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Mixture Model on Development of Bivariate Product Distribution and its Properties

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In the study, some bivariate distributions were developed from mixture model offspring, using the Independent (Product) distribution approach. These developments are categorized under the IID and II_nD: where the Bivariate Exponential distribution, Bivariate Lindley distribution and Bivariate Juchez distribution are constructed as IIDs; and Bivariate Exponential-Lindley distribution, Bivariate Exponential-Juchez distribution and Bivariate Lindley-Juchez distribution as (II_nDs). The properties of these distributions which involve: the shape of the bivariate PDFs, moments, moment generating function, mean, covariance and coefficient of correlation, maximum likelihood estimator, reliability analysis, renewal property and

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probability patterns; are studied across the distributions. Finally, under renewal properties, functions are derived which can model two-dimensional queuing and renewal processes, for events where the arrival and service times are dependent.

Keywords: Mixture model; bivariate derivations; product distribution; renewal property.

1 Introduction

Distribution development is one such that constantly evolves as events unfold. These developments are theoretically categorized in to three kinds namely: univariate, bivariate and multivariate representations. The background knowledge of the univariate distributions gives validation to the development of bivariate distributions; since they naturally form the marginal or conditional distributions. Johnson [1] did a comprehensive and updated work on continuous univariate distributions. Specifically, mixture model which recently received an avalanche use has played a huge part in the recent univariate innovations. Examples are Exponential and Lindley distribution as developed in the works of Epstein [2] and Lindley [3]. Recently, Ghitany [4] in the application of Lindley distribution modeled the waiting times (in minutes) of 100 bank customers. Echebiri [5] developed Juchez distribution which fits better as a probability distribution for modeling live-streaming data; other univariate distribution types are Frechet, Weibull, Pareto, Sujatha, Odoma, Shanker, Pranav, Aradhana, Amarendra, Devya and Shambhu distributions. However univariate distributions by default do not capture sufficiently all real life phenomena, that is, cases where it is inevitable to model bivariate outcomes; hence this necessitated development of bivariate distributions.

In the development of bivariate distributions the structure of the two random variables are cases that determine intended methodological approach. Some variables are dependent while others are independent; For instance, the height of a person could also betray a lot of information about the individual's weight. At the other hand, the person's age does not affect the wrist circumference. This notion, spells out in clear terms the concepts of bivariate structures: whether dependent structure or independent. In other words, it could be described as "correlated and uncorrelated" class of random variables.

Some of the well-known, classic bivariate distributions are bivariate normal, bivariate-t, bivariate log-normal, bivariate gamma, bivariate extreme value, bivariate Birnbaum-Saunders distributions, bivariate skew normal distribution, bivariate geometric skew normal distribution etc., with the usual purpose of modeling the two marginal and finding the association between them. Now, the development, study and applications of bivariate distributions are one of the vital areas of research in statistics field. Hutchinson & Lai [6] and recently: Arnold & Sarabia [7], Kotz & Nadarajah [8], Kotz & Nadarajah [9], Nelsen [10] among many, have published several papers and books on bivariate theory. Extensive reviews have been done by Lai [11], Lai [12] and Sarabia & Gomez [13] in the bivariate construction methods; where the scope covers both for discrete and continuous bivariate distributions. Product distribution, Marginal transformation, Copula Method, Method of Mixing and Compounding, Trivariate Reduction Method, Frailty Approach and Conditional Specification Method make up the different handy bivariate construction methods as seen in their reviews.

Product distribution represents a multivariate statistical distribution whose j^{th} marginal distributions are independent component probability density functions; The component distributions may be continuous or discrete, univariate or multivariate, and is given as:

$$g(x_1, \dots, x_n) = \prod_{i=1}^{n} f_i(x_i) = f_1(x_1) f_2(x_2) \dots f_n(x_n) \tag{1}$$

In as much as the study of product distributions backdated to 1940s, the first thorough treatment of the topic was a paper by Springer and Thompson [14]. This is followed by an improvement in both theoretical and algorithmic study, and the extensive use of Monte Carlo theory and other numerical methods. Mixing and Compounding Method as another way for bivariate construction, is majorly used to particularly combine two different bivariate distributions. If F_1 *and* F_2 are two bivariate distribution functions, then the new type derivation is given by

$$F(x_1, x_2) = \theta F_1(x_1, x_2) + (1-\theta) F_2(x_1, x_2), \quad 0 \le \theta \le 1 \tag{2}$$

This was employed by Mardia [15] in the derivation of bivariate families of distributions. Copula distribution is also another method for developing multivariate distributions. Unlike the product counterpart mentioned above, it has different developmental approaches that describe the dependence between variables. The various copula predefined kernels are used for the parameterization, and also they allow for the investigation of the different degrees of these dependence. Some examples are Archimedean kernel, Clayton kernels, Gaussian kernel, Plackett kernel, Independent or Product kernel and Frank kernel. Gumbel [16] and Gumbel [17] used Farlie-Gumbel-Morgenstern (FGM) copula for the development of Gumbel bivariate exponential distribution and bivariate logistic distributions respectively, and it is given as

$$F(x_1 x_2) = x_1 x_2 (1 + \theta[1 - x_1][1 - x_2]) \tag{3}$$

Another approach to copula method is Sklar's theorem which adopts product kernel that combines the marginal cumulative distribution functions (CDF) unto n-dimensional copula C such that for all real $x_1, x_2, x_3, \dots, x_n$:

$$F(x_1, x_2, x_3, \dots, x_n) = C[F(x_1), F(x_2), \dots, F(x_n)] \tag{4}$$

Studying through the literature, Marshal Olkin survival copula method as predominantly used in the development of bivariate Dagum distribution, Muhammed [18], bivariate Lindley distributions based on stress and shock models, Oliveira [19], bivariate exponentiated Frétchet distribution, Saboor [20], Marshall–Olkin bivariate Weibull distribution, Kundu [21], bivariate generalized exponential, Kundu [22] etc, like many other methods, follow dependent structure or measurement. In other words, the random variables are correlated; and this area has been exhaustively developed using identical (same) distribution in their bivariate development. Inasmuch as very many life bivariate outcomes are dependent, the independent bivariate structure which has received minute exploration should be developed as well for the modeling of the uncorrelated bivariate outcomes.

To this end, as unconventional to the recent developments, this paper aims at derivation of bivariate distributions using product distribution method, under a full parametric approach, with the assumption that random variables are uncorrelated. In addition to the usual identical bivariate development, different marginal distributions are considered; and some properties of the derived models are explored as well.

For the arrangement structure of the paper, the antecedent abstract and introduction is preceded by the mixture model bivariate development: the baseline distribution review, methodical derivations using the Product Distribution for the Independence approach. Finally the derivations of some relevant properties follow suit, as occasion serves.

2 Mixture Model Bivariate Development

2.1 Baseline distributions

Lindley and Juchez distributions are derived from the composition of exponential and gamma distributions with suitable mixing probabilities; where the gamma distribution is characterized by a constant scale parameter θ: and shape parameter $\alpha = 2$ for Lindley distribution; and two different shape parameters: $\alpha = 2$ and 4 for Juchez distribution. The PDFs of Lindley and Juchez distributions are derived from the mixture model, Lindsay [23].

$$f(x) = \sum_{i=1}^{k} d_i g_i \,; \text{ where } \sum_{i=1}^{k} d_i = 1, \; d_i > 0 \tag{5}$$

with their corresponding CDFs , and given as:

$$lind(x) = \frac{\theta^2}{\theta+1} \, (1 + x)e^{-\theta x} \,, \; x > 0, \; \theta > \; 0 \tag{6}$$

$$Lind(x) = \; 1 - \left(\frac{\theta + 1 + \; \theta x}{\theta + 1}\right) e^{-\theta x}$$

$$juc(x) = \frac{\theta^4}{\theta^3+\theta^2+6} \, (1 + x + \; x^3)e^{-\theta x} \,, \; x > 0, \; \theta > \; 0 \tag{7}$$

$$Juc(x) = 1 - \left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x}$$

In addition, we include the Exponential distribution, which is not a mixture model offspring; however, it is a vital baseline distribution in most of those mixture developments.

$$exp(x) = \theta e^{-\theta x},\ x > 0, \theta > 0 \qquad (8)$$
$$Exp(x) = 1 - e^{-\theta x}$$

2.2 Product distribution (Independent approach)

As earlier hinted, bivariate development is usually composite of a single marginal distribution; however, in this development, we incorporate different distributions as independent marginal distributions, in addition to the status quo. All the bivariate mathematical constructions and their properties are verified and or derived with the assistance of both mathematical and statistical software known as Mathematica, Wolfram [24] and R, R Core Team [25]. More so, the bivariate derivations here are categorized under these two different phenomena:

- Independent and Identical distributions (IID) and
- Independent and Non - Identical distributions (II_nD)

2.2.1 Independent and identical distributions (IID)

These simply represent bivariate combinations of the same distribution. So, under this category, Bivariate Exponential Distribution, Bivariate Lindley Distribution and Bivariate Juchez Distribution are derived together with their individual multivariate extensions.

Let $X_1, X_2, X_3, \dots, X_n$ be multivariate random variables, with marginal distributions $f_1(x_1), f_2(x_2), \dots, f_n(x_n)$, then the product distribution is given by:

$$g(x_1, x_2, \dots, x_n) = \prod_i^n f_i(x_i) = f_1(x_1) f_2(x_2) \dots f_n(x_n) \qquad (9)$$

Now, if $X_1 \sim Exp(x_1; \theta_1)$ $\quad and \quad$ $X_2 \sim Exp(x_2; \theta_2)$, then the bivariate pdf derivation is

$$g(x_1 x_2) = \left[\theta_1\, e^{-(\theta_1 x_1)}\right] \times \left[\theta_2\, e^{-(\theta_2 x_2)}\right] = \theta_1\, \theta_2\, e^{-(\theta_1 x_1 + \theta_2 x_2)} \qquad (10)$$

and the multivariate extension is given as :

$$g(x_1, x_2, x_3 \dots x_n) = \left(\prod_{i=1}^n \theta_i\right) \left[e^{-[\sum_{i=1}^n \theta_i x_i]}\right] \qquad (11)$$

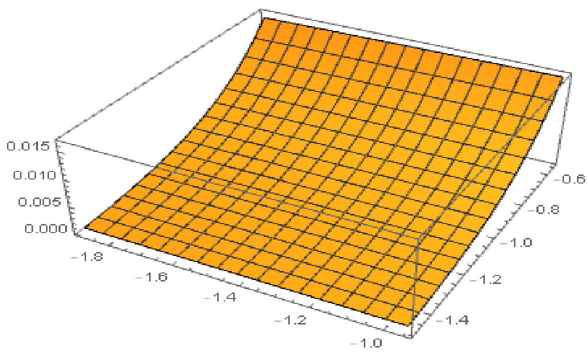

Fig. 1. The shape of the PDF of the Bivariate exponential distribution

If $X_1 \sim Lind(x_1; \theta_1)$ $\quad and \quad$ $X_2 \sim Lind(x_2; \theta_2)$, then the bivariate Lindley derivation is given as:

$$g(x_1x_2) = \left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]\ e^{-(\theta_1x_1+\theta_2x_2)} \tag{12}$$

and the multivariate extension is given as :

$$g(x_1,x_2,x_3\ldots x_n) = \frac{\prod_{i=1}^{n}\theta_i^2}{\prod_{i=1}^{n}\gamma_i}\ \{\prod_{i=1}^{n}\alpha_i\}\left[e^{-[\sum_{i=1}^{n}\theta_i x_i]}\right] \tag{13}$$

where

$$\begin{Bmatrix} \gamma_1 = (\theta_1+1) \\ \gamma_2 = (\theta_2+1) \\ \gamma_3 = (\theta_3+1) \\ \cdots\ \cdots \\ \vdots \\ \cdots\ \cdots \\ \gamma_n = (\theta_n+1) \end{Bmatrix} \ and \ \begin{Bmatrix} \alpha_1 = 1+x_1 \\ \alpha_2 = 1+x_2 \\ \alpha_3 = 1+x_3 \\ \cdots\ \cdots \\ \vdots \\ \cdots\ \cdots \\ \alpha_n = 1+x_n \end{Bmatrix}$$

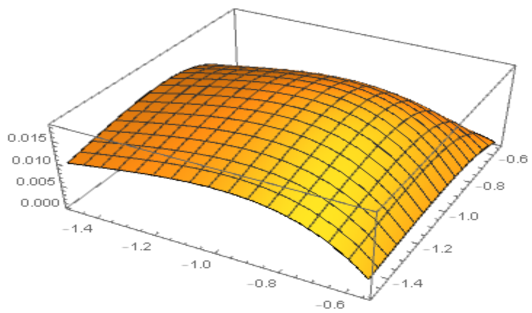

Fig. 2. The shape of the PDF of the Bivariate Lindley distribution

More so, if $X_1 \sim Juc(x_1;\theta_1)$ $\quad and \quad$ $X_2 \sim Juc(x_2;\theta_2)$, then the bivariate Lindley derivation is given as:

$$g(x_1x_2) = \left[\frac{\theta_1^4\ \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+x_1^2+x_1^3)(1+x_2^2+x_2^3)]\ e^{-(\theta_1x_1+\theta_2x_2)} \tag{14}$$

The multivariate extension is given as:

$$g(x_1,x_2,x_3\ldots x_n) = \frac{\prod_{i=1}^{n}\theta_i^4}{\prod_{i=1}^{n}a_i}\ \{\prod_{i=1}^{n}\varphi_i\}\left[e^{-[\sum_{i=1}^{n}\theta_i x_i]}\right] \tag{15}$$

where

$$\begin{Bmatrix} a_1 = \theta_1^3+\theta_1^2+6 \\ a_2 = \theta_2^3+\theta_2^2+6 \\ a_3 = \theta_3^3+\theta_3^2+6 \\ \cdots\ \cdots\ \cdots \\ \vdots \\ \cdots\ \cdots\ \cdots \\ a_n = \theta_n^3+\theta_n^2+6 \end{Bmatrix} \ and \ \begin{Bmatrix} \varphi_1 = 1+x_1^2+x_1^3 \\ \varphi_2 = 1+x_2^2+x_2^3 \\ \varphi_3 = 1+x_3^2+x_3^3 \\ \cdots\ \cdots\ \cdots \\ \vdots \\ \cdots\ \cdots\ \cdots \\ \varphi_n = 1+x_n^2+x_n^3 \end{Bmatrix}$$

The bivariate cumulative distribution function (CDF) for the IID is derived using mathematical software, Wolfram [24], and given thus:

$$G(x_1x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f(s,t)\ dt\ ds \tag{16}$$

The CDF of the bivariate exponential distribution is given as

$$G(x_1x_2) = \int_0^{x_2}\int_0^{x_1}\theta_1\theta_2\ e^{-(\theta_1 s+\theta_2 t)}\ dt\ ds \tag{17}$$

$$G(x_1 x_2) = e^{-(\theta_1 x_1 + \theta_2 x_2)} \tag{18}$$

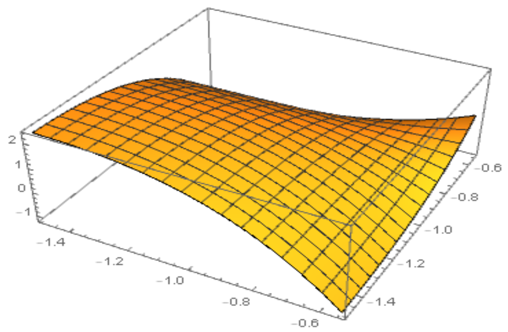

Fig. 3. The shape of the PDF of the Bivariate Juchez distribution

The CDF of the bivariate Lindley distribution is given as

$$G(x_1 x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+s)(1+t)]\, e^{-(\theta_1 s+\theta_2 t)}\, dt\, ds \tag{19}$$

$$G(x_1 x_2) = \left[1 + \frac{\theta_1 x_1+\theta_2 x_2+\theta_1\theta_2 x_1+\theta_1\theta_2 x_2+\theta_1\theta_2 x_1 x_2}{1+\theta_1+\theta_2+\theta_1\theta_2}\right] e^{-(\theta_1 x_1+\theta_2 x_2)} \tag{20}$$

The CDF of the bivariate Juchez distribution is given as

$$G(x_1 x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\left[\frac{\theta_1^4\ \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+s^2+s^3)(1+t^2+t^3)]\, e^{-(\theta_1 s+\theta_2 t)}\, dt\, ds \tag{21}$$

$$\rightarrow G(x_1 x_2) = \begin{bmatrix} 1 + \dfrac{\theta_1^3 x_1+3\theta_1^2 x_1^2+\theta_1^3 x_1^3}{6+\theta_1^2+\theta_1^3} + \dfrac{\theta_2^3 x_2+3\theta_2^2 x_2^2+\theta_2^3 x_2^3}{6+\theta_2^2+\theta_2^3} \\ + \dfrac{(\theta_1^3 x_1+3\theta_1^2 x_1^2+\theta_1^3 x_1^3)(\theta_2^3 x_2+3\theta_2^2 x_2^2+\theta_2^3 x_2^3)}{(6+\theta_1^2+\theta_1^3)(6+\theta_2^2+\theta_2^3)} \end{bmatrix} e^{-(\theta_1 x_1+\theta_2 x_2)} \tag{22}$$

To verify the validity of the IIDs, it suffices to state that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1 x_2)\, dx_1 dx_2 = 1 \tag{23}$$

Throughout the study, the equation in (23) is run in wolfram mathematica using the algorithm: $\{Integrate[f(x_1 x_2), \{x, 0, Infinity\}, \{y, 0, Infinity\}]\}$ for the verification of the properness or validity of the PDFs. The output: $\{\ ConditionalExpression[1, Re[a, b] > 0\}$ implies that the PDF is valid or proper; and can further be used for any relevant computations.

Let $X_1 X_2 \sim Exp(\theta)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1 x_2)\, dx_1 dx_2 = \int_0^{\infty}\int_0^{\infty} \theta_1\ \theta_2\ e^{-(\theta_1 x_1+\theta_2 x_2)}\, dx_1 dx_2 \tag{24}$$

$$\theta_1\ \theta_2 \int_0^{\infty}\int_0^{\infty} e^{-(\theta_1 x_1+\theta_2 x_2)}\, dx_1 dx_2 = \theta_1\ \theta_2\ \left[\frac{1}{\theta_1\ \theta_2}\right] = 1 \tag{25}$$

$X_1 X_2 \sim Lind(\theta)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1 x_2)\, dx_1 dx_2 = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]\, e^{-(\theta_1 x_1+\theta_2 x_2)}\, dx_1 dx_2 \tag{26}$$

$$\left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right]\int_0^\infty\int_0^\infty[(1+x_1)(1+x_2)]\,e^{-(\theta_1x_1+\theta_2x_2)}\,dx_1dx_2$$
$$=\left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right]\left[\frac{(\theta_1+1)(\theta_2+1)}{\theta_1^2\ \theta_2^2}\right]=1 \tag{27}$$

$X_1X_2 \sim Juc(\theta)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_1x_2)\,dx_1dx_2 = \int_0^\infty\int_0^\infty\left[\frac{\theta_1^4\ \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+x_1^2+x_1^3)(1+x_2^2+x_2^3)] * e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 \tag{28}$$

$$\left[\frac{\theta_1^4\ \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right]\int_0^\infty\int_0^\infty[(1+x_1^2+x_1^3)(1+x_2^2+x_2^3)]e^{-(\theta_1x_1+\theta_2x_2)}\,dx_1dx_2$$
$$=\left[\frac{\theta_1^4\ \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right]\left[\frac{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}{\theta_1^4\ \theta_2^4}\right]=1 \tag{29}$$

Independent and Non - Identical distributions (II_nD)

Here, we project bivariate combinations of different distributions following same product model in equation (9); say, Bivariate Exponential-Lindley Distribution, Bivariate Exponential-Juchez Distribution, Bivariate Lindley-Juchez Distribution.

Let $X_1\sim Exp(x_1;\theta_1)$ *and* $X_2\sim Lind(x_2;\theta_2)$, then the bivariate Exponential-Lindley derivation is given thus

$$g(x_1x_2)=\left[\frac{\theta_1\ \theta_2^2}{(\theta_2+1)}\right](1+x_2)\,e^{-(\theta_1x_1+\theta_2x_2)} \tag{30}$$

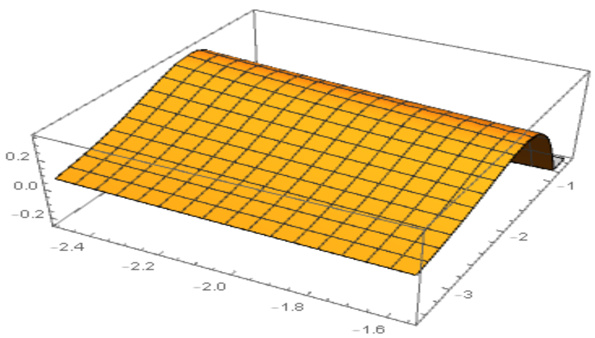

Fig. 4. The shape of the PDF of the Bivariate Exponential-Lindley distribution

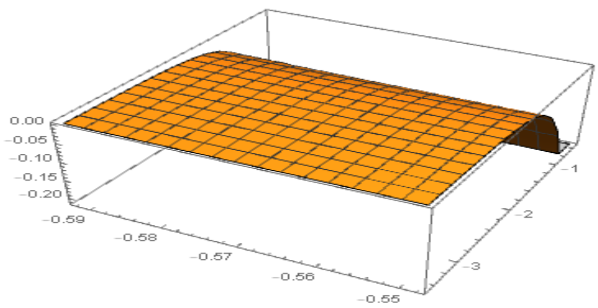

Fig. 5. The shape of the PDF of the Bivariate Exponential-Juchez distribution

Let $X_1\sim Exp(x_1;\theta_1)$ *and* $X_2\sim Juc(x_2;\theta_2)$, then the bivariate Exponential-Juchez derivation is given as

$$g(x_1x_2)=\left[\frac{\theta_1\ \theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\,e^{-(\theta_1x_1+\theta_2x_2)} \tag{31}$$

Let $X_1 \sim Lind(x_1;\theta_1)$ *and* $X_2 \sim Juc(x_2;\theta_2)$, then the bivariate Lindley-Juchez derivation is

$$g(x_1 x_2) = \left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1+x_1)(1+x_2^2+x_2^3)\, e^{-(\theta_1 x_1+\theta_2 x_2)} \tag{32}$$

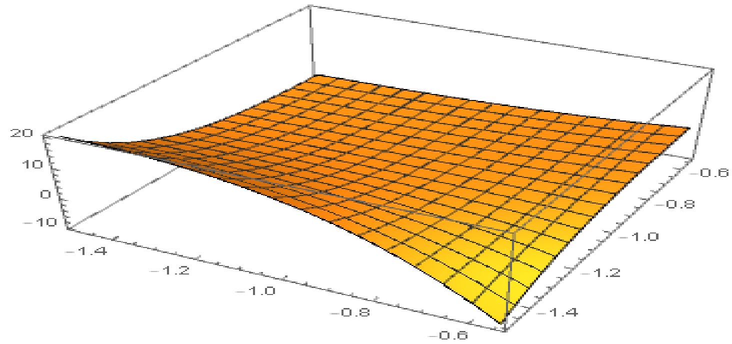

Fig. 6. The shape of the PDF of the Bivariate Lindley-Juchez Distribution

Let $X_1 \sim Exp(x_1;\theta_1)$, $X_2 \sim Lind(x_2;\theta_2)$ *and* $X_3 \sim Juc(x_3;\theta_3)$, then the multivariate Exponential-Lindley-Juchez derivation is

$$f(x_1 x_2 x_3) = \left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+x_2)(1+x_3^2+x_3^3)\, e^{-(\theta_1+\theta_2 x_2+\theta_3 x_3)} \tag{33}$$

From equation (16) the bivariate cumulative distribution function (CDF) for the II_nD is derived thus:

The CDF of the bivariate Exponential- Lindley distribution is given as

$$G(x_1 x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\left[\frac{\theta_1\ \theta_2^2}{(\theta_2+1)}\right](1+t)\, e^{-(\theta_1 s+\theta_2 t)}\, dt\, ds \tag{34}$$

$$\rightarrow \qquad G(x_1 x_2) = \left[1+\frac{\theta_2 x_2}{1+\theta_2}\right] e^{-(\theta_1 x_1+\theta_2 x_2)} \tag{35}$$

The CDF of the bivariate Exponential-Juchez distribution is given as

$$G(x_1 x_2) = \int_0^{x_2}\int_0^{x_1}\left[\frac{\theta_1\,\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+t^2+t^3)\, e^{-(\theta_1 s+\theta_2 t)}\, ds\, dt \tag{36}$$

$$\rightarrow \qquad G(x_1 x_2) = \left[1+\frac{6\theta_2 x_2+\theta_2^3 x_2+3\theta_2^2 x_2^2+\theta_2^3 x_2^3}{6+\theta_2^2+\theta_2^3}\right] e^{-(\theta_1 x_1+\theta_2 x_2)} \tag{37}$$

The CDF of the bivariate Lindley-Juchez distribution is given as

$$G(x_1 x_2) = \int_0^{x_2}\int_0^{x_1}\left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1+s)(1+t^2+t^3)\ e^{-(\theta_1 s+\theta_2 t)}\, ds\, dt \tag{38}$$

$$\rightarrow G(x_1 x_2) = \left[1+\frac{\begin{matrix}\theta_2^3+\theta_2^2+6+12\theta_2 x_2+2\theta_2^3 x_2+6\theta_2^2 x_2^2+2\theta_2^3 x_2^3+6\theta_2\theta_1 x_2+3\theta_2^2\theta_1 x_2^2\\ \theta_2^3\theta_1 x_2+\theta_2^3\theta_1 x_2^3+6\theta_1 x_1+6\theta_2\theta_1 x_2 x_1+\theta_2^2\theta_1 x_1+3\theta_2^2\theta_1 x_2^2 x_1\\ +\theta_2^3\theta_1 x_1+\theta_2^3\theta_1 x_2 x_1+\theta_2^3\theta_1 x_2^3 x_1\end{matrix}}{6+\theta_2^2+\theta_2^3+6\theta_1+\theta_2^2\theta_1+\theta_2^3\theta_1}\right] e^{-(\theta_1 x_1+\theta_2 x_2)} \tag{39}$$

The CDF of the Multivariate extension: Exponential-Lindley-Juchez distribution is given as

$$G(x_1x_2x_3) = \int_0^{x_3}\int_0^{x_2}\int_0^{x_1}\left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+s)(1+t+t^3)\ e^{-(\theta_1 r+\theta_2 s+\theta_3 t)}\ drdsdt \tag{40}$$

$$\rightarrow \qquad G(x_1x_2x_3) = -\left[1+\frac{\begin{array}{c}6\theta_3x_3+3x_3^2+\theta_3^3x_3+\theta_3^3x_3^3+6\theta_2\theta_3x_3+3\theta_2x_3^3+\\ \theta_2\theta_3^3x_3+\theta_2\theta_3^3x_3^3+6\theta_2x_2+6\theta_2\theta_3x_2x_3+b\theta_3^2x_2+\\ 3\theta_2x_2x_3^3+\theta_2\theta_3^3x_2+\theta_2\theta_3^3x_2x_3+\theta_2\theta_3^3x_2x_3^3\end{array}}{6+6\theta_2+\theta_3^2+\theta_2\theta_3^2+\theta_3^3+\theta_2\theta_3^3}\right]e^{-\theta_1x_1-\theta_2x_2-\theta_3x_3} \tag{41}$$

To verify the validity of the II_nDs, we recall equation (23):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)\ dx_1dx_2\ = 1$$

Recall that this wolfram mathematica algorithm: $\{\ \text{Integrate}[f(x_1x_2), \{x, 0, \text{Infinity}\}, \{y, 0, \text{Infinity}\}]\ \}$ is employed for the verification of the properness or validity of the PDFs.

Let $X_1X_2 \sim Exp - Lind(\theta)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)\ dx_1dx_2\ = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1\ \theta_2^2}{(\theta_2+1)}\right](1+x_2)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 \tag{42}$$

$$\left[\frac{\theta_1\ \theta_2^2}{(\theta_2+1)}\right]\int_0^{\infty}\int_0^{\infty}(1+x_2)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 = \left[\frac{\theta_1\ \theta_2^2}{(\theta_2+1)}\right]\left[\frac{(\theta_2+1)}{\theta_1\ \theta_2^2}\right] = 1 \tag{43}$$

$X_1X_2 \sim Exp - Juc(\theta)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)\ dx_1dx_2\ = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1\ \theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 \tag{44}$$

$$\left[\frac{\theta_1\ \theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right]\int_0^{\infty}\int_0^{\infty}(1+x_2^2+x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 = \left[\frac{\theta_1\ \theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right]\left[\frac{(\theta_2^3+\theta_2^2+6)}{\theta_1\ \theta_2^4}\right] = 1 \tag{45}$$

$X_1X_2 \sim Lind - Juc(\theta)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)\ dx_1dx_2\ = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1+x_1)(1+x_2^2+x_2^3)e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 \tag{46}$$

$$\left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\ \theta_2^2+6)}\right]\int_0^{\infty}\int_0^{\infty}(1+x_1)(1+x_2^2+x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2$$

$$\left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right]\left[\frac{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}{\theta_1^2\theta_2^4}\right] = 1 \tag{47}$$

$X_1X_2\ X_3 \sim Exp - Lind - Juc(\theta)$

$$\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2x_3)\ dx_1dx_2dx_3\ = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+x_2)(1+x_3^2+x_3^3) \tag{48}$$

$$*\ e^{-(\theta_1+\theta_2x_2\,+\,\theta_3x_3)}\ dx_1dx_2dx_3$$

$$\left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\ \theta_3^2+6)}\right]\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}(1+x_2)(1+x_3^2+x_3^3)\ e^{-(\theta_1x_1\,+\theta_2x_2\,+\,\theta_3x_3)}$$

$$= \left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right]\left[\frac{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}{\theta_1\theta_2^2\theta_3^4}\right] = 1 \tag{49}$$

3 Bivariate Properties for the Product Distributions

3.1 Conditional Bivariate Development

If X_1 and X_2 have a joint probability density function $f(x_1 x_2)$, then the conditional pdf of X, given that $X_2 = x_2$, is defined for all values of x_2 such that $(x_2) > 0$, by

$$\square(x_1|x_2) = \frac{f(x_1 x_2)}{f(x_2)} = f(x_1) \tag{50}$$

$$\square(x_2|x_1) = \frac{f(x_1 x_2)}{f(x_1)} = f(x_2) \tag{51}$$

By implication, the conditional properties for the independent identical and non-identical distributions are given thus:

$$h_{a_1:a_2}(x_1|x_2) = \frac{f_{a_1:a_2}(x_1 x_2)}{f_{a_1:a_2}(x_2)} = f_{a_1:a_2}(x_1) \tag{52}$$

$$h_{a_1:a_2}(x_2|x_1) = \frac{f_{a_1:a_2}(x_1 x_2)}{f_{a_1:a_2}(x_1)} = f_{a_1:a_2}(x_2) \tag{53}$$

$$h_{a:b}(x_1|x_2) = \frac{f_{a:b}(x_1 x_2)}{f_{a:b}(x_2)} = f_{a:b}(x_1) \tag{54}$$

$$h_{a:b}(x_2|x_1) = \frac{f_{a:b}(x_1 x_2)}{f_{a:b}(x_1)} = f_{a:b}(x_2) \tag{55}$$

where a_1: a_2 and $a : b$ implies independent identical and non-identical distributions respectively.

To further exemplify this, we select one, each of both IIDs and II_nDs; say Bivariate Lindley distribution and Exponential Juchez distribution.

$$h_{Lind1:Lind2}(x_1|x_2) = \frac{\left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]\ e^{-(\theta_1 x_1+\theta_2 x_2)}}{\left[\frac{\theta_2^2}{(\theta_2+1)}\right][1+x_2]\ e^{-(\theta_2 x_2)}} = \left[\frac{\theta_1^2}{(\theta_1+1)}\right][1+x_1]\ e^{-(\theta_1 x_1)} \tag{56}$$

$$h_{Lind2:Lind1}(x_2|x_1) = \frac{\left[\frac{\theta_1^2\ \theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]\ e^{-(\theta_1 x_1+\theta_2 x_2)}}{\left[\frac{\theta_1^2}{(\theta_1+1)}\right][1+x_1]\ e^{-(\theta_1 x_1)}} = \left[\frac{\theta_2^2}{\theta_2+1}\right][1+x_2]\ e^{-(\theta_2 x_2)} \tag{57}$$

$$h_{Exp:Juc}(x_1|x_2) = \frac{\left[\frac{\theta_1\ \theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_1 x_1+\theta_2 x_2)}}{\left[\frac{\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_2 x_2)}} = \theta_1 e^{-(\theta_1 x_1)} \tag{58}$$

$$h_{Juc:Exp}(x_2|x_1) = \frac{\left[\frac{\theta_1\ \theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_1 x_1+\theta_2 x_2)}}{\theta_1\ e^{-(\theta_1 x_1)}} = \left[\frac{\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_2 x_2)} \tag{59}$$

These show that for independent joint distribution, the conditional distribution equals a marginal of the bivariate distribution. This implies that the conditional expectation equals the mean of the marginal distribution.

3.2 Moments

The r^{th} moment of the Bivariate and Multivariate distributions is obtained thus:

$$E(x^r x^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\ f(x_1 x_2)\ dx_1 dx_2 \tag{60}$$

$$E(x^r x^s x^t) = \int_0^\infty \int_0^\infty \int_0^\infty x_1^r x_2^s\, x_3^t\, f(x_1 x_2 x_3)\;\; dx_1 dx_2 dx_3 \tag{61}$$

$$E(x^r x^s x^t \dots\, x^z) = \int_0^\infty \int_0^\infty \int_0^\infty \dots \int_0^\infty x_1^r x_2^s\, x_3^t \dots x_n^z\, f(x_1 x_2 x_3 \dots\, x_n)\;\; dx_1 dx_2 dx_3 \dots\, x_n \tag{62}$$

$X_1 X_2 \sim Exp(\theta)$

$$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\;\; \theta_1 \theta_2\, e^{-(\theta_1 x_1 + \theta_2 x_2)}\, dx_1 dx_2 \tag{63}$$

$$= \frac{\Gamma(1+r)\Gamma(1+s)}{\theta_1^r \theta_2^s};\quad r, s\; > \; -1\,, \qquad \theta_1 \theta_2 > 0$$

$$\rightarrow \quad E(x_1^{r=1} x_2^{s=1}) = \frac{1}{\theta_1 \theta_2} = \mu_{x_1\, x_2} \tag{64}$$

$X_1 X_2 \sim Lind(\theta)$

$$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\;\; \frac{\theta_1^2 \theta_2^2}{(1+\theta_1)(1+\theta_2)} (1 + x_1)\,(1 + x_2)e^{-(\theta_1 x_1 + \theta_2 x_2)}\, dx_1 dx_2 \tag{65}$$

$$= \frac{(1 + \theta_1 + r)(1 + \theta_2 + s)\Gamma(1+r)\Gamma(1+s)}{\theta_1^r \theta_2^s\, (1+\theta_1)(1+\theta_2)};\quad r, s\; > \; -1\,, \qquad \theta_1 \theta_2 > 0$$

$$\rightarrow\; E(x_1^{r=1} x_2^{s=1}) = \frac{(2+\theta_1)(2+\theta_2)}{\theta_1 \theta_2 (1+\theta_1)(1+\theta_2)} = \mu_{x_1\, x_2} \tag{66}$$

$X_1 X_2 \sim Juc(\theta)$

$$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\;\; \frac{\theta_1^4 \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)} (1 + x_1\; + x_1^3)\,(1 + x_2\; + x_2^3)e^{-(\theta_1 x_1 + \theta_2 x_2)}\, dx_1 dx_2 \tag{67}$$

$$= \frac{\left(\theta_1^3 + \theta_1^2(1+r) + (1+r)(2+r)(3+r)\right)\left(\theta_2^3 + \theta_2^2(1+s) + (1+s)(2+s)(3+s)\right)\Gamma(1+r)\Gamma(1+s)}{\theta_1^r \theta_2^s\, (\theta_1^3 + \theta_1^2 + 6)(\,\theta_2^3 + \theta_2^2 + 6)};r, s > -1\,, \theta_1 \theta_2 > 0$$

$$\rightarrow \qquad E(x_1^{r=1} x_2^{s=1}) = \frac{(\theta_1^3+2\theta_1^2+24)(\theta_2^3+2\theta_2^2+24)}{\theta_1 \theta_2 (\theta_1^3+\theta_1^2+6)(\,\theta_2^3+\theta_2^2+6)} = \mu_{x_1\, x_2} \tag{68}$$

$X_1 X_2 \sim Exp - Lind(\theta)$

$$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\;\; \frac{\theta_1 \theta_2^2}{(1+\theta_2)}\; (1 + x_2)e^{-(\theta_1 x_1 + \theta_2 x_2)}\, dx_1 dx_2 \tag{69}$$

$$= \frac{(1 + \theta_2 + s)\Gamma(1+r)\Gamma(1+s)}{\theta_1^r \theta_2^s\, (1+\theta_2)};\quad r, s\; > \; -1\,, \qquad \theta_1 \theta_2 > 0$$

$$\rightarrow \qquad E(x_1^{r=1} x_2^{s=1}) = \frac{2+\theta_2}{\theta_1 \theta_2 (1+\theta_2)} \tag{70}$$

$X_1 X_2 \sim Exp - Juc(\theta)$

$$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\;\; \frac{\theta_1 \theta_2^4}{(\,\theta_2^3+\theta_2^2+6)}\; (1 + x_2\; + x_2^3)e^{-(\theta_1 x_1 + \theta_2 x_2)}\, dx_1 dx_2 \tag{71}$$

$$= \frac{\left(\theta_1^3 + \theta_1^2(1+r) + (1+r)(2+r)(3+r)\right)\Gamma(1+r)\Gamma(1+s)}{\theta_1^r \theta_2^s\, (\,\theta_2^3 + \theta_2^2 + 6)};r, s > -1\,,\; \theta_1 \theta_2 > 0$$

$$\rightarrow \quad E(x_1^{r=1} x_2^{s=1}) = \frac{(\theta_2^3+2\theta_2^2+24)}{\theta_1 \theta_2 (\,\theta_2^3+\theta_2^2+6)} \tag{72}$$

$X_1 X_2 \sim Lind - Juc(\theta)$

$$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s\;\; \frac{\theta_1^2 \theta_2^4}{(1+\theta_1)(\,\theta_2^3+\theta_2^2+6)}\; (1 + x_1)(1 + x_2\; + x_2^3)e^{-(\theta_1 x_1 + \theta_2 x_2)}\, dx_1 dx_2 \tag{73}$$

$$= \frac{(1+\theta+r)\big(\theta_2^3+\theta_2^2(1+s)+(1+s)(2+s)(3+s)\big)\Gamma(1+r)\Gamma(1+s)}{\theta_1^r\theta_2^s\,(\theta_1^3+\theta_1^2+6)(\,\theta_2^3+\theta_2^2+6)}; r,s>-1\,,\theta_1\theta_2>0$$

$$E(x_1^{r=1}x_2^{s=1}) = \frac{(2+\theta_1)(\theta_2^3+2\theta_2^2+24)}{\theta_1\theta_2(1+\theta_1)(\,\theta_2^3+\theta_2^2+6)} \tag{74}$$

$X_1X_2X_3 \sim Exp - Lind - Juc(\theta)$

$$E(x_1^r x_2^s x_3^t) = \int_0^\infty\int_0^\infty\int_0^\infty x_1^r x_2^s x_3^t\ \frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\,\theta_3^3+\theta_3^2+6)}\ (1+x_1)(1+x_2+x_2^3)e^{-(\theta_1x_1+\theta_2x_2+\theta_3x_3)}\,dx_1dx_2dx_3 \tag{75}$$

$$= \frac{(1+\theta+r)\big(\theta_2^3+\theta_2^2(1+s)+(1+s)(2+s)(3+s)\big)\Gamma(1+r)\Gamma(1+s)}{\theta_1^r\theta_2^s\theta_3^t\,(1+\theta_2)(\,\theta_3^3+\theta_3^2+6)}; r,s>-1\,,\theta_1\theta_2>0$$

$$\rightarrow\ E(x_1^{r=1}x_2^{s=1}x_3^{t=1}) = \frac{(2+\theta_2)(\theta_3^3+2\theta_3^2+24)}{\theta_1\theta_2\theta_3(1+\theta_2)(\,\theta_3^3+\theta_3^2+6)} = \mu_{x_1x_2x_3} \tag{76}$$

3.3 Mean, covariance and coefficient of variation

Having obtained the r^{th} moment of the bivariate, we hence recall that evaluation: $E(x^r x^s)]_{r,s=1}$ and $E(x^r x^s x^t \dots x^z)]_{r,s,t,\ \dots\ ,z=1}$ are the means of the various derived distributions. The covariance and correlation of a bivariate distribution are given as:

$$Cov(X_1X_2) = E(X_1X_2) - E(X_1)E(X_2). \tag{77}$$

$$Cor(X_1X_2) = \frac{Cov(x_1x_2)}{\sqrt{Var(X)Var(Y)}} \tag{78}$$

The covariance of both bivariate IIDs and II_nDs are derived following the evaluations $E(x_1^{r=1}x_2^{s=1})$ as obtained in the bivariate moments of the distributions.

$$E(X_1X_2) - E(X_1)E(X_2) = Cov(X_1X_2)$$

For $X_1X_2 \sim Exp(\theta)$: $\frac{1}{\theta_1\theta_2} - \frac{1}{\theta_1}\frac{1}{\theta_2} = 0$ (79)

$$X_1X_2 \sim Lind(\theta):\quad \frac{(2+\theta_1)(2+\theta_2)}{\theta_1\theta_2(1+\theta_1)(1+\theta_2)} - \frac{(2+\theta_1)}{\theta_1(1+\theta_1)}\ \frac{(2+\theta_2)}{\theta_2(1+\theta_2)} = 0 \tag{80}$$

$$X_1X_2 \sim Juc(\theta):\quad \frac{(\theta_1^3+2\theta_1^2+24)}{\theta_1(\theta_1^3+\theta_1^2+6)} - \frac{(\theta_2^3+2\theta_2^2+24)}{\theta_2(\,\theta_2^3+\theta_2^2+6)} = 0 \tag{81}$$

$$X_1X_2 \sim Exp - Lind(\theta):\ \frac{2+\theta_2}{\theta_1\theta_2(1+\theta_2)} - \frac{1}{\theta_1}\ \frac{2+\theta_2}{\theta_2(1+\theta_2)} = 0 \tag{82}$$

$$X_1X_2 \sim Exp - Juc(\theta):\ \frac{(\theta_2^3+2\theta_2^2+24)}{\theta_1\theta_2(\,\theta_2^3+\theta_2^2+6)} - \frac{1}{\theta_1}\ \frac{(\theta_2^3+2\theta_2^2+24)}{\theta_2(\,\theta_2^3+\theta_2^2+6)} = 0 \tag{83}$$

$$X_1X_2 \sim Lind - Juc(\theta):\ \frac{(2+\theta_1)(\theta_2^3+2\theta_2^2+24)}{\theta_1\theta_2(1+\theta_1)(\,\theta_2^3+\theta_2^2+6)} - \frac{(2+\theta_1)}{\theta_1(1+\theta_1)}\ \frac{(\theta_2^3+2\theta_2^2+24)}{\theta_2(\,\theta_2^3+\theta_2^2+6)} = 0 \tag{84}$$

Consequently, the developed models are independent, and following calculations: $Cov(X_1, X_2) = 0$. This implies that the coefficient of variation is zero as well, for all the distributions in this study. More so, the correlation coefficient $\rho = \frac{Cov(x_1x_2)}{\sigma_{x_1}\sigma_{x_2}} = 0$, by the same implication.

3.4 Moment generating function

The moment generating function of Bivariate or Multivariate Distribution is derived thus:

$$M_{x_1x_2}(t_1,t_2) \ = \ E\ (\ e^{t_1x_1+t_2x_2}\ ) \ = \ \int_0^\infty \int_0^\infty e^{t_1x_1+t_2x_2}\ f(x_1x_2)\ dx_1dx_2 \tag{85}$$

$$M_{x_1x_2x_3}(t_1,t_2,t_3) \ = \ E\ (\ e^{t_1x_1+t_2x_2+t_3x_3}\ ) \ = \ \int_0^\infty \int_0^\infty e^{t_1x_1+t_2x_2}\ f(x_1x_2)\ dx_1dx_2 \tag{86}$$

$X_1X_2{\sim}Exp(\theta)$

$$M_{x_1x_2}(t_1,t_2) \ = \frac{\theta_1\theta_2}{(\theta_1-t_1)(\theta_2-t_1)} \tag{87}$$

$X_1X_2{\sim}Lind(\theta)$

$$M_{x_1x_2}(t_1,t_2) \ = \ \frac{\theta_1^2\theta_2^2(1+\theta_1-t_1)(1+\theta_2-t_2)}{(1+\theta_1)(1+\theta_2)(\theta_1-t_1)^2(\theta_2-t_2)^2} \tag{88}$$

$X_1X_2{\sim}Juc(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1^4\theta_2^4(6+(\theta_1-t_1)^2+(\theta_1-t_1)^3)(6+(\theta_2-t_2)^2(1+\theta_2-t_2))}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)(\theta_1-t_1)^4(\theta_2-t_2)^4} \tag{89}$$

$X_1X_2{\sim}Exp - Lind(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1\theta_2^2(1+\theta_2-t_2)}{(1+\theta_2)(\theta_1-t_1)(\theta_2-t_2)^2} \tag{90}$$

$X_1X_2{\sim}Exp - Juc(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1\theta_2^4(6+(\theta_2-t_2)^2+(\theta_2-t_2)^3)}{(\theta_2^3+\theta_2^2+6)(\theta_1-t_1)(\theta_2-t_2)^4} \tag{91}$$

$X_1X_2{\sim}Juc - Lind(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1^4\theta_2^2(6+(\theta_1-t_1)^2+(\theta_1-t_1)^3)(1+\theta_2-t_2)}{(\theta_1^3+\theta_1^2+6)(1+\theta_2)(\theta_1-t_1)^4(\theta_2-t_2)^2} \tag{92}$$

$X_1X_2X_3{\sim}Exp - Lind - Juc(\theta)$

$$M_{x_1x_2x_3}(t_1,t_2,t_3) = \frac{\theta_1\theta_2^2\theta_3^4(1+\theta_2-t_2)(6+(\theta_3-t_2)^2(1+\theta_3-t_3))}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)(\theta_1-t_1)(\theta_2-t)^2(\theta_3-t_3)^4} \tag{93}$$

3.5 Maximum likelihood estimator

For brevity purposes we estimate the parameters of Exponential-Lindley-Juchez distribution only. However, further derivations could be made for the parameters of every other independent distribution.

Let(X_{1i}, X_{2i}, X_{3i}) $i = 1,2,3, \dots, n$, be random vectors from Exponential-Lindley-Juchez multivariate distribution, the maximum likelihood estimator (MLE) is obtained thus:

$$\mathrm{L}f(x_1x_2x_3,\theta) = \left(\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3 + \ \theta_3^2+6)}\right)^n \prod_{i=1}^{n} (1+x_{i2})(1+x_{i3}^2+x_{i3}^3)$$
$$e^{-(\theta_1\sum_{i=1}^n x_{i_1} + \theta_2\sum_{i=1}^n x_{i_2} + \theta_3\sum_{i=1}^n x_{i_3})} \tag{94}$$

$$ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3) = nln\theta_1 + 2nln\theta_2 + 4nln\theta_3 - nln(1+\theta_2) - \ nln(\ \theta_3^3 + \ \theta_3^2 + 6)$$
$$+\sum_{i=1}^{n} \ln(1+x_{i3}^2+x_{i3}^3 + \ x_{i2} + x_{i2}x_{i3}^2 + x_{i2}x_{i3}^3)$$
$$-(\theta_1\sum_{i=1}^n x_{i_1} \ + \theta_2\sum_{i=1}^n x_{i_2} \ + \ \theta_3\sum_{i=1}^n x_{i_3}) \tag{95}$$

In estimation of the parameters, the estimator is maximized at

$\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1} = 0$, $\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2} = 0$ and $\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3} = 0$, then

$$\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1} = \frac{n}{\theta_1} - \sum_{i=1}^n x_{i_1} \ = 0 \tag{96}$$

$$\rightarrow \qquad \hat{\theta}_1 = \frac{1}{\bar{x}_1}$$

$$\frac{\partial \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2} = \frac{2n}{\theta_2} - \frac{n}{1+\theta_2} - \sum_{i=1}^{n} x_{i_2} = 0 \tag{97}$$

$$\frac{\partial \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_3} = \frac{4n}{\theta_3} - \frac{n(3\theta_3^2 + 2\theta_3)}{\theta_3^3 + \theta_3^2 + 6} - \sum_{i=1}^{n} x_{i_3} = 0 \tag{98}$$

The likelihood equations in (96), (97) and (98) can easily be solved iteratively using Fisher's scoring method due to the closed form equations obtained. Fisher's scoring method is a form of Newton's method used in resolving for MLE's numerically: $(\hat{\theta}_i - \theta_i) = \frac{I'(\theta_i)}{I''(\theta_i)} = \frac{\frac{\partial \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_i}}{\frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_i \partial \theta_j}}$. We have thus:

$$\frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1 \partial \theta_3} = \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2 \partial \theta_3} = 0 \tag{99}$$

$$\frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1^2} = -\frac{n}{\theta_1^2} \tag{100}$$

$$\frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2^2} = -\frac{2n}{\theta_2^2} + \frac{n}{(1+\theta_2)^2} \tag{101}$$

$$\frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_3^2} = -\frac{4n}{\theta_3^2} + \frac{(2\theta_3 + 3\theta_3^2)^2}{(6+\theta_3^2+\theta_3^3)^2} - \frac{2+6\theta_3}{6+\theta_3^2+\theta_3^2} \tag{102}$$

Resolving the following matrix equations, the solutions of MLE ($\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\theta}_3$) for $f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)$ are obtained:

$$\begin{bmatrix} \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1^2} & \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1 \partial \theta_3} \\ \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2^2} & \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2 \partial \theta_3} \\ \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_3 \partial \theta_1} & \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_3 \partial \theta_1} & \frac{\partial^2 \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_3^2} \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 = \theta_{01} \\ \hat{\theta}_2 = \theta_{02} \\ \hat{\theta}_3 = \theta_{03} \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_1} \\ \frac{\partial \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_2} \\ \frac{\partial \ln \mathrm{L} f(x_1 x_2 x_3, \theta_1 \theta_2 \theta_3)}{\partial \theta_3} \end{bmatrix} \tag{103}$$

where θ_{01}, θ_{02} and θ_{03} are initial values of $\hat{\theta}_1 \hat{\theta}_2$ and $\hat{\theta}_3$.

3.6 Bivariate reliability analysis

The Reliability function models the probability that a component will survive after a given time; whereas, hazard function is the likelihood that a system will terminate after a given period of time or cycle. The reliability $R(x, \theta)$ and hazard rate models $H(x, \theta)$ of bivariate distributions are given as:

$$R(x_1 x_2, \theta) = P(X_1 \geq x_1, X_2 \geq x_2) = \int_{x_1}^{\infty} \int_{x_2}^{\infty} f(t_1, t_2)\, dt_1 dt_2 \tag{104}$$

$$R(x_1, x_2, x_3, \theta) = P(X_1 \geq x_1, X_2 \geq x_2, X_3 \geq x_3) = \int_{x_1}^{\infty} \int_{x_2}^{\infty} \int_{x_3}^{\infty} f(t_1, t_2, t_3)\, dt_1 dt_2 dt_3 \tag{105}$$

$$\rightarrow \qquad R(x_1, x_2, \theta) = 1 - G(X_1) - G(X_2) + G(X_1, X_2) \tag{106}$$

$$R(x_1, x_2, x_3, \theta) = 1 - G(X_1) - G(X_2) - G(X_3) + G(X_1, X_2, X_3) \tag{107}$$

We consider (random selection of) two bivariate: say, Exponential-Exponential from IID and Exponential-Lindley; and the Exponential-Lindley-Juchez multivariate distributions from II_nD.

$$R_{Exp-Exp}(x_1, x_2, \theta) = 1 - \left[1 - e^{-\theta_1 x_1}\right] - \left[1 - e^{-\theta_2 x_2}\right] + \left[e^{-(\theta_1 x_1 + \theta_2 x_2)}\right] \tag{108}$$

$$R_{Exp-Lind}(x_1, x_2, \theta) = 1 - \left[1 - e^{-\theta_1 x_1}\right] - \left[1 - \left(\frac{\theta_2 + 1 + \theta_2 x_2}{\theta_2 + 1}\right) e^{-\theta_2 x_2}\right] + \left[\left(1 + \frac{\theta_2 x_2}{1+\theta_2}\right) e^{-(\theta_1 x_1 + \theta_2 x_2)}\right] \tag{109}$$

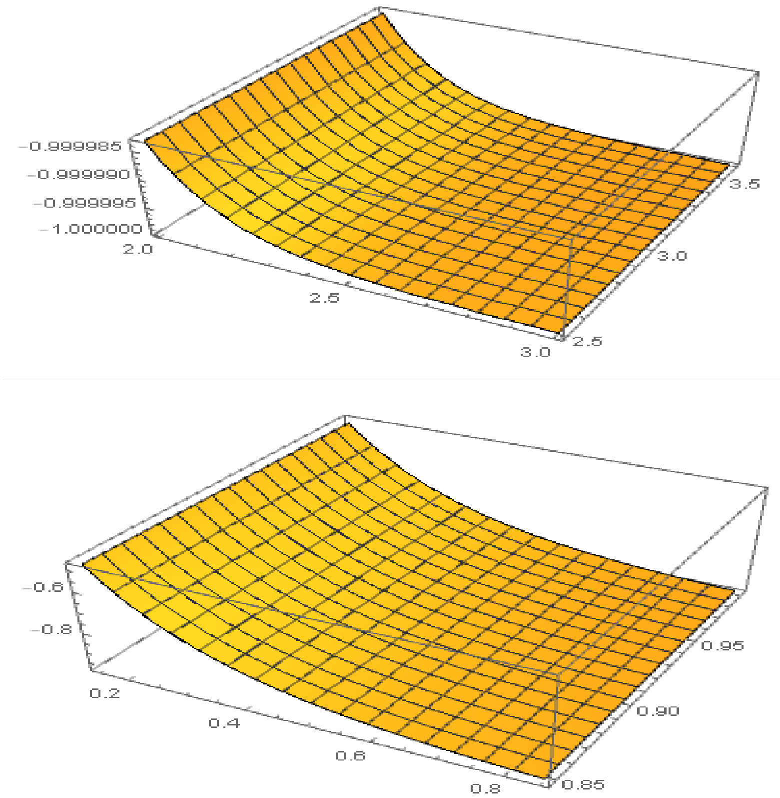

Fig. 7. (a)and (b). The shape of the reliability function of the bivariate exponential-exponential distribution and the bivariate exponential-lindley distribution

$$R_{Exp-Lind-Juc}(x_1, x_2, x_3, \theta) = 1 - \left[1 - e^{-\theta_1 x_1}\right] - \left[1 - \left(\frac{\theta_2 + 1 + \theta_2 x_2}{\theta_2 + 1}\right) e^{-\theta_2 x_2}\right] - \left[1 - \left(1 + \frac{\theta x\left[\theta^2 + \theta^2 x^2 + 3\theta x + 6\right]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x}\right] \tag{110}$$

$$+\left[-\left(1 + \frac{\begin{array}{c}6\theta_3 x_3 + 3x_3^2 + \theta_3^3 x_3 + \theta_3^3 x_3^3 + 6\theta_2\theta_3 x_3 + 3\theta_2 x_3^3 + \\ \theta_2\theta_3^3 x_3 + \theta_2\theta_3^3 x_3^3 + 6\theta_2 x_2 + 6\theta_2\theta_3 x_2 x_3 + b\theta_3^2 x_2 + \\ 3\theta_2 x_2 x_3^3 + \theta_2\theta_3^3 x_2 + \theta_2\theta_3^3 x_2 x_3 + \theta_2\theta_3^3 x_2 x_3^3\end{array}}{6 + 6\theta_2 + \theta_3^2 + \theta_2\theta_3^2 + \theta_3^3 + \theta_2\theta_3^3}\right) e^{-\theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3}\right] \tag{111}$$

$$\rightarrow \qquad H(x_1 x_2, \theta) = \frac{f(x_1 x_2, \theta)}{R(x_1 x_2, \theta)} \qquad and \qquad H(x_1, x_2, x_2, \theta) = \frac{f(x_1, x_2, x_2, \theta)}{R(x_1, x_2, x_2, \theta)} \tag{112}$$

3.7 Renewal property

In the modeling of two-dimensional queuing and renewal processes, in scenarios, where the arrival and service times are dependent; bivariate distributions can be employed. The renewal properties are obtained with the help of Laplace transformation: $E[\exp(-t_1x_1 - t_2x_2)]$.

Let X_1X_2 be a random vector with joint PDF $f(x_1x_2)$ then the function is derived thus:

$$\varphi_{X_1X_2}(t_1, t_2) = E[\exp(-t_1x_1 - t_2x_2)] \tag{113}$$

$$= \int_0^\infty \int_0^\infty \exp(-t_1x_1 - t_2x_2)\ f(x_1x_2)\ dx_1dx_2 \tag{114}$$

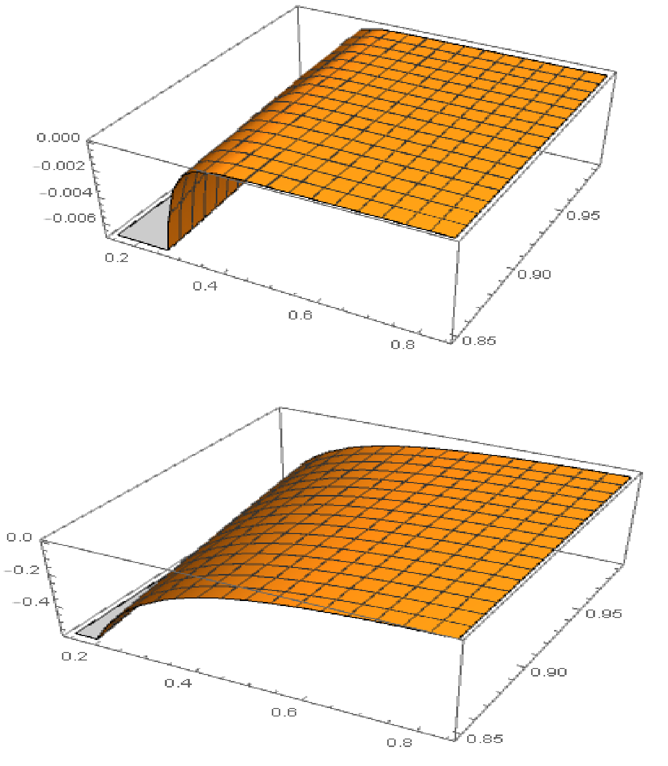

Fig. 8 (a)and (b). The shape of the hazard function of the bivariate exponential-exponential distribution and the bivariate exponential-lindley distribution

Concisely, the bivariate distributions randomly chosen for this representation are Bivariate Exponential-Juchez, Bivariate Lindley-Lindley and Bivariate Juchez-Lindley distributions:

$$\varphi_{Exp-Juc}(t_1, t_2) = \frac{\theta_1\theta_2^4(6+(\theta_2+t_2)^2+(\theta_2+t_2)^3)}{(\theta_1-t_1)(6+\theta_2^2+\theta_2^3)(\theta_2+t_2)^4}, \quad [\theta_2 + t_2] > 0 \tag{115}$$

$$\varphi_{Lind-Lind}(t_1, t_2) = \frac{(1+\theta_1+t_1)(1+\theta_2-t_2)}{(\theta_1+t_1)^2(\theta_2-t_2)^2}, \qquad [\theta_1 + t_1] > 0 \tag{116}$$

$$\varphi_{Juc-Lind}(t_1, t_2) = \frac{\theta_1^4\theta_2^2(6+(\theta_1+t_1)^2+(\theta_1+t_1)^3)(1+\theta_2-t)}{(6+\theta_1^2+\theta_1^3)(1+\theta_2)(\theta_1+t_1)^4(\theta_2-t_2)^2}, \quad [\theta_1 + t_1] > 0 \tag{117}$$

Equations (102)(103)&(104) are the two-dimensional queuing and renewal process models.

Table 1. Statistical Table for the Bivariate Juchez and Bivariate Lindley-Juchez distribution

	Bivariate Juchez		$(\theta_1 = 0.5, \theta_2 = 0.5)$		**Bivariate Lindley-Juchez**			
$X_1 \| X_2$	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
1	0.000318	0.007077	0.001209	0.001633	0.003606	0.003281	0.002654	0.002011
2	0.000707	0.001574	0.002690	0.003632	0.008021	0.007297	0.005902	0.004474
3	0.001209	0.002690	0.004599	0.006208	0.013710	0.012474	0.010087	0.007648
4	0.001633	0.003632	0.006208	0.008381	0.018509	0.016840	0.013618	0.010325
5	0.001881	0.004182	0.007149	0.009651	0.021314	0.019391	0.015682	0.011890
6	0.001942	0.004318	0.007381	0.009965	0.022007	0.020022	0.016192	0.012276
7	0.001854	0.004123	0.007047	0.009513	0.021009	0.019114	0.015457	0.011719
8	0.001669	0.003711	0.006344	0.008565	0.018914	0.017208	0.013916	0.010551
9	0.001436	0.003193	0.005458	0.007368	0.016272	0.014805	0.011973	0.009077
10	0.001191	0.002650	0.004529	0.006114	0.013502	0.012284	0.009934	0.007532
11	0.000960	0.002135	0.003649	0.004926	0.010878	0.009897	0.008004	0.006069
12	0.000755	0.001678	0.002869	0.003873	0.008554	0.007782	0.006294	0.004772
13	0.000581	0.001293	0.002210	0.002984	0.006589	0.005994	0.004847	0.003675
14	0.000440	0.000978	0.001673	0.002258	0.004987	0.004537	0.003669	0.002782
15	0.000328	0.000729	0.001246	0.001683	0.003717	0.003382	0.002735	0.002074
16	0.000241	0.000537	0.000917	0.001238	0.002735	0.002488	0.002012	0.001525
17	0.000175	0.000390	0.000667	0.000901	0.001989	0.001809	0.001463	0.001109
18	0.000126	0.000281	0.000480	0.000648	0.001431	0.001302	0.001053	0.000798
19	0.000090	0.000200	0.000342	0.000462	0.001021	0.000929	0.000751	0.000569
20	0.000063	0.000142	0.000242	0.000326	0.000722	0.000657	0.000531	0.000403

Table 2. Statistical Table for the Bivariate Exponetial and Bivariate Exponetial-Lindley distribution

	Bivariate Exponential		$(\theta_1 = 0.5, \theta_2 = 0.5)$		**Bivariate Exponential-Lindley**			
$X_1 \| X_2$	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
1	0.091970	0.055782	0.033838	0.020521	0.061313	0.057825	0.045111	0.034202
2	0.055783	0.038338	0.020521	0.012447	0.037188	0.033833	0.027362	0.020744
3	0.038338	0.020521	0.012446	0.007549	0.022555	0.020521	0.016596	0.012582
4	0.020513	0.012446	0.007549	0.004579	0.013680	0.012446	0.010066	0.007631
5	0.012447	0.007549	0.004578	0.002777	0.008297	0.007549	0.006105	0.004629
6	0.007549	0.004578	0.002777	0.001684	0.005032	0.004579	0.003703	0.002807
7	0.004578	0.002777	0.001684	0.001021	0.003052	0.002777	0.002459	0.001703
8	0.002777	0.001684	0.001021	0.000619	0.001851	0.001684	0.001362	0.001033
9	0.001684	0.001021	0.000619	0.000376	0.001229	0.001022	0.000826	0.000626
10	0.001022	0.000619	0.000376	0.000228	0.000681	0.000620	0.000501	0.000379
11	0.000619	0.000376	0.000228	0.000138	0.000413	0.000376	0.000303	0.000230
12	0.000375	0.000228	0.000138	0.000083	0.000250	0.000228	0.000184	0.000139
13	0.000227	0.000138	0.000083	0.000051	0.000152	0.000138	0.000111	0.000084
14	0.000138	0.000083	0.000051	0.000031	0.000092	0.000084	0.000068	0.000051
15	0.000084	0.000051	0.000031	0.000019	0.000056	0.000051	0.000041	0.000031
16	0.000051	0.000031	0.000019	0.000011	0.000034	0.000031	0.000025	0.000019
17	0.000031	0.000019	0.000011	0.000007	0.000021	0.000019	0.000015	0.000011
18	0.000019	0.000011	0.000007	0.000004	0.000012	0.000011	0.000009	0.000007
19	0.000011	0.000007	0.000004	0.000003	0.000008	0.000007	0.000006	0.000004
20	0.000007	0.000004	0.000003	0.000002	0.000005	0.000004	0.000003	0.000003

3.8 Bivariate probability patterns

In this section, we limit the generation of statistical table to just two bivariate distributions for brevity purposes; hence the choice of the distribution selection is purely random. This is with the intent to investigate the trend or the shape of the distributions; and at same time test whether the probability axiom: $0 \leq P(x_1, x_2) \leq 1$ is consistent across various parametric and variable values. The values are obtained by making numerical substitution for $\theta_1, \theta_2, X_1, X_2$ in the chosen Bivariate PDFs.

In Tables 1-3, the probability patterns is examined; and very explicitly, the output validate the probability axioms, where the range of values obtained are within 0 and 1. By carefully examining, vertically across the variables we notice a monotonic downward movement; in other words, as X increases the probability values tend to zero. This consequently suggests that the bivariate distributions are unimodal and the probability outcomes, asymptotic.

Table 3. Statistical Table for the Bivariate Lindley and Bivariate Exponential-Juchez distribution

	Bivariate Lindley			$(\boldsymbol{\theta_1 = 0.5}, \boldsymbol{\theta_2 = 0.5})$		**Bivariate Exponential-Juchez**		
$\boldsymbol{X_1 \| X_2}$	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
1	0.040875	0.037188	0.030074	0.028013	0.005410	0.012031	0.020565	0.027764
2	0.037188	0.033838	0.027362	0.020745	0.003281	0.007297	0.012474	0.016839
3	0.030075	0.027362	0.022127	0.016776	0.001990	0.004426	0.007566	0.010214
4	0.022801	0.020744	0.016776	0.012719	0.001207	0.002684	0.004588	0.006195
5	0.016596	0.015098	0.012210	0.009257	0.000732	0.001628	0.002783	0.003757
6	0.011743	0.010684	0.008640	0.006550	0.000444	0.000987	0.001688	0.002279
7	0.008140	0.007405	0.005989	0.004541	0.000269	0.000599	0.001024	0.001382
8	0.005554	0.005053	0.004087	0.003098	0.000163	0.000363	0.000621	0.000838
9	0.003743	0.003405	0.002754	0.002088	0.000099	0.000220	0.000377	0.000508
10	0.002497	0.002272	0.001837	0.001393	0.000060	0.000133	0.000228	0.000308
11	0.001653	0.001503	0.001215	0.000921	0.000036	0.000081	0.000139	0.000187
12	0.001086	0.000988	0.000799	0.000605	0.000022	0.000049	0.000084	0.000113
13	0.000709	0.000645	0.000521	0.000396	0.000013	0.000029	0.000051	0.000069
14	0.000461	0.000419	0.000339	0.000257	0.000008	0.000018	0.000031	0.000042
15	0.000298	0.000272	0.000219	0.000166	0.000005	0.000010	0.000019	0.000025
16	0.000192	0.000175	0.000141	0.000107	0.000003	0.000006	0.000011	0.000015
17	0.000123	0.000122	0.000091	0.000068	0.000002	0.000004	0.000007	0.000009
18	0.000079	0.000072	0.000058	0.000044	0.000001	0.000002	0.000004	0.000006
19	0.000050	0.000046	0.000037	0.000028	0.0000007	0.000001	0.000003	0.000003
20	0.000032	0.000029	0.000024	0.000018	0.0000004	9×10^{-07}	0.000002	0.000002

4 Conclusion

The paper aimed at developing different bivariate distributions, using mixture model offspring as the baseline or marginal distributions. Product distribution approach was used for the model developmental constructions of bivariate Independent Identical Distributions (IIDs) and Independent Non-Identical Distributions (II_nD). Under IIDs, A New Bivariate Exponential, Bivariate Lindley, Bivariate Juchez Distributions are constructed; whereas, Bivariate Exponential-Lindley, Bivariate Exponential-Juchez and Bivariate Lindley-Juchez Distributions are derived under (II_nDs). The validity or the properness of the PDFs, the verification of the statistical properties and all the mathematical bottlenecks were, carried out in both Mathematica and R software. Some properties considered are: the shape of the bivariate PDFs, moments, moment generating function, mean, covariance, coefficient of correlation, maximum likelihood estimator, reliability analysis, renewal property and probability patterns.

The probability patterns confirm with the shape of the PDFs that the various bivariate distributions, as derived in this study, are unimodal and non-normal. The conditional distributions are equal to the marginal, due to independence; by implication, the conditional expectations also are equal to the mean of the marginal distributions. More so, the covariance was calculated to be zero; and it then follows that coefficient of correlation is zero. Finally the models derived under renewal properties could be used to model two-dimensional queuing and renewal processes, for situations where the arrival and service times are dependent.

5 Future Work

Having adopted product distribution in the development of bivariate models, using the category of distributions from mixture distribution, other developmental approaches like Marginal transformation, Copula Method, Method of Mixing and Compounding, Trivariate Reduction Method, Frailty Approach and Conditional

Specification Method, could be explored as well; and the performance comparison among other counterpart bivariate models could be done to test for better fit across models with respect to various bivariate data emanating from different fields of life.

Competing Interests

Authors have declared that no competing interests exist.

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