{|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\Upsilon\omega)| - |\tau^*(\omega^*,\Upsilon\omega^*)|\big|, \frac{|\tau^*(\omega,\Upsilon\omega)|+|\tau^*(\omega^*,\Upsilon\omega^*)|}{2}\right\} \\ &= \lambda \max\left\{|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\omega)| - |\tau^*(\omega^*,\omega^*)|\big|, \frac{|\tau^*(\omega,\omega)|+|\tau^*(\omega^*,\omega^*)|}{2}\right\} \\ &= \lambda|\tau^*(\omega,\omega^*)|\end{aligned}$$

which implies that $(1-\lambda)|\tau^*(\omega,\omega^*)| \leq 0$. This is possible only when $|\tau^*(\omega,\omega^*)| = 0$, since $\lambda < 1$. Hence, $\tau^*(\omega,\omega^*) = 0$ and then,

$$\tau^*(\omega,\omega^*) = \tau^*(\omega,\omega) = \tau^*(\omega^*, \omega^*)$$

By (τ_1^*), we have $\omega = \omega^*$. Consequently, Υ has unique fixed point ω.

Now, we give an example in support of our results.

***Example* 3.7** Let $\mathcal{C} = (-\infty, 0]$ and define $\tau^*: \mathcal{C} \times \mathcal{C} \to (-\infty, \infty)$ by $\tau^*(\theta,\vartheta) = \max\{\theta,\vartheta\}$. It is easy to check that $(\mathcal{C},\tau^*)$ is a complete DPM space. Define $\Upsilon: \mathcal{C} \to \mathcal{C}$ as $\Upsilon\theta = \frac{\theta}{2}, \forall\, \theta \in \mathcal{C}$. Further, for all $\theta,\vartheta \in \mathcal{C}$ with $\theta \geq \vartheta$ and $\lambda = \frac{1}{2}$, we have

$$\begin{aligned}|\tau^*(\Upsilon\theta,\Upsilon\vartheta)| &= \left|\max\left\{\frac{\theta}{2},\frac{\vartheta}{2}\right\}\right| = \left|\frac{\theta}{2}\right| \\ &\leq \frac{1}{2}\left\{|\theta| + \left|\left|\frac{\theta}{2}\right| - \left|\frac{\vartheta}{2}\right|\right|\right\} \\ &= \frac{1}{2}\left\{|\max\{\theta,\vartheta\}| + \left|\left|\max\left\{\theta,\frac{\theta}{2}\right\}\right| - \left|\max\left\{\vartheta,\frac{\vartheta}{2}\right\}\right|\right|\right\} \\ &= \lambda\left[|\tau^*(\theta,\vartheta)| + \big||\tau^*(\theta,\Upsilon\theta)| - |\tau^*(\vartheta,\Upsilon\vartheta)|\big|\right]\end{aligned}$$

Cleary, (3.1) is satisfied. Also

$$\begin{aligned}|\tau^*(\mathcal{T}\theta,\mathcal{T}\vartheta)| &= \left|\max\left\{\frac{\theta}{2},\frac{\vartheta}{2}\right\}\right| = \left|\frac{\theta}{2}\right| \\ &\leq \frac{1}{2}\max\left\{|\theta| + \left|\left|\frac{\theta}{2}\right| - \left|\frac{\vartheta}{2}\right|\right|, \frac{\left|\frac{\theta}{2}\right|+\left|\frac{\vartheta}{2}\right|}{2}\right\} \\ &= \frac{1}{2}\left\{|\max\{\theta,\vartheta\}| + \left|\left|\max\left\{\theta,\frac{\theta}{2}\right\}\right| - \left|\max\left\{\vartheta,\frac{\vartheta}{2}\right\}\right|\right|, \frac{\left|\max\left\{\theta,\frac{\theta}{2}\right\}\right|+\left|\max\left\{\vartheta,\frac{\vartheta}{2}\right\}\right|}{2}\right\}\end{aligned}$$

$$= \lambda \left\{ |\tau^*(\theta, \vartheta)| + \big||\tau^*(\theta, \Upsilon\theta)| - |\tau^*(\vartheta, \Upsilon\vartheta)|\big|, \frac{|\tau^*(\theta,\Upsilon\theta)| + |\tau^*(\vartheta,\Upsilon\vartheta)|}{2} \right\}$$

for $\lambda = \frac{1}{2}$ and $\forall\, \theta, \vartheta \in \mathcal{C}$ with $\theta \geq \vartheta$. Cleary, (3.3) is satisfied. In the view of Theorem 3.4 and Theorem 3.5, Υ has a unique fixed point in $\mathcal{C}$, indeed $\Upsilon 0 = 0$.

Now, we present some coincidence point and common fixed-point theorems for dualistic E_Δ-contraction, dualistic rational E_Δ-contraction and generalized dualistic E_Δ-contraction, and deductions.

***Definition* 3.8** Let $(\mathcal{C}, \tau^*)$ be a DPM space and Υ, Δ be two self-mappings on . We say the mapping Υ, a dualistic E_Δ-contraction, if for all distinct $\theta, \vartheta \in \mathcal{C}$, there exists number $\lambda \in [0,1)$ such that

$$|\tau^*(\Upsilon\theta, \Upsilon\vartheta)| \leq \lambda\big[|\tau^*(\Delta\theta, \Delta\vartheta)| + \big||\tau^*(\Delta\theta, \Upsilon\theta)| - |\tau^*(\Delta\vartheta, \Upsilon\vartheta)|\big|\big]. \tag{3.21}$$

***Definition* 3.9** Let $(\mathcal{C}, \tau^*)$ be a DPM space and Υ, Δ be two self-mappings on . We say the mapping Υ, a dualistic rational E_Δ-contraction, if for all distinct $\theta, \vartheta \in \mathcal{C}$, there exist numbers $\lambda_1, \lambda_2 \in [0,1)$ with $\lambda_1 + \lambda_2 < 1$ such that

$$\begin{aligned} |\tau^*(\Upsilon\theta, \Upsilon\vartheta)| \leq \lambda_1 &\big[|\tau^*(\Delta\theta, \Delta\vartheta)| + \big||\tau^*(\Delta\theta, \Upsilon\theta)| - |\tau^*(\Delta\vartheta, \Upsilon\vartheta)|\big|\big] \\ &+ \lambda_2 \frac{[1 + |\tau^*(\Delta\theta,\Upsilon\theta)|]|\tau^*(\Delta\vartheta,\Upsilon\vartheta)|}{1 + |\tau^*(\Delta\theta,\Delta\vartheta)|}. \end{aligned} \tag{3.22}$$

***Definition* 3.10** Let $(\mathcal{C}, \tau^*)$ be a DPM space. and Υ, Δ be two self-mappings on $\mathcal{C}$. We say the mapping Υ, a generalized dualistic E_Δ-contraction, if for all distinct $\theta, \vartheta \in \mathcal{C}$, there exists a number $\lambda \in [0,1)$ such that

$$|\tau^*(\Upsilon\theta, \Upsilon\vartheta)| \leq \lambda \max \left\{ \begin{matrix} |\tau^*(\Delta\theta, \Delta\vartheta)| + \big||\tau^*(\Delta\theta, \Upsilon\theta)| - |\tau^*(\Delta\vartheta, \Upsilon\vartheta)|\big|, \\ \frac{|\tau^*(\Delta\theta,\Upsilon\theta)| + |\tau^*(\Delta\vartheta,\Upsilon\vartheta)|}{2} \end{matrix} \right\}. \tag{3.23}$$

***Theorem* 3.11** Let $(\mathcal{C}, \tau^*)$ be a complete DPM space and $\Upsilon, \Delta: \mathcal{C} \to \mathcal{C}$ be two mappings such that

(1) $\Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$,
(2) Υ is a dualistic E_Δ-contraction.

If $\Upsilon(\mathcal{C})$ or $\Delta(\mathcal{C})$ is a complete subspace of $\mathcal{C}$, then Υ and Δhave a coincidence point. Further, if Υ, Δ are weakly compatible mappings, then Υ and Δ have a unique common fixed point.

Proof. Let θ_0 be an arbitrary point in $\mathcal{C}$. Since $\Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$, we can find $\theta_1 \in \mathcal{C}$ such that $\Upsilon\theta_0 = \Delta\theta_1$. In general, θ_i is chosen such that $\Upsilon\theta_i = \Delta\theta_{i+1}$ for $i = 0,1,2, \ldots.$ If $\Upsilon\theta_i = \Upsilon\theta_{i-1} = \Delta\theta_i$ for some $i \in \mathbb{N}$, then $\vartheta = \Upsilon\theta_i = \Upsilon\theta_{i-1} = \Delta\theta_i$ is a point of coincidence of Υ and Δ. Suppose that $\Upsilon\theta_i \neq \Upsilon\theta_{i-1}$ and thus $\Delta\theta_i \neq \Delta\theta_{i+1}$ for all $i \in \mathbb{N}$. By the dualistic E_Δ-contraction condition (3.21), we obtain

$$\begin{aligned} |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq \lambda\big[|\tau^*(\Delta\theta_i, \Delta\theta_{i+1})| + \big||\tau^*(\Delta\theta_i, \Upsilon\theta_i)| - |\tau^*(\Delta\theta_{i+1}, \Upsilon\theta_{i+1})|\big|\big] \\ &= \lambda\big[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|\big|\big]. \end{aligned} \tag{3.24}$$

If $|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| < |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|$ for some i, from (3.24), we have

$$\begin{aligned} |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq \lambda[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|] \\ &= \lambda|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| \end{aligned}$$

which is a contradiction. Hence, $|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \geq |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|$ and so from (3.24), we have

$$\begin{aligned} |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq \lambda[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|] \\ &= \lambda[2|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|]. \end{aligned}$$

The last inequality gives

$$|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| \le \frac{2\lambda}{1+\lambda}|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| = c|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)|.$$

where $c = \frac{2\lambda}{1+\lambda}$. From this, we can write,

$$\begin{aligned}|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\le c|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \\ &\le c^2|\tau^*(\Upsilon\theta_{i-2}, \Upsilon\theta_{i-1})| \\ &\le \cdots \le c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|. \end{aligned} \tag{3.25}$$

Now, consider the self-distance

$$\begin{aligned}|\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| &\le \lambda\big[|\tau^*(\Delta\theta_i, \Delta\theta_i)| + \big||\tau^*(\Delta\theta_i, \Upsilon\theta_i)| - |\tau^*(\Delta\theta_i, \Upsilon\theta_i)|\big|\big] \\ &= \lambda\big[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_{i-1})| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)|\big|\big] \\ &= \lambda|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_{i-1})| \end{aligned} \tag{3.26}$$

Similarly,

$$|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_{i-1})| \le \lambda|\tau^*(\Upsilon\theta_{i-2}, \Upsilon\theta_{i-2})|$$

The inequality (3.26) implies that

$$|\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| \le \lambda^2|\tau^*(\Upsilon\theta_{i-2}, \Upsilon\theta_{i-2})|$$

Proceeding further in a similar way, we get

$$|\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| \le \lambda^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)|$$

The equation implies (2.2) that

$$\begin{aligned} d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_{i+1}) &\le |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| - \tau^*(\Upsilon\theta_i, \Upsilon\theta_i) \\ &\le |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| + |\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| \\ &\le c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \end{aligned}$$

Now, for $j > i$, we have

$$\begin{aligned} d_{\tau^*}\big(\Upsilon\theta_i, \Upsilon\theta_j\big) &\le d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_{i+1}) + d_{\tau^*}(\Upsilon\theta_{i+1}, \Upsilon\theta_{i+2}) + \cdots + d_{\tau^*}\big(\Upsilon\theta_{j-1}, \Upsilon\theta_j\big) \\ &\le c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \\ &+c^{i+1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda^{i+1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \\ &+ \cdots + c^{j-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda^{j-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \\ &= (c^i + c^{i+1} + \cdots + c^{j-1})|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| \\ &+(\lambda^i + \lambda^{i+1} + \cdots + \lambda^{j-1})|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \\ &\le \frac{c^i}{1-c}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \frac{\lambda^i}{1-\lambda}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \end{aligned}$$

The last inequality gives

$$d_{\tau^*}\big(\Upsilon\theta_i, \Upsilon\theta_j\big) \le \frac{c^i}{1-c}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \frac{\lambda^i}{1-\lambda}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_0)| \tag{3.27}$$

We conclude that $\lim_{i,j\to\infty} d^s_{\tau^*}\big(\Upsilon\theta_i, \Upsilon\theta_j\big) = \lim_{i,j\to\infty} \max\big\{d_{\tau^*}\big(\Upsilon\theta_i, \Upsilon\theta_j\big), d_{\tau^*}\big(\Upsilon\theta_j, \Upsilon\theta_i\big)\big\} = 0$, thus, $\{\Upsilon\theta_i\}$ is a Cauchy sequence in $(\mathcal{C}, d^s_{\tau^*})$. Since $(\mathcal{C}, \tau^*)$ is a complete DPM space, by Lemma 2.7(2), $(\mathcal{C}, d^s_{\tau^*})$ is a complete metric space. Consequently, there exists an element $\omega \in \Upsilon(\mathcal{C}) \subset \mathcal{C}$ such that such that $\Upsilon\theta_i \to \omega$ as $i \to \infty$, that is $\lim_{i\to\infty} d_{\tau^*}(\Upsilon\theta_i, \omega) = 0$ and by Lemma 2.7(3), we know that

$$\tau^*(\omega, \omega) = \lim_{i\to\infty} \tau^*(\Upsilon\theta_i, \omega) = \lim_{i,j\to\infty} \tau^*\big(\Upsilon\theta_i, \Upsilon\theta_j\big). \tag{3.28}$$

Since, $\lim_{i\to\infty} d_{\tau^*}(\theta_i, \omega) = 0$, by (2.2) and (3.28), we have

$$\tau^*(\omega,\omega) = \lim_{i\to\infty} \tau^*(\Upsilon\theta_i, \omega) = \lim_{i,j\to\infty} \tau^*\big(\Upsilon\theta_i, \Upsilon\theta_j\big) = 0. \tag{3.29}$$

This shows that $\{\Upsilon\theta_i\}$ is a Cauchy sequence converging to $\omega \in (\mathcal{C},\tau^*)$. As $\omega \in \Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$, there exists $\sigma \in \mathcal{C}$ such that $\omega = \Delta\sigma$ and by (3.29), we have $\tau^*(\Delta\sigma\,,\Delta\sigma\,) = 0$. By condition (3.21), we have

$$\begin{aligned}|\tau^*(\Delta\theta_{i+1},\Upsilon\sigma)| &= |\tau^*(\Upsilon\theta_i,\Upsilon\sigma)|\\ &\le \lambda\big[|\tau^*(\Delta\theta_i,\Delta\sigma)| + \big||\tau^*(\Delta\theta_i,\Upsilon\theta_i)| - |\tau^*(\Delta\sigma,\Upsilon\sigma)|\big|\big]\\ &= \lambda\big[|\tau^*(\Upsilon\theta_{i-1},\Delta\sigma)| + \big||\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Delta\sigma,\Upsilon\sigma)|\big|\big]\end{aligned}$$

Applying limit as $i \to \infty$ and using equation (3.29), we have

$$|\tau^*(\Delta\sigma,\Upsilon\sigma)| \le \lambda\big[|\tau^*(\Delta\sigma,\Delta\sigma)| + \big||\tau^*(\Delta\sigma,\Delta\sigma)| - |\tau^*(\Delta\sigma,\Upsilon\sigma)|\big|\big] = \lambda|\tau^*(\Delta\sigma,\Upsilon\sigma)|,$$

which implies that $|\tau^*(\Delta\sigma,\Upsilon\sigma)| = 0$, because $\lambda < 1$ and then $\tau^*(\Delta\sigma,\Upsilon\sigma) = 0$. Again from (3.21), we have

$$|\tau^*(\Upsilon\sigma,\Upsilon\sigma)| \le \lambda\big[|\tau^*(\Delta\sigma,\Delta\sigma)| + \big||\tau^*(\Delta\sigma,\Upsilon\sigma)| - |\tau^*(\Delta\sigma,\Upsilon\sigma)|\big|\big] = 0.$$

Since $\lambda < 1$, and $\tau^*(\Delta\sigma,\Delta\sigma) = 0, \tau^*(\Delta\sigma,\Upsilon\sigma) = 0$, we get $|\tau^*(\Upsilon\sigma,\Upsilon\sigma)| = 0$. Hence, $\tau^*(\Upsilon\sigma,\Upsilon\sigma) = 0$, and

$$\tau^*(\Delta\sigma,\Delta\sigma) = \tau^*(\Upsilon\sigma,\Upsilon\sigma) = \tau^*(\Delta\sigma,\Upsilon\sigma)$$

By using axiom (τ_1^*), we have $\Delta\sigma = \Upsilon\sigma$. Thus, $\omega = \Delta\sigma = \Upsilon\sigma$ is a point of coincidence of Υ and Δ. Since Υ and Δ are weakly compatible mappings, $\omega = \Delta\sigma = \Upsilon\sigma$ implies $\Upsilon\omega = \Upsilon\Delta\sigma = \Delta\Upsilon\sigma = \Delta\omega$. By (3.21), we get

$$|\tau^*(\Upsilon\sigma,\Upsilon\omega)| \le \lambda\big[|\tau^*(\Delta\sigma,\Delta\omega)| + \big||\tau^*(\Delta\sigma,\Upsilon\omega)| - |\tau^*(\Delta\sigma,\Upsilon\omega)|\big|\big] = |\tau^*(\Upsilon\sigma,\Upsilon\omega)|$$

Thus, $\tau^*(\Upsilon\sigma,\Upsilon\omega) = 0 = \tau^*(\Upsilon\sigma,\Upsilon\sigma) = \tau^*(\Upsilon\omega,\Upsilon\omega)$. Hence, $\omega = \Delta\omega = \Upsilon\omega$, that is ω is common fixed point of Υ and Δ. To prove the uniqueness of ω, suppose that there exists another common fixed point ω^* of Υ and Δ; we prove that $\omega = \omega^*$. By (3.21), we obtain

$$\begin{aligned}|\tau^*(\omega,\omega^*)| &= |\tau^*(\Upsilon\omega,\Upsilon\omega^*)|\\ &\le \lambda\big[|\tau^*(\Delta\omega,\Delta\omega^*)| + \big||\tau^*(\Delta\omega,\Upsilon\omega)| - |\tau^*(\Delta\omega^*,\Upsilon\omega^*)|\big|\big]\\ &= \lambda\big[|\tau^*(\omega,\omega^*)| + \big||\tau^*(\omega,\omega)| - |\tau^*(\omega^*,\omega^*)|\big|\big]\\ &= \lambda|\tau^*(\omega,\omega^*)|\end{aligned}$$

which implies that $(1-\lambda)|\tau^*(\omega,\omega^*)| \le 0$. This is possible only when $|\tau^*(\omega,\omega^*)| = 0$, since $\lambda < 1$. Hence, $\tau^*(\omega,\omega^*) = 0$ and then,

$$\tau^*(\omega,\omega^*) = \tau^*(\omega,\omega) = \tau^*(\omega^*\,,\omega^*\,)$$

By (τ_1^*), we have $\omega = \omega^*$. Consequently, Υ and Δ have a unique common fixed point ω.

***Theorem* 3.12** Let $(\mathcal{C},\tau^*)$ be a complete DPM space and $\Upsilon,\Delta\colon \mathcal{C} \to \mathcal{C}$ be two mappings such that

(1) $\Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$,
(2) Υ is a dualistic rational E_Δ- contraction.

If $\Upsilon(\mathcal{C})$ or $\Delta(\mathcal{C})$ is a complete subspace of $\mathcal{C}$, then Υ and Δhave a coincidence point. Further, if Υ,Δ are weakly compatible mappings, then Υ and Δ have a unique common fixed point.

Proof Following the steps of proof of Theorem 3.11, we construct the sequence $\{\theta_i\}$ by iterating

$$\Upsilon\theta_0 = \Delta\theta_1, \Upsilon\theta_i = \Delta\theta_{i+1} \text{ for } i = 0,1,2, \ldots.$$

where $\theta_0 \in \mathcal{C}$ is arbitrary point. By the dualistic rational E_Δ-contraction condition (3.22), we obtain

$$\begin{aligned}
|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq \lambda_1\big[|\tau^*(\Delta\theta_i, \Delta\theta_{i+1})| + \big||\tau^*(\Delta\theta_i, \Upsilon\theta_i)| - |\tau^*(\Delta\theta_{i+1}, \Upsilon\theta_{i+1})|\big|\big] \\
&\quad + \lambda_2 \tfrac{[1+|\tau^*(\Delta\theta_i,\Upsilon\theta_i)|]|\tau^*(\Delta\theta_{i+1},\Upsilon\theta_{i+1})|}{1+|\tau^*(\Delta\theta_i,\Delta\theta_{i+1})|}. \\
&= \lambda_1\big[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|\big|\big] \\
&\quad + \lambda_2 \tfrac{[1+|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|]|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|}{1+|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|} \\
&= \lambda_1\big[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|\big|\big] + \lambda_2|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| \quad (3.30)
\end{aligned}$$

If $|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| < |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|$ for some i, from (3.30), we have

$$\begin{aligned}
|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq \lambda_1[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|] + \lambda_2|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| \\
&= (\lambda_1 + \lambda_2)|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|
\end{aligned}$$

which is a contradiction. Hence, $|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \geq |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|$ and so from (3.30), we have

$$\begin{aligned}
|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq \lambda_1[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|] + \lambda_2|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| \\
&= \lambda_1[2|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|] + \lambda_2|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})|.
\end{aligned}$$

The last inequality gives

$$|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| \leq \frac{2\lambda_1}{1+\lambda_1-\lambda_2}|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| = c|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)|.$$

where $c = \frac{2\lambda_1}{1+\lambda_1-\lambda_2}$. From this, we can write,

$$\begin{aligned}
|\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| &\leq c|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \\
&\leq c^2|\tau^*(\Upsilon\theta_{i-2}, \Upsilon\theta_{i-1})| \\
&\leq \cdots \leq c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|. \qquad (3.31)
\end{aligned}$$

Now, consider the self-distance

$$\begin{aligned}
|\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| &\leq \lambda_1\big[|\tau^*(\Delta\theta_i, \Delta\theta_i)| + \big||\tau^*(\Delta\theta_i, \Upsilon\theta_i)| - |\tau^*(\Delta\theta_i, \Upsilon\theta_i)|\big|\big] + \lambda_2 \tfrac{[1+|\tau^*(\Delta\theta_i,\Upsilon\theta_i)|]|\tau^*(\Delta\theta_i,\Upsilon\theta_i)|}{1+|\tau^*(\Delta\theta_i,\Delta\theta_i)|} \\
&= \lambda_1\big[|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_{i-1})| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)|\big|\big] + \lambda_2|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \\
&= (\lambda_1 + \lambda_2)|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| = \lambda|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \qquad (3.32)
\end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2$. The inequality (3.31) implies that

$$|\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| \leq \lambda c^{i-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|$$

The equation implies (2.2) that

$$\begin{aligned}
d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_{i+1}) &\leq |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| - \tau^*(\Upsilon\theta_i, \Upsilon\theta_i) \\
&\leq |\tau^*(\Upsilon\theta_i, \Upsilon\theta_{i+1})| + |\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| \\
&\leq c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda c^{i-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|
\end{aligned}$$

Now, for $j > i$, we have

$$\begin{aligned}
d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_j) &\leq d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_{i+1}) + d_{\tau^*}(\Upsilon\theta_{i+1}, \Upsilon\theta_{i+2}) + \cdots + d_{\tau^*}(\Upsilon\theta_{j-1}, \Upsilon\theta_j) \\
&\leq c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda c^{i-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| \\
&\quad + c^{i+1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda c^i|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| \\
&\quad + \cdots + c^{j-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda c^{j-2}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|
\end{aligned}$$

$$= (c^i + c^{i+1} + \cdots + c^{j-1})|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|$$
$$+\lambda(c^{i-1} + c^i + \cdots + c^{j-2})|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|$$
$$\leq \frac{c^i}{1-c}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda\frac{c^{i-1}}{1-c}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|$$

The last inequality gives

$$d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_j) \leq \frac{c^i}{1-c}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| + \lambda\frac{c^{i-1}}{1-c}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)| \tag{3.33}$$

We conclude that $\lim_{i,j\to\infty} d^s_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_j) = \lim_{i,j\to\infty} \max\{d_{\tau^*}(\Upsilon\theta_i, \Upsilon\theta_j), d_{\tau^*}(\Upsilon\theta_j, \Upsilon\theta_i)\} = 0$, thus, $\{\Upsilon\theta_i\}$ is a Cauchy sequence in $(\mathcal{C}, d^s_{\tau^*})$. Since $(\mathcal{C}, \tau^*)$ is a complete DPM space, by Lemma 2.7(2), $(\mathcal{C}, d^s_{\tau^*})$ is a complete metric space. Consequently, there exists an element $\omega \in \Upsilon(\mathcal{C}) \subset \mathcal{C}$ such that such that $\Upsilon\theta_i \to \omega$ as $i \to \infty$, that is $\lim_{i\to\infty} d_{\tau^*}(\Upsilon\theta_i, \omega) = 0$ and by Lemma 2.7(3), we know that

$$\tau^*(\omega, \omega) = \lim_{i\to\infty} \tau^*(\Upsilon\theta_i, \omega) = \lim_{i,j\to\infty} \tau^*(\Upsilon\theta_i, \Upsilon\theta_j). \tag{3.34}$$

Since, $\lim_{i\to\infty} d_{\tau^*}(\theta_i, \omega) = 0$, by (2.2) and (3.34), we have

$$\tau^*(\omega, \omega) = \lim_{i\to\infty} \tau^*(\Upsilon\theta_i, \omega) = \lim_{i,j\to\infty} \tau^*(\Upsilon\theta_i, \Upsilon\theta_j) = 0. \tag{3.35}$$

This shows that $\{\Upsilon\theta_i\}$ is a Cauchy sequence converging to $\omega \in (\mathcal{C}, \tau^*)$. As $\omega \in \Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$, there exists $\sigma \in \mathcal{C}$ such that $\omega = \Delta\sigma$ and by (3.35), we have $\tau^*(\Delta\sigma, \Delta\sigma) = 0$. By condition (3.22), we have

$$|\tau^*(\Delta\theta_{i+1}, \Upsilon\sigma)| = |\tau^*(\Upsilon\theta_i, \Upsilon\sigma)|$$
$$\leq \lambda_1\big[|\tau^*(\Delta\theta_i, \Delta\sigma)| + \big||\tau^*(\Delta\theta_i, \Upsilon\theta_i)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|\big]$$
$$+\lambda_2\frac{[1+|\tau^*(\Delta\theta_i, \Upsilon\theta_i)|]|\tau^*(\Delta\sigma, \Upsilon\sigma)|}{1+|\tau^*(\Delta\theta_i, \Delta\sigma)|}$$
$$= \lambda_1\big[|\tau^*(\Upsilon\theta_{i-1}, \Delta\sigma)| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|\big]$$
$$+\lambda_2\frac{[1+|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)|]|\tau^*(\Delta\sigma, \Upsilon\sigma)|}{1+|\tau^*(\Upsilon\theta_{i-1}, \Delta\sigma)|}$$

Applying limit as $i \to \infty$ and using equation (3.35), we have

$$|\tau^*(\Delta\sigma, \Upsilon\sigma)| \leq \lambda_1\big[|\tau^*(\Delta\sigma, \Delta\sigma)| + \big||\tau^*(\Delta\sigma, \Delta\sigma)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|\big] + \lambda_2|\tau^*(\Delta\sigma, \Upsilon\sigma)|,$$
$$= \lambda_2|\tau^*(\Delta\sigma, \Upsilon\sigma)|$$

which implies that $|\tau^*(\Delta\sigma, \Upsilon\sigma)| = 0$, because $\lambda_2 < 1$ and then $\tau^*(\Delta\sigma, \Upsilon\sigma) = 0$. Again from (3.22), we have

$$|\tau^*(\Upsilon\sigma, \Upsilon\sigma)| \leq \lambda_1|\tau^*(\Delta\sigma, \Delta\sigma)|$$

Since $\lambda_1 < 1$, we get $|\tau^*(\Upsilon\sigma, \Upsilon\sigma)| = 0$. Hence, $\tau^*(\Upsilon\sigma, \Upsilon\sigma) = 0$, and

$$\tau^*(\Delta\sigma, \Delta\sigma) = \tau^*(\Upsilon\sigma, \Upsilon\sigma) = \tau^*(\Delta\sigma, \Upsilon\sigma)$$

By using axiom (τ_1^*), we have $\Delta\sigma = \Upsilon\sigma$. Thus, $\omega = \Delta\sigma = \Upsilon\sigma$ is a point of coincidence of Υ and Δ. Since Υ and Δ are weakly compatible mappings, $\omega = \Delta\sigma = \Upsilon\sigma$ implies $\Upsilon\omega = \Upsilon\Delta\sigma = \Delta\Upsilon\sigma = \Delta\omega$. By (3.22), we get

$$|\tau^*(\Upsilon\sigma, \Upsilon\omega)| \leq \lambda_1|\tau^*(\Upsilon\sigma, \Upsilon\omega)|$$

Thus, $\tau^*(\Upsilon\sigma, \Upsilon\omega) = 0 = \tau^*(\Upsilon\sigma, \Upsilon\sigma) = \tau^*(\Upsilon\omega, \Upsilon\omega)$. Hence, $\omega = \Delta\omega = \Upsilon$. To prove the uniqueness of ω, suppose that there exists another common fixed point ω^* of Υ and Δ; we prove that $\omega = \omega^*$. By (3.22), we obtain

$$|\tau^*(\omega, \omega^*)| = |\tau^*(\Upsilon\omega, \Upsilon\omega^*)|$$

$$\leq \lambda_1\left[|\tau^*(\Delta\omega,\Delta\omega^*)| + \left||\tau^*(\Delta\omega,\Upsilon\omega)| - |\tau^*(\Delta\omega^*,\Upsilon\omega^*)|\right|\right] + \lambda_2 \frac{(1+|\tau^*(\Delta\omega,\Upsilon\omega)|)|\tau^*(\Delta\omega^*,\Upsilon\omega^*)|}{1+|\tau^*(\Delta\omega,\Delta\omega^*)|}$$
$$= \lambda_1|\tau^*(\omega,\omega^*)|$$

which implies that $(1-\lambda_1)|\tau^*(\omega,\omega^*)| \leq 0$. This is possible only when $|\tau^*(\omega,\omega^*)| = 0$, since $\lambda < 1$. Hence, $\tau^*(\omega,\omega^*) = 0$ and then,

$$\tau^*(\omega,\omega^*) = \tau^*(\omega,\omega) = \tau^*(\omega^*,\omega^*)$$

By (τ_1^*), we have $\omega = \omega^*$. Consequently, Υ and Δ have a unique common fixed point ω.

***Theorem* 3.13** Let $(\mathcal{C},\tau^*)$ be a complete DPM space and $\Upsilon,\Delta:\mathcal{C}\to\mathcal{C}$ be two mappings such that

(1) $\Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$,
(2) Υ is a dualistic generalized E_Δ-contraction.

If $\Upsilon(\mathcal{C})$ or $\Delta(\mathcal{C})$ is a complete subspace of $\mathcal{C}$, then Υ and Δ have a coincidence point. Further, if Υ,Δ are weakly compatible mappings, then Υ and Δ have a unique common fixed point.

Proof Following the steps of proof of Theorem 3.11, we construct the sequence $\{\theta_i\}$ by iterating

$$\Upsilon\theta_0 = \Delta\theta_1, \Upsilon\theta_i = \Delta\theta_{i+1} \text{ for } i = 0,1,2,\ldots.$$

where $\theta_0 \in \mathcal{C}$ is arbitrary point. By the dualistic rational E_Δ-contraction condition (3.23), we obtain

$$\begin{aligned}|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})| &\leq \lambda \max\left\{|\tau^*(\Delta\theta_i,\Delta\theta_{i+1})| + \left||\tau^*(\Delta\theta_i,\Upsilon\theta_i)| - |\tau^*(\Delta\theta_{i+1},\Upsilon\theta_{i+1})|\right|, \frac{|\tau^*(\Delta\theta_i,\Upsilon\theta_i)|+|\tau^*(\Delta\theta_{i+1},\Upsilon\theta_{i+1})|}{2}\right\}\\ &= \lambda \max\left\{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| + \left||\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|\right|, \frac{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|+|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|}{2}\right\}\end{aligned} \tag{3.36}$$

If $|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| < |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|$ for some i, from (3.36), we have

$$\begin{aligned}|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})| &\leq \lambda \max\{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| + |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|, |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|\}\\ &= \lambda|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|\end{aligned}$$

which is a contradiction. Hence, $|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| \geq |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|$ and so from (3.36), we have

$$\begin{aligned}|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})| &\leq \lambda \max\left\{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| + |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|, \frac{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|+|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|}{2}\right\}\\ &\leq \lambda \max\{2|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|, |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|\}\\ &= \lambda[2|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})|].\end{aligned}$$

The last inequality gives

$$|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})| \leq \frac{2\lambda}{1+\lambda}|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| = c|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|.$$

where $c = \frac{2\lambda}{1+\lambda}$. From this, we can write,

$$\begin{aligned}|\tau^*(\Upsilon\theta_i,\Upsilon\theta_{i+1})| &\leq c|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|\\ &\leq c^2|\tau^*(\Upsilon\theta_{i-2},\Upsilon\theta_{i-1})|\\ &\leq \cdots \leq c^i|\tau^*(\Upsilon\theta_0,\Upsilon\theta_1)|.\end{aligned} \tag{3.37}$$

Now, consider the self-distance

$$\begin{aligned}|\tau^*(\Upsilon\theta_i,\Upsilon\theta_i)| &\leq \lambda \max\left\{|\tau^*(\Delta\theta_i,\Delta\theta_i)| + \left||\tau^*(\Delta\theta_i,\Upsilon\theta_i)| - |\tau^*(\Delta\theta_i,\Upsilon\theta_i)|\right|, \frac{|\tau^*(\Delta\theta_i,\Upsilon\theta_i)|+|\tau^*(\Delta\theta_i,\Upsilon\theta_i)|}{2}\right\}\\ &= \lambda \max\left\{\begin{matrix}|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_{i-1})| + \left||\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)| - |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|\right|,\\ \frac{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|+|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|}{2}\end{matrix}\right\}\\ &= \lambda \max\{|\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_{i-1})|, |\tau^*(\Upsilon\theta_{i-1},\Upsilon\theta_i)|\}\end{aligned}$$

$$\leq \lambda|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| \tag{3.38}$$

Above inequality implies that

$$|\tau^*(\Upsilon\theta_i, \Upsilon\theta_i)| \leq \lambda c^{i-1}|\tau^*(\Upsilon\theta_0, \Upsilon\theta_1)|$$

As already elaborated in the proof of Theorem 3.12, the classical procedure leads to $\{\Upsilon\theta_i\}$ is a Cauchy sequence in $(\mathcal{C}, d_{\tau^*}^s)$. Since $(\mathcal{C}, \tau^*)$ is a complete DPM space, by Lemma 2.7(2), $(\mathcal{C}, d_{\tau^*}^s)$ is a complete metric space. Consequently, there exists an element $\omega \in \Upsilon(\mathcal{C}) \subset \mathcal{C}$ such that such that $\Upsilon\theta_i \to \omega$ as $i \to \infty$, that is $\lim_{i\to\infty} d_{\tau^*}(\Upsilon\theta_i, \omega) = 0$ and by Lemma 2.7(3), we know that

$$\tau^*(\omega, \omega) = \lim_{i\to\infty} \tau^*(\Upsilon\theta_i, \omega) = \lim_{i,j\to\infty} \tau^*(\Upsilon\theta_i, \Upsilon\theta_j). \tag{3.39}$$

Since, $\lim_{i\to\infty} d_{\tau^*}(\theta_i, \omega) = 0$, by (2.2) and (3.39), we have

$$\tau^*(\omega, \omega) = \lim_{i\to\infty} \tau^*(\Upsilon\theta_i, \omega) = \lim_{i,j\to\infty} \tau^*(\Upsilon\theta_i, \Upsilon\theta_j) = 0. \tag{3.40}$$

This shows that $\{\Upsilon\theta_i\}$ is a Cauchy sequence converging to $\omega \in (\mathcal{C}, \tau^*)$. As $\omega \in \Upsilon(\mathcal{C}) \subset \Delta(\mathcal{C})$, there exists $\sigma \in \mathcal{C}$ such that $\omega = \Delta\sigma$ and by (3.40), we have $\tau^*(\Delta\sigma, \Delta\sigma) = 0$. By condition (3.23), we have

$$\begin{aligned}|\tau^*(\Delta\theta_{i+1}, \Upsilon\sigma)| &= |\tau^*(\Upsilon\theta_i, \Upsilon\sigma)| \\ &\leq \lambda \max\left\{|\tau^*(\Delta\theta_i, \Delta\sigma)| + \big||\tau^*(\Delta\theta_i, \Upsilon\theta_i)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|, \frac{|\tau^*(\Delta\theta_i, \Upsilon\theta_i)| + |\tau^*(\Delta\sigma, \Upsilon\sigma)|}{2}\right\} \\ &= \lambda \max\left\{|\tau^*(\Upsilon\theta_{i-1}, \Delta\sigma)| + \big||\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|, \frac{|\tau^*(\Upsilon\theta_{i-1}, \Upsilon\theta_i)| + |\tau^*(\Delta\sigma, \Upsilon\sigma)|}{2}\right\}\end{aligned}$$

Applying limit as $i \to \infty$ and using equation (3.40), we have

$$\begin{aligned}|\tau^*(\Delta\sigma, \Upsilon\sigma)| &\leq \lambda \max\left\{|\tau^*(\Delta\sigma, \Delta\sigma)| + \big||\tau^*(\Delta\sigma, \Delta\sigma)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|, \frac{|\tau^*(\Delta\sigma, \Delta\sigma)| + |\tau^*(\Delta\sigma, \Upsilon\sigma)|}{2}\right\} \\ &= \lambda|\tau^*(\Delta\sigma, \Upsilon\sigma)|\end{aligned}$$

which implies that $|\tau^*(\Delta\sigma, \Upsilon\sigma)| = 0$, because $\lambda < 1$ and then $\tau^*(\Delta\sigma, \Upsilon\sigma) = 0$. Again from (3.23), we have

$$|\tau^*(\Upsilon\sigma, \Upsilon\sigma)| \leq \lambda_1|\tau^*(\Delta\sigma, \Delta\sigma)|$$

Since $\lambda < 1$, we get $|\tau^*(\Upsilon\sigma, \Upsilon\sigma)| = 0$. Hence, $\tau^*(\Upsilon\sigma, \Upsilon\sigma) = 0$, and

$$\tau^*(\Delta\sigma, \Delta\sigma) = \tau^*(\Upsilon\sigma, \Upsilon\sigma) = \tau^*(\Delta\sigma, \Upsilon\sigma)$$

By using axiom (τ_1^*), we have $\Delta\sigma = \Upsilon\sigma$. Thus, $\omega = \Delta\sigma = \Upsilon\sigma$ is a point of coincidence of Υ and Δ. Since Υ and Δ are weakly compatible mappings, $\omega = \Delta\sigma = \Upsilon\sigma$ implies $\Upsilon\omega = \Upsilon\Delta\sigma = \Delta\Upsilon\sigma = \Delta\omega$. By (3.23), we get

$$\begin{aligned}|\tau^*(\Upsilon\sigma, \Upsilon\omega)| &\leq \lambda \max\left\{|\tau^*(\Delta\sigma, \Delta\sigma)| + \big||\tau^*(\Delta\sigma, \Upsilon\sigma)| - |\tau^*(\Delta\sigma, \Upsilon\sigma)|\big|, \frac{|\tau^*(\Delta\sigma, \Upsilon\sigma)| + |\tau^*(\Delta\sigma, \Upsilon\sigma)|}{2}\right\} \\ &= \lambda|\tau^*(\Delta\sigma, \Delta\sigma)|\end{aligned}$$

Thus, $\tau^*(\Upsilon\sigma, \Upsilon\omega) = 0 = \tau^*(\Upsilon\sigma, \Upsilon\sigma) = \tau^*(\Upsilon\omega, \Upsilon\omega)$. Hence, $\omega = \Delta\omega = \Upsilon\omega$, that is ω is common fixed point of Υ and Δ. To prove the uniqueness of ω, suppose that there exists another common fixed point ω^* of Υ and Δ; we prove that $\omega = \omega^*$. By (3.23), we obtain

$$\begin{aligned}|\tau^*(\omega, \omega^*)| &= |\tau^*(\Upsilon\omega, \Upsilon\omega^*)| \\ &\leq \lambda \max\left\{|\tau^*(\Delta\omega, \Delta\omega^*)| + \big||\tau^*(\Delta\omega, \Upsilon\omega)| - |\tau^*(\Delta\omega^*, \Upsilon\omega^*)|\big|, \frac{|\tau^*(\Delta\omega, \Upsilon\omega)| + |\tau^*(\Delta\omega^*, \Upsilon\omega^*)|}{2}\right\} \\ &= \lambda|\tau^*(\omega, \omega^*)|\end{aligned}$$

which implies that $(1-\lambda)|\tau^*(\omega,\omega^*)| \leq 0$. This is possible only when $|\tau^*(\omega,\omega^*)| = 0$, since $\lambda < 1$. Hence, $\tau^*(\omega,\omega^*) = 0$ and then,

$$\tau^*(\omega,\omega^*) = \tau^*(\omega,\omega) = \tau^*(\omega^*,\omega^*)$$

By (τ_1^*), we have $\omega = \omega^*$. Consequently, Υ and Δ have a unique common fixed point ω.

***Example* 3.14** If we take $\Upsilon\theta = \frac{\theta}{4}, \forall\, \theta \in \mathcal{C}$ and $\Delta\theta = \frac{\theta}{2}, \forall\, \theta \in \mathcal{C}$ in Example 3.7. Then, for all $\theta, \vartheta \in \mathcal{C}$ with $\theta \geq \vartheta$ and $\lambda = \frac{1}{2}$, conditions (3.21) and (3.23) are satisfied. In view of Theorem 3.11 and 3.13, Υ and Δ have a unique common fixed point 0.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Fulga A, Proca A.: A new Generalization of Wardowski Fixed Point Theorem in Complete Metric Spaces. Adv. Theory Nonlinear Anal. Its Appl. 2017; 1:57–63.

[2] Dass BK, Gupta S. An extension of Banach contraction principle through rational expressions, Indian J. Pure Appl. Math. 1975; 6:1455-1458.

[3] Matthews SG. Partial Metric Topology, Ann. New York Acad. Sci. 1994;728:183-197.

[4] O'Neill SJ. partial metric, valuations and domain theory, Ann. New York Acad. Sci. 1996;806:304-315.

[5] Fulga A. Karapınar E. Revisiting of some outstanding metric fixed point theorems via E-contraction. Analele Univ. Ovidius-Constanta-Ser. Mat. 2018;26:73–97.

[6] Karapınar E, Fulga A, Aydi H. Study on Pata E-contractions. Adv. Differ. Equ. 2020;539. DOI:10.1186/s13662-020-02992-4.

[7] Aydi H, Nashine HK, Samet B, Yazidi H. Coincidence and common fixed-point results in partially ordered cone metric spaces and applications to integral equations, Nonlinear Analysis. 2011;74:6814-6825

[8] Fisher B, Khan MS. Common Fixed Points and Constant Mappings, Studia Sci. Math. Hungar. 1978; 11:467-470.

[9] Shagari et. al.: Common fixed points of L-fuzzy Maps for Meir-Keeler type contractions. J. Adv. Math. Stud.m. 2019;12(2):218-229.

[10] Tiwari SK, Dubey RP, Dubey AK. Cone Metric Spaces and Common Fixed-Point Theorems for Generalized Jaggi and Das–Gupta Contractive Mapping. Int. j. of Mathematical Archive. 2013;4(10):93-100.

[11] Wen X, Huang XJ. Common fixed-point theorem under contractions in partial metric spaces, J. Comput. Anal. Appl. 2011;13(3):583-589.

[12] Altun I, Sola F, Simsek H. Generalized contractions on partial metric spaces, Topology Appl. 2010; 157(18):2778-2785.

[13] Arshad M, Nazam M, Beg I. Fixed point theorems in ordered dualistic partial metric spaces, Korean J. Math. 2016;24(2):169-179

[14] Nazam M, Arshad M, Abbas M. Some fixed-point results for dualistic rational contractions, Appl. Gen. Topol. 2016;17:199-209.

[15] Nazam M, Arshad M. Some fixed-point results in ordered dualistic partial metric spaces, Trans. A. Razmadze Math. Instit. 2018;172:498–509.

[16] Alqahtani B, Fulga A, Karapinar E, Rakocevic V. Contractions with rational inequalities in the extended b-metric space. J. Inequal. Appl. 2019;220.

[17] Huang H, Singh YM, Khan MS, Radenovic S. Rational type contractions in extended b-metric spaces. Symmetry. 2021;13:614.

[18] Karapinar E, Fulga A. Contraction in Rational Forms in the Framework of Super Metric Spaces, MPDI, Mathematic. 2022;10;3077:1-12.

[19] Nazam M, Park MC, Mahmood H. On a fixed-point theorem with application to functional equations, Open Math. 2019;17:1724-1736.

[20] Oltra S, Valero O. Banach's fixed point theorem for partial metric spaces, Rend. Ist. Mat. Univ. Trieste. 2004;36:17-26.

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